Publications of the Astronomical Society of the Pacific, **122**:215–223, 2010 February

Effects of Nonuniform Input Spectra on Signal-to-Noise Ratio in Wide-Bandwidth Digital Correlation

Shing Kwong Wong[1,2,3] and Tzihong Chiueh[1,2,3,4]

Received 2009 October 26; accepted 2009 December 22; published 2010 January 20

ABSTRACT. In a low-bit sampling digital correlator for wide-bandwidth interferometry observations, nonuniform spectra of the analog input can degrade the correlator efficiency. In this work we evaluate this issue in detail, particularly for correlators having fine spectral resolution. We find the degradation to be due to nonlinear transfer of noise among different frequency channels, thereby altering the per channel signal-to-noise ratio (S/N) in an unfavorable manner with low-power channels having worse S/N and high-power channels, better S/N. (The favorable S/N in high-power channels arise primarily from effective oversampling.) To the leading order, the favorable and unfavorable S/N at different channels can largely cancel and the S/N degradation occurs as a second-order effect. However, when the two input spectra for correlation deviate from each other, such a cancellation mechanism may be suppressed.

Online material: color figures

1. INTRODUCTION

Since the first invention by Weinreb (1961), digital correlators have been gaining increasing popularity in radio astronomy applications for more than three decades. This is partly due to the rapid development of fast digital signal processors, such as FPGA and ASIC, and partly due to the demand for flexible maneuvers of measured data, such as very long delay correlation and fringe stopping, etc. The particular observations targeting at the radio continuum are even required to cover a large bandwidth to acquire as much astronomical source information as possible. For an astronomical source with no spectral line feature, the detection signal-to-noise ratio (S/N) normally scales as the square root of the bandwidth. It is for such applications modern receivers are designed to have tens of gigahertz bandwidth and digital correlators are to process data at a several gigahertz speed.

However, wide-bandwidth receivers often suffer from nonuniform spectral response when the signal is transmitted through the front-end electronics to the back-end digital processor. For example, the multiple stages of amplification may severely distort the analog input spectrum, the cables and filters tend to be more lossy for high frequency than for low frequency thereby creating a spectral slope, etc. In digital systems, the system spectrum can also be affected by the sampler frequency response. While the spectral slope may be corrected by a slope equalizer, nonmonotonic spectral distortion is often difficult to correct. Therefore nonuniform data spectra are a common problem for wide-bandwidth digital correlators. We are so motivated to address this issue in this article from the perspective of degradation of S/N.

Enhanced noises pertinent to digitization has been studied in the past, and most of these studies are pioneered from engineer's perspectives of signal integrity for large signals (Widrow 1956, 1961; Widrow et al. 1996). However, signals tend to be orders of magnitude smaller than noise in astronomy observations, and it is only possible to recover signals statistically. It is therefore of great importance in astronomical observation that the statistical property of noise is understood as fully as possible in order to optimize the retrieval of small signals. Several earlier works have addressed this issue in the context of digital sampling and correlation for astronomical data (Cole 1968; Cooper 1970; Roberts 1997; Kouwenhoven & Voute 2001). However, very few works discussed the spectral distortion upon digitization and how the signal retrieval for a broad band system is affected by such distortion. Even for the few works studying spectral distortion due to digitization, crude approximation, valid for high-bit digitization, has been adopted and the nonlinearity of low-bit digitization is not fully explored (Bennett 1948; Sripad & Snyder 1977; Gary 1990). The effects of spectral slope on sensitivity have been addressed previously in publicly available memos (Lamb 2001; Thompson & Emerson 2005).

In this work, we study the issue of digitized spectral distortion in a systematic way, with an attempt to uncover the nonlinear power transfer mechanism of digitization and to evaluate

[1] Department of Physics, National Taiwan University, 106, Taipei, Taiwan; d92222012@ntu.edu.tw, chiuehth@phys.ntu.edu.tw.

[2] Center for Theoretical Sciences, National Taiwan University, 106, Taipei, Taiwan.

[3] LeCosPA, National Taiwan University, 106, Taipei, Taiwan.

[4] Center for Quantum Sciences and Engineering, National Taiwan University, 106, Taipei, Taiwan.

the adverse effects of nonuniform input spectrum quantitatively. The paper is organized as follows. In § 2, we provide a basic analysis for digitization of input that consists of a weak correlated signal in the presence of strong noise, and for optimization of S/N for broadband sources. Section 4 explicitly derives the optimized S/N for noise that obeys the Gaussian distribution. We quantify the degradation of nonflat spectrum for various cases in § 5. When high-frequency channels are highly suppressed compared to low-frequency channels, the system's effective bandwidth is reduced and it becomes an oversampling system; the performance gain of oversampling is addressed in § 6. Section 7 discusses the performance degradation when the respective spectra of two inputs are different. Finally, the conclusion is given in § 8.

2. BASIC ANALYSIS

The digital samplers of a correlation system receive input analog electric fields from two distinct receivers, Rx1 and Rx2. In each receiver, independent white noise from various sources is introduced into the signal prior to its arrival at the correlator. We decompose the inputs from both receivers into uncorrelated white noise and the correlated signals, where the correlated signals are those that the correlator will measure whereas the uncorrelated white noise is to be integrated out. The Fourier components of the analog inputs are written as follows: the Gaussian white noise, $\tilde{w}_{(1)}(\nu)$ and $\tilde{w}_{(2)}(\nu)$, and the correlated signals, $\tilde{u}_{(1)}(\nu)$ and $\tilde{u}_{(2)}(\nu)$. Although not necessary, it would be a good practice to include a decomposition between w and u, since they are of different physical origins. In astronomy interferometer systems, u is the signal from the radio source on the sky, whereas w arises primarily from the noise of the first-stage amplifier of the receiver.

Practically, the measurement is taken for a finite time interval of integration. For simplicity we adopt an approximately equivalent approach to assume that the signals are periodic for the integration time interval and the period is normalized to 1. Thus the frequency is discrete, $\nu \in \mathbf{Z}$. The astronomical radio sources have constant flux, and for these random phase signals, the statistical behavior of the signals is invariant with respect to time, and the correlation of the signals only exists between identical frequencies, i.e.,

$$\langle \tilde{u}_{(1)}(\nu)\tilde{u}_{(2)}(\nu')\rangle = \tilde{c}_{(12)}(\nu)\delta(\nu+\nu'). \tag{1}$$

In reality, the signal has a finite spectral bandwidth, and both active and passive components on the transmission paths of signals can alter the signal spectral shapes. We model the analog front-end system as a linear response function followed by the digitization transfer function acting on discrete points in the time domain. The front-end linear response function is a function of frequency $\tilde{R}_{(i)}(\nu)$ acting on the electric fields. The analog input can be written as

$$\tilde{A}_{(i)}(\nu) = [\tilde{w}_{(i)}(\nu) + \tilde{u}_{(i)}(\nu)]\tilde{R}_{(i)}(\nu). \tag{2}$$

Here the response functions $\tilde{R}_{(i)}(\nu)$ of different receiver front ends are not necessarily the same. The variance of phase distortion on the signals due to the receiver front-end response $R_{(i)}$ can be calibrated out provided that the correlator has sufficient frequency resolution, but a moderate difference between the magnitude of the spectral response will cause nontrivial effect and may further degrade the efficiency of the correlator. We will discuss this issue in § 7. Until then we shall assume $\tilde{R}_{(i)}(\nu)$ to be the same for the two analog inputs for the convenience of our discussion.

The two samplers of the two receivers generate data sets at a finite sampling rate. To retrieve the maximum bandwidth of the signals, the sampling rate Γ should be at least twice the system bandwidth. In time sequence, the input is sampled at discrete points q as

$$A_{(i)}(q) = \sum_{\nu} \tilde{A}_{(i)}(\nu) e^{2\pi i \nu q}, \tag{3}$$

where $q \in \{\frac{0}{\Gamma}, \frac{1}{\Gamma}, \frac{2}{\Gamma}, \ldots, \frac{\Gamma-1}{\Gamma}\}$ and the summation sums over relevant modes in both positive and negative ranges : $\nu = 1, \ldots, Nq-1$, and $-1, \ldots, -(Nq-1)$, where DC and the Nyquist mode correspond to $\nu = 0$, Nq.

For a digital sampling correlator system, the digitization transfer function translates the sampled analog signals into a finite set of digital levels, in accordance to a digitization function F, where digitized signals $d_{(i)}(q) = F(A_{(i)}(q))$. For a 1 bit digitization sampler, we have

$$F(x) = \mathrm{sgn}(x) = \begin{cases} +1 & \text{for } x > 0 \\ \text{undetermined} & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases} \tag{4}$$

Since $x = 0$ has a measure of zero, we ignore this case in the forthcoming discussions. In many applications, particularly in astronomy, the correlated signal u has a much smaller amplitude than w, giving $\tilde{c}_{(12)}(\nu) \ll 1$. It is useful to define the zeroth-order analog amplitudes,

$$A_{0(i)}(q) = A_{(i)}(q)|_{u=0}. \tag{5}$$

A measurement is performed by accumulating the correlated data for a finite time. What is done by the correlator is actually taking the products between sampled data of the same frequency from the two samplers and linearly combining the products of all frequencies. Different choices of combination coefficients optimize measurements of different physical quantities. For a given measured quantity, there exists a set of optimized coefficients $\tau_{qq'}$, where the optimized measurement $D_{(12)} = \sum_{q_1 q_2} \tau_{q_1 q_2} d_{(1)}(q_1) d_{(2)}(q_2)$. The expectation value of the measured result is simply the ensemble average of D_{12}, i.e., $\mathbf{S} = \langle D_{(12)}\rangle$. We will discuss our method to optimize the coefficient set τ in the next section.

Expanding around the point where $u_{(1)} = u_{(2)} = 0$ gives us the Taylor series of the expectation value over u. As the digitization function F is odd, the zeroth-order terms in u will be averaged out. By taking the first order term in u we get

$$\begin{aligned}\mathbf{S} \approx & \sum_{q_1 q_2} \tau_{q_1 q_2} \sum_{\nu_1 \nu_2} \langle \tilde{u}_{(1)}(\nu_1) \tilde{R}(\nu_1) e^{2\pi i \nu_1 q_1} \tilde{u}_{(2)}(\nu_2) \tilde{R}(\nu_2) e^{2\pi i \nu_2 q_2} \rangle \\ & \times \langle F'(A_{0(1)}(q_1)) \rangle \langle F'(A_{0(2)}(q_2)) \rangle \\ = & \sum_{q_1 q_2} \tau_{q_1 q_2} \Phi^2 \sum_{\nu} \tilde{c}_{(12)}(\nu) |\tilde{R}|^2(\nu) e^{2\pi i \nu (q_1 - q_2)}, \end{aligned} \tag{6}$$

where $\Phi = \langle F'(A_0(q)) \rangle$. It is important to note that ideal digitization is not only a linear operation for small-amplitude signals, but it also retains the original signal spectral shape.

Although the expansion of F into Taylor series seems odd since the derivatives of digitization transfer functions diverge, the divergence is actually integrable. The derivative is a δ-function, but the ensemble average keeps it finite.

On the other hand, the square variance of the measured correlation noise is the zeroth-order ensemble average over D_{12}^2,

$$\begin{aligned}\mathbf{N}^2 = \langle D_{0(12)}^2 \rangle = & \sum_{q_1 q_2 q_3 q_4} \tau_{q_1 q_2} \tau_{q_3 q_4} \langle F(A_{0(1)}(q_1)) F(A_{0(1)}(q_3)) \rangle \\ & \times \langle F(A_{0(2)}(q_2)) F(A_{0(2)}(q_4)) \rangle \\ = & \sum_{q_1 q_2 q_3 q_4} \tau_{q_1 q_2} \tau_{q_3 q_4} \chi(q_1 - q_3) \chi(q_2 - q_4), \end{aligned} \tag{7}$$

where $\chi(q) = \langle F(A_0(q')) F(A_0(q' + q)) \rangle$, the autocorrelation functions of the input noise after digitization. In contrast to small-amplitude signals, digitization is a highly nonlinear operation to the large-amplitude noise, which can greatly distort the input spectrum when not uniform.

We may further expand χ into the power spectrum of the digitized signals, $\chi(q) = \sum_\nu \tilde{\chi}(\nu) e^{2\pi i \nu q}$. The signal and noise expectation values thus are decomposed into summations in Fourier space,

$$\mathbf{N^2} = \sum_{\nu_1 \nu_2} \tilde{\tau}^*(\nu_1, \nu_2) \tilde{\tau}(\nu_1, \nu_2) \tilde{\chi}(\nu_1) \tilde{\chi}(\nu_2) \tag{8}$$

and

$$\mathbf{S} = \sum_{\nu_1 \nu_2} \tilde{\tau}(\nu_1, \nu_2) \tilde{\rho}_c(\nu_1) \delta(\nu_1 - \nu_2) \Phi^2, \tag{9}$$

where $\tilde{\rho}_c(\nu) = \tilde{c}(\nu) |\tilde{R}|^2(\nu)$. Note that the formulation is simplified in Fourier space due to the time translation symmetry of the statistical properties. We again stress that the spectral shape of the noise gets distorted and nonlinearly mixed upon digitization, whereas that of the small signal can remain intact as a result of linear operation.

3. OPTIMIZATION OF THE S/N

For a measurement with a set of optimization parameters τ_i and the signal β_i, where i is the index of discrete frequency channel, the signal expectation value is a linear combination of the parameters,

$$\mathbf{S}(\tau) = \beta^{\mathrm{T}} \tau. \tag{10}$$

On the other hand, the noise square variance is a quadratic function of τ,

$$\mathbf{N}^2(\tau) = \tau^{\mathrm{T}} \gamma \tau, \tag{11}$$

where $\gamma = \gamma^{\mathrm{T}}$.

To optimize the S/N, a proper set of τ should be chosen. The S/N maximization condition is given as

$$d\left(\frac{\mathbf{S^2}}{\mathbf{N^2}}\right)(\tau) = 0, \tag{12}$$

thus

$$2\frac{d\mathbf{S}}{\mathbf{S}} = \frac{d\mathbf{N^2}}{\mathbf{N^2}}. \tag{13}$$

Solving this equation provides the relation between the optimization parameters τ and the measured γ and β. Note that the scale of the parameter vector τ is not relevant since the S/N is dimensionless. Therefore, the optimized S/N is $(\sqrt{\mathbf{S^2}/\mathbf{N^2}})_{\mathrm{op}} = (\beta \gamma^{-1} \beta)^{-\frac{1}{2}}$.

For the correlator system discussed in this article, the optimized τ is therefore obtained as

$$\tilde{\tau}(\nu) \propto \tilde{\chi}^{-2}(\nu) \tilde{\rho}_c(\nu), \tag{14}$$

and

$$\mathrm{S/N_{op}} = \sqrt{\sum_\nu |\tilde{\rho}_c|^2(\nu) \tilde{\chi}^{-2}(\nu) \Phi^4}. \tag{15}$$

In the continuum observation with an interferometer, a correlated signal generally has both pure real and imaginary parts across the full band, and $\tilde{c}(\nu)$ appearing in $\tilde{\rho}_c$ is a constant complex number over the positive frequency and the complex conjugate over the negative frequency.

4. POWER SPECTRUM OF DIGITIZED GAUSSIAN NOISE AND CORRELATION TRANSFER FUNCTION

To calculate the power spectrum $\tilde{\chi}$, explicit evaluation of the ensemble average is necessary. The ensemble average $\langle\rangle$ is performed by integrations over all $\tilde{w}(\nu)$ with Gaussian probabilities,

$$\langle\rangle = \frac{1}{(2\pi)^{\frac{\Gamma}{2}}} \int e^{-\tilde{w}_0^2/2} d\tilde{w}_0 \int e^{-\tilde{w}_{Ny}^2/2} d\tilde{w}_{Ny} \Pi_{\nu=1}^{Ny-1} \times \iint d\tilde{w}_\nu^* d\tilde{w}_\nu e^{-\tilde{w}_\nu^* * \tilde{w}_\nu}. \quad (16)$$

Further decomposing the complex $\tilde{w}$ into real and imaginary parts, i.e., $\tilde{w}_x = \sqrt{2}\,\mathrm{Re}(\tilde{w})$, $\tilde{w}_y = \sqrt{2}\,\mathrm{Im}(\tilde{w})$, transforms the integrations into a Γ dimensional Gaussian integral. The distribution function in this integration phase space is spherically symmetric, normalized by a chosen basis. With this symmetry property the dimension of the integration can be largely reduced and the problems simplified. As an example, for vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ with unit length, the integration $G(\mathbf{v_1}, \mathbf{v_2}) = \langle F(\mathbf{v_1} \cdot \mathbf{x}) F(\mathbf{v_2} \cdot \mathbf{x})\rangle$ reduces to a two-dimensional integral on the plane where $\mathbf{v_1}$ and $\mathbf{v_2}$ lie. Furthermore, the integration depends only on the angle between the two vectors. That is, $G(\mathbf{v_1}, \mathbf{v_2}) = g(\mathbf{v_1} \cdot \mathbf{v_2})$.

The 1 bit digitization function $F(x) = \mathrm{sgn}(x)$, and therefore

$$g(\mathbf{v_1} \cdot \mathbf{v_2}) = \frac{1}{(2\pi)} \int d^2x e^{-x^2/2} \mathrm{sgn}(\mathbf{v_1} \cdot \mathbf{x}) \mathrm{sgn}(\mathbf{v_2} \cdot \mathbf{x}) = 2[\sin^{-1}(\mathbf{v_1} \cdot \mathbf{v_2})]/\pi. \quad (17)$$

In this 1 bit case, the scale of the response function is irrelevant. So we have normalized the response function as $\sum_\nu |\tilde{R}|^2(\nu) = \sum_\nu \tilde{\rho}(\nu) = 1$. With the example here, the noise autocorrelation function can be explicitly derived as

$$\chi(q) = \langle F\left(\sum_\nu \tilde{w}(\nu)\tilde{R}(\nu)\right) F\left(\sum_\nu \tilde{w}(\nu)\tilde{R}(\nu) e^{2\pi i\nu q}\right)\rangle = g\left(\sum_\nu \tilde{\rho}(\nu) e^{2\pi i\nu q}\right), \quad (18)$$

where the vector components of $\mathbf{v_1}$ and $\mathbf{v_2}$ are $\tilde{w}(\nu)\tilde{R}(\nu)$ and $\tilde{w}(\nu)\tilde{R}(\nu)e^{2\pi i\nu q}$, respectively. A plot of $g(x)$ for different digitization bit numbers is shown in Figure 1. The same plot has been given previously for signal correlation efficiency of different digitization bits (Coles 1968), but in our case this plot serves as the basis for explaining the spectral distortion.

Note that the digitization transfer function $F(x)$ is usually defined in a normalized space, even in the case of higher sampling precision. The normalization redefines the input amplitude so that the expected rms of input noise at the sampling points equals unity, therefore validating our discussion of the normalized input power. The value of $F(x)$ can further be rescaled to meet the constraint that $g(1) = 1$ and $g(-1) = -1$, as shown in Figure 1. This normalization ensures that the total noise power before and after digitization is the same.

The ideal efficiency of the digitized signal is Φ^2, where $\Phi = \langle F'(\sum_\nu \tilde{w}(\nu)\tilde{R}(\nu))\rangle$. For a normalized vector $\mathbf{v}$, the integration

$$\Phi(\mathbf{v}) \equiv \langle F'(\mathbf{v} \cdot \mathbf{x})\rangle = (\sqrt{2\pi})^{-1} \int dx e^{-x^2/2} F'(x), \quad (19)$$

independent of $\mathbf{v}$. Using the property of the Gaussian integral, we obtain the signal efficiency, $\Phi^2 = 2/\pi$ for 1 bit digitization.

This signal efficiency basically reflects that digitization cannot capture the full strength of the correlated signal, and the loss of signal is $1 - 2/\pi = 36.5\%$. Despite that, digitization does not distort the spectrum of the small-amplitude signals. By contrast, the total power of noise is preserved upon digitization in our normalization, but the noise spectrum is distorted. For the particular 1 bit digital correlator, the results can be summarized as follow. With a Gaussian noise spectrum $\tilde{\rho}$, the spectrum after digitization can be obtained by the operation

$$\tilde{\chi} = FT\left[\left(\frac{2}{\pi}\right) \sin^{-1}(FT^*(\tilde{\rho}))\right] \quad (20)$$

for a normalized $\rho(0)$, where FT and FT^* are the forward and backward Fourier transformation operators respectively. The optimized wideband S/N of the correlator is

$$\left(\frac{2}{\pi}\right)(\Gamma)^{\frac{1}{2}} \sqrt{\langle(\tilde{\rho}\tilde{\chi}^{-1})^2\rangle_\nu}. \quad (21)$$

The first factor in the formula represents the ideal digitization efficiency. The digital samplers can not fully capture the signals and result in this factor. Digitization is irreversible, and for the 1 bit case, the efficiency is about 63.5%. The second factor is the input bandwidth of the system and is the most important factor of the correlation gain. Higher sampling rates provide higher S/N up to the input bandwidth, an important issue to be further

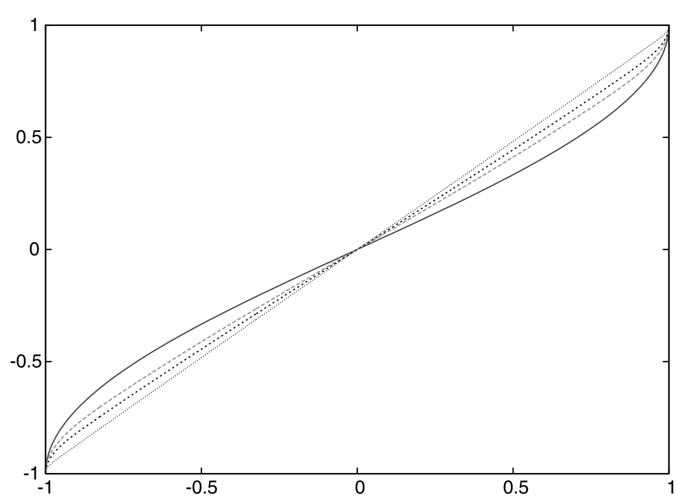

FIG. 1.—Correlation transfer functions $g(x)$ of different digitization bits. The *solid line* is for 1 bit, 2 level system, *long-dashed line* for 1.5 bit, 3 level system, *short-dashed line* for 2 bit, 4 level system, and *dotted line* for 3 bit, 8 level system. See the electronic edition of the *PASP* for a color version of this figure.

elaborated. The third factor represents effects due to the nonflatness of the input spectrum. With a nonflat input spectrum, the noise spectral shape can be distorted by digitization, while the total noise power remains the same. Thus for bands with relatively small input power, the digitization noise can increase, resulting in a reduced S/N. For bands with high input power, the conclusion reverses and a relatively high S/N results. The overall S/N is, however, reduced when the input spectrum is not uniform as a result of noise power transfer across different frequencies that makes the originally uniform S/N in different frequency channels nonuniform. We shall illustrate this transfer mechanism in the following case studies.

5. CASE STUDIES

Nonlinear transfer yields noise leakage from one frequency channel to another. This process can be modeled as the growth and decay processes per frequency channel, and we adopt a nonlinear Green's function approach to investigate each process separately. The Green's function for the decay process consists of a δ-function source whose power can leak into other frequency channels upon digitization. The Green's function of the growth process consists of a deep and narrow notch, which is to be filled up to a certain level due to leakage from other channels upon digitization. Armed with the understanding of these two fundamental processes, we can proceed to study the consequence of spectral nonuniformity of various situations.

5.1. Narrowband Input Spectrum

Suppose the input has narrow bandwidth at a single frequency, with the normalized power spectrum $\tilde{\rho}(\nu) = \frac{1}{2}[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)]$. The input signal autocorrelation function is thus $\cos(2\pi\tau)$, where τ is the delay time. The autocorrelation function after digitization transforms to $\frac{2}{\pi}\sin^{-1}(\cos(2\pi\tau))$, which is a periodic triangle function with a period of $\frac{1}{\nu}$. The spectral shape of this function is described as follows.

The digitizer acts as a harmonics generator so that the resulting output spectrum consists of a series of all odd harmonics spectral lines of the input signals. The harmonics run beyond the Nyquist frequency and round up to fill the spectrum. For a 1 bit sampler, the powers of the first, third, fifth, …harmonics are $\frac{8}{\pi^2}, \frac{8}{9\pi^2}, \frac{8}{25\pi^2}, \ldots$, respectively. The peaks decay in power law as ν^{-2}. An example is shown in Figure 2. The factor $\frac{8}{\pi^2}$ is a consequence of the integration $\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos(x)g(x)dx = 2\int_0^{\frac{\pi}{2}}\cos(x)(\frac{2}{\pi})\sin^{-1}(\cos(2\pi\nu))dx$, which is an important factor in the discussion in § 6.

We also note that the out-of-band power leakage is generally distributed nonuniformly across the spectral domain. This can be understood from the Fourier transformation of 1 bit $g(\cos(x))$ shown in Figure 1. When the same input is digitized with more bits, a greater range within $-1 < x < 1$ can be approximated as a linear function, $g(x) \simeq x$, except near $x = \pm 1$ where the nonlinearity of digitization is pronounced. In the limit of infinite bits, the two ends at $x = \pm 1$ become δ-functions of vanishing measure, and not only is the digitization noise white, but it also has infinitely small power. Therefore, the noise leakage is generally nonuniformly distributed over the whole spectral domain, and it is more so for lower bit sampling.

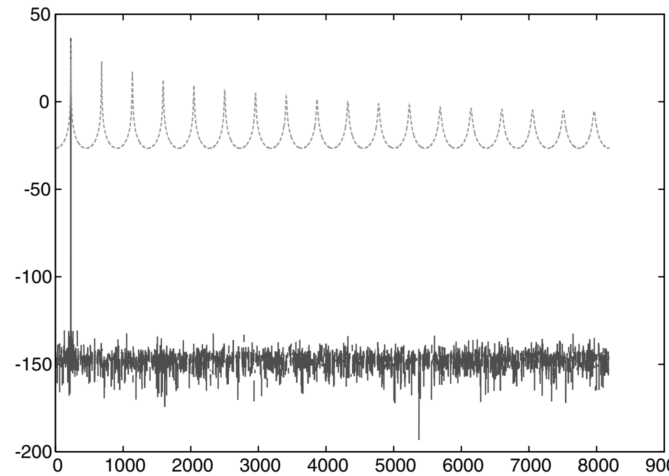

FIG. 2.—Digitized power spectrum of a narrowband input. The *solid line* indicates the input narrowband power spectrum. The *dashed line* shows the digitized spectrum with all the harmonics modes. The vertical axis is in units of dB. See the electronic edition of the *PASP* for a color version of this figure.

5.2. Broadband Flat Spectrum with a Frequency Notch

The input spectrum is modeled as a flat spectrum with a notch at $\nu = \nu_0$, where $\tilde{\rho}(\nu) = \frac{1}{\Gamma}\{\Gamma - [\delta(\nu - \nu_0) + \delta(\nu + \nu_0)]\}$. Let Γ be large enough so that total power is unaffected by the notch, and moreover the peak amplitude of δ-function is taken to be Γ to ensure a positive-definite $\tilde{\rho}(\nu)$. The autocorrelation function composes of a sharp peak at zero delay and a cosine function with a small amplitude $\frac{2}{\Gamma}$ in the delay space. The digitization raises the cosine function to an amplitude $g'(0) = \frac{4}{\pi\Gamma}$ but leaves the zero-delay component unchanged. The resulting digitization power spectrum is therefore a flat spectrum with a dip of $\frac{2}{\pi}$ amplitude at the notch frequency. Figure 3 shows the input and output spectra. The output spectrum reflects the ideal 1 bit sampling efficiency with an overall flat spectrum. The narrowband notch is filled with leakage noise to a power density $1 - \frac{2}{\pi} \simeq -4.4$ dB of other flat channels after digitization.

5.3. S/N with Nonflat Input Spectrum

From the preceding discussion, we found that noise can increase at some frequency due to the digitization power leakage from other frequency channels of higher power. Therefore, S/N will decrease in these weak frequency channels. If the S/N of a particular frequency channel is to decrease by 3 dB, the effective bandwidth of the channel will shrink to $\frac{1}{4}$ of the original, and this channel will become quite inefficient in contributing to the total S/N. However, for a high-power channel, its S/N will, on the

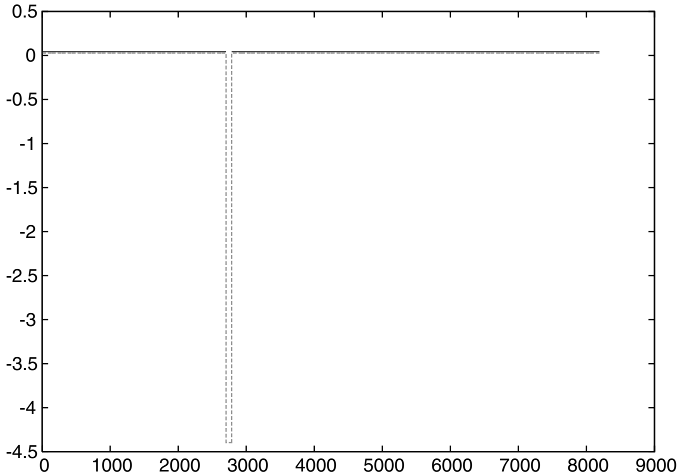

FIG. 3.—Digitized power spectrum of an input broadband flat spectrum with a narrow deep notch. The *solid line* indicates the input spectrum and the *dashed line* is the digitized spectrum. The vertical axis is in units of dB. See the electronic edition of the *PASP* for a color version of this figure.

contrary, increase due to leakage of the in-band noise into low-power channels.

When an observation of radio continuum is considered where the input has roughly the same S/N over all channels, the overall output S/N is the optimal sum of all digitized channels. We model the nonflat spectrum as one that consists of a high-power band with power density 10 dB larger than that in the remaining bands. Different bandwidths and central frequencies of the high-power band are chosen to calculate the optimized S/N with 1 bit digitization. All frequencies are normalized to the full frequency bandwidth. The results are given in Table 1, where the S/N is compared with the ideal 1 bit correlator system with flat input spectrum.

In contrast to the oversampling case to be discussed in § 6, the results here depend on the central frequency of the high-power band to a lesser degree. This is because the input spectrum is less singular. Our results demonstrate that a nonflat input spectrum degrades the overall S/N only moderately in most cases, because the gain in a high-power band and the loss in a low-power band can largely average out to some degree. The worst case happens when the high-power band is roughly 20% to 25% of the whole sampling bandwidth, where 20% of S/N degradation is observed. But when input spectrum has a wider high-power bandwidth, the degradation can be reduced.

We also calculate the case for input power spectra linearly decreasing in logarithmic scale from DC to the Nyquist frequency with various slopes. The results are given in Table 2. In systems with higher digitization bit number, the S/N degradation effect arising from the nonflat spectrum is less pronounced; this is illustrated in Table 3, which shows the absolute efficiency of the correlator with different bit digitization numbers. The two tables illustrate that a sloped input spectrum does not seriously degrade the overall S/N in a wideband digital correlator, regardless of the sampling bit number. The explanation is similar to that in the previous case. This conclusion is to be contrasted to a wideband analog correlator with no frequency resolution. A nonflat spectrum yields worse correlation efficiency for an analog correlator than for a 1 bit digital correlator when compared with their respective performances with a flat input spectrum. Despite having no digitization error, such an analog correlator cannot be optimized in accordance to the desired frequency weighting scheme give in § 4. At input spectral slopes of 3 dB, 5 dB, and 10 dB, the efficiencies of the 4 bit, 3 bit and 2 bit correlator all exceed the analog one. The efficiency of the 1 bit correlator cannot exceed the analog one due to a limit to the oversampling effect, which will be described in the next section.

6. OVERSAMPLING

We next consider an oversampling system where the input bandwidth spans only up to a frequency significantly below the Nyquist frequency. The input spectrum may be modeled as a top-hat function for simplicity. In the output spectrum after digitization, the noise is scattered into other bands outside the input band, so that the in-band noise is reduced compared with that without oversampling. This important property of digitalization can greatly reduce the impact of adverse effects of a nonflat input spectrum as seen in the last section. An extreme case for illustration is derived in § 5.1, where the input with narrow bandwidth is sampled with a high-bandwidth 1 bit sampler, and the input bandwidth is narrower than the correlator can resolve. The output in-band noise power is $\frac{8}{\pi^2}$ of the input noise power, whereas the power of the correlated component remains unchanged. Therefore, the S/N is increased by a factor of $\frac{\pi^2}{8} \simeq 0.9$ dB, and the correlator S/N is $(\frac{2}{\pi})(\frac{\pi^2}{8}) = \frac{\pi}{4} = 78.5\%$

TABLE 1
S/N OF DIFFERENT HIGH-POWER INPUT BANDWIDTHS AND CENTRAL FREQUENCIES COMPARED TO A FLAT SPECTRUM FOR 1 BIT DIGITIZATION

Bandwidth	Center Frequency		
	$0.0 + \mathrm{BW}/2$	0.25	0.5
0.02	95.24%	95.25%	95.26%
0.05	89.95%	89.92%	89.96%
0.10	84.65%	84.56%	84.65%
0.15	82.13%	82.02%	82.13%
0.20	81.26%	81.18%	81.26%
0.25	81.40%	81.42%	81.39%
0.30	82.16%	82.32%	82.15%
0.35	83.32%	83.60%	83.32%
0.40	84.73%	85.02%	84.73%
0.50	87.94%		87.96%
0.60	91.22%		91.22%
0.70	94.08%		94.08%
0.80	96.45%		96.45%

TABLE 2

S/N of Different Input Spectral Slopes Compared to a Flat Spectrum for 1 bit Digitization

Slope	S/N
−3 dB	99.35294%
−5 dB	98.25092%
−7 dB	96.70301%
−10 dB	93.76675%
−15 dB	88.03347%
−20 dB	82.16897%
−25 dB	76.72094%
−30 dB	71.87791%

compared to the ideal, infinite-bit correlation. This value of S/N is to be contrasted to the 63.5% S/N for the full-band 1 bit correlation.

In reality, the bandwidth is not infinitely narrow. For signals with a narrow but finite bandwidth, the S/N obtained from the oversampling scheme may be even better than that of the single-tone case discussed earlier, provided that the frequency resolution of the correlator is finer than the bandwidth of the input signal. This gain is due to the frequency broadening effect in the digitized noise spectrum, as a result of the scattering of in-band noise via beating between positive and negative in-band frequencies. For example, an input band between 2 to 4 Ghz can scatter the in-band power by the third harmonic term to 1 Ghz via the triple beating $(2, 2, -3)$ Ghz, and to 5 Ghz via the triple beating $(4, 4, -3)$ Ghz. Such a mechanism is impossible for the single-tone input. Having this extra scattering, the in-band digitized noise of a finite-bandwidth input is therefore smaller than the digitized noise of the single-tone input, which is $8/\pi^2$ of the original noise power with a S/N $(2/\pi)(\pi^2/8) = 78.6\%$ of the infinite-resolution case. An example of the oversampled output spectrum is shown in Figure 4. The convex top-hat spectrum signifies the result of the in-band, positive-negative frequency beating of noise, and the total in-band digitization S/N is 86.2% of the infinite-resolution case, which is 7.5% higher than the single-tone case.

However, for input signals with a wider bandwidth, this result can reverse, because noise from other odd harmonics can leak into the input band thereby enhancing the in-band noise. This effect is particularly severe when the input central frequency is near DC, Nyquist, and half of the Nyquist frequency, where the harmonics are easily folded into the input band. A demonstration is given by comparing Figure 5 with Figure 4, where the input spectrum is centered at half of the Nyquist as opposed to at a quarter of the Nyquist. In Figure 5, the in-band digitization S/N is reduced to 80% of the ideal case.

Below we explicitly calculate the S/N upon 1 bit digitization compared to the input S/N of different bandwidths and different central frequencies. Again, all frequencies are normalized to the sampling frequency. The results are given in Table 4. In comparison with Table 1, which has a similar setup, we find that results in Table 1 have slightly higher performance than those in Table 4 due to the contribution from the low-power band. For the case where bandwidth is 0.5 and central frequency is 0.25, it can be seen that the $(-10$ dB$)$ low-power band contributes less than 2% to S/N, in comparison to 8% for the case where this low-power band is perfectly equalized before sampling.

As an alternative to a 1 bit interleaving sampling scheme where two samplers of half Nyquist frequency are in use, one may consider to completely give up the low-power band and samples the data at half of the original Nyquist frequency with a 1.5 bit, two-level digitization scheme. A 1.5 bit sampling correlator has an 81% S/N compared to the infinite-resolution correlator. Since half of the full bandwidth is sampled, the S/N is $81\%/\sqrt{2} = 57.3\%$ of an ideal full-band correlator, which is better than the 1 bit case under consideration by almost 2% in S/N.

Due to the freedom to choose normalization, there is an alternative viewpoint on the quantization noise. We may consider the S/N reduction of a digital correlator as an effect of add-on digitization noises rather than degradation of signal. In this normalization the small-amplitude signal is unaffected by digitization and retains its input spectral shape. For the flat input spectrum case, the total noise power increases by a factor $\pi/2$. By contrast, for a narrowband noise the increase of noise power in the same frequency band is just $4/\pi$. But if we sum up the power scattered into other harmonic bands for this case, we retrieve the total power of $\pi/2$ times the input power. Hence we can separate the digitization noise as an in-band noise with $4/\pi - 1 \simeq 27\%$ of input noise power and the out-of-band noises of $\pi/2 - 4/\pi \simeq 30\%$ of the input noise power. For input

TABLE 3

Comparison of Correlator Efficiency Versus Spectral Slopes among Different Digitization Bits

Bit Number	Spectral Slope						
	0 dB	−3 dB	−5 dB	−10 dB	−15 dB	−20 dB	−30 dB
1 bit	63.66%	63.25%	62.55%	59.69%	56.04%	52.31%	45.76%
2 bit	88.12%	87.77%	87.17%	84.39%	80.10%	75.01%	65.73%
3 bit	96.26%	96.12%	95.87%	94.62%	92.22%	88.46%	78.62%
4 bit	98.85%	98.80%	98.72%	98.28%	97.34%	95.56%	88.26%
Analog	100.00%	98.08%	95.00%	84.30%	73.73%	65.25%	53.75%

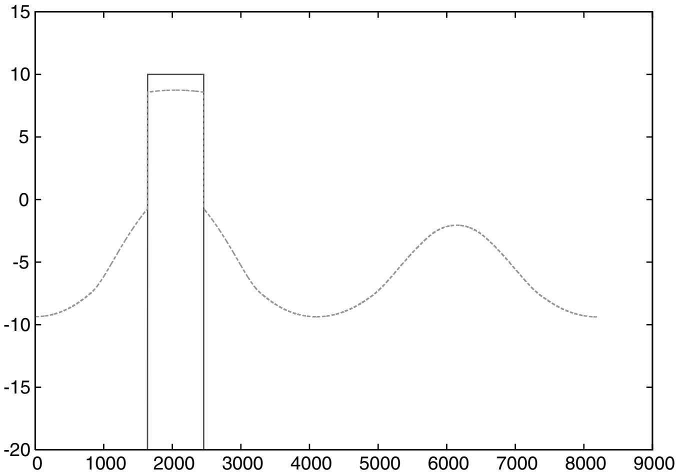

FIG. 4.—Oversampling transforms an offcenter input spectrum (*solid line*) of finite bandwidth to the output spectrum (*dashed line*). See the electronic edition of the *PASP* for a color version of this figure.

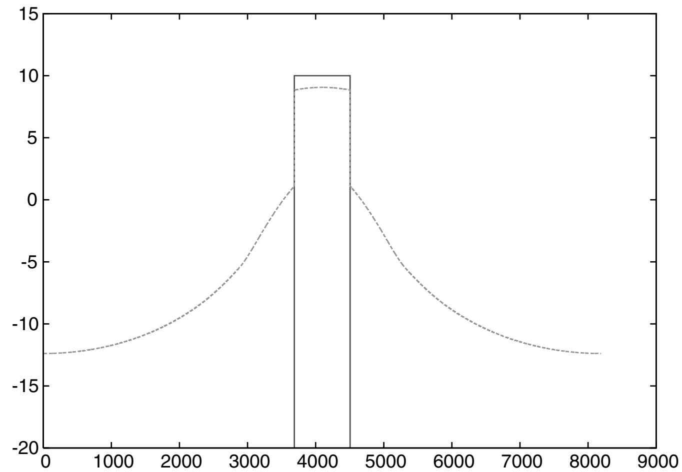

FIG. 5.—Oversampling transforms an input spectrum (*solid line*) of finite-bandwidth centered at half of the Nyquist frequency to the output spectrum (*dashed line*), where the in-band noise is obviously higher than that shown in Fig. 4. See the electronic edition of the *PASP* for a color version of this figure.

noise of finite bandwidth, the digitized noise spectrum can be broadened and eventually covers the whole frequency domain, where the in-band and out-of-band noises are mixed together. The average power density of this add-on noise spectrum is $\pi/2 - 1$ divided by the oversampling ratio. Moreover, the add-on noise can be roughly uniformly distributed in the frequency range below the sampling frequency, more so when the input bandwidth is closer to the sampling frequency.

Our quantitative results can also be understood qualitatively from the Taylor expansion of the $\sin^{-1}(x)$ function,

$$\sin^{-1}(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \ldots \qquad (22)$$

The first term gives the original input noise power, and all higher harmonics terms contribute to the add-on digitization noise. Multiplication of functions in the time domain is equivalent to the convolutions in the frequency domain. Thus, we get all odd harmonics from these high-order terms, which scatter digitization noise over the entire frequency range. The convolution also mixes positive in-band frequencies with negative in-band frequency and scatters them out of band. This particular mixing mechanism is only possible for finite bandwidth input but impossible for narrowband input. Such an extra scattering of add-on noise is the reason behind why the finite-bandwidth input may have better S/N than the narrowband input. In general, it is the competition between noise leakage to and noise injection from other bands that determines the final in-band S/N, and when the input bandwidth becomes so wide as to be close to the sampling frequency the S/N will decrease.

7. CORRELATION S/N BETWEEN INPUTS OF DIFFERENT SPECTRAL SHAPES

The preceding discussions are based on the assumption of identical spectral response $\tilde{R}_{(i)}(\nu)$ for both receivers. In this section we will consider the more general case, where the two inputs of a correlator inherit different spectral shapes, and examine the impact on the S/N.

Suppose the two receiver front ends have their spectral responses $\tilde{R}_{(1)}(\nu)$ and $\tilde{R}_{(2)}(\nu)$. Following the analysis in § 2, similar results to equation (8) and equation (9) are obtained, with the correlation spectrum $\tilde{\rho}_c(\nu) = \tilde{c}(\nu)\tilde{R}_{(1)}(\nu)\tilde{R}^*_{(2)}(\nu)$, and the digitized noise power spectra $\tilde{\chi}_{(1)}(\nu_1)$ and $\tilde{\chi}_{(2)}(\nu_2)$ are now distinctly different. The phase difference between $\tilde{R}_{(1)}$ and $\tilde{R}_{(2)}$ is compensated by the optimization parameters, and the optimum S/N thereby depends only on the two input spectral shapes. That is, in a 1-bit case, the S/N is

$$\left(\frac{2}{\pi}\right)(\Gamma)^{\frac{1}{2}}\sqrt{\langle(\tilde{\rho}_{(1)}\tilde{\chi}^{-1}_{(1)})(\tilde{\rho}_{(2)}\tilde{\chi}^{-1}_{(2)})\rangle_\nu}, \qquad (23)$$

where $\tilde{\chi}_{(i)}$ is obtained from the input spectra $\tilde{\rho}_{(i)}$ through equation (20).

The result is simply a replacement of the single signal and noise digitized spectra in equation (21) by the geometric mean of the respective corresponding spectra of two receivers. Due to the nonlinear mixing in equation (20), the resulting S/N will depart from the naively expected S/N that is proportional to the geometric mean of the two receivers' responses. We explicitly demonstrate this effect with several different spectral slopes. Table 5 shows the S/N of a 1 bit correlator for two receivers of different spectral slopes compared to that of the ideal flat input, where the averages and differences of slopes are in logarithmic scale. In cases where two spectral slopes deviate more than

TABLE 4

Comparison of In-band S/N for Oversampling Inputs of Different Bandwidths and Central Frequencies with reference to the Ideal, Infinite-Resolution Case

Bandwidth	Central Frequency				
	$0.0 + \mathrm{BW}/2$	0.125	0.25	0.375	0.5
0.1 ……	80.44%	86.27%	86.11%	86.41%	80.42%
0.2 ……	79.84%	82.11%	84.76%	84.63%	79.85%
0.3 ……	78.86%		82.40%	82.49%	78.85%
0.4 ……	77.50%		79.14%	79.98%	77.51%
0.5 ……	75.79%		75.79%	77.13%	75.79%

TABLE 5

Comparison of Correlator Efficiency Degradation When the Two Inputs have Different Spectral Slope

Slope Difference	Average Slope			
	0 dB	−10 dB	−20 dB	−30 dB
0 dB ……….	100.00%	93.77%	82.17%	71.88%
4 dB ……….	99.48%	93.42%	82.02%	71.82%
8 dB ……….	97.93%	92.37%	81.57%	71.63%
12 dB ………	95.44%	90.66%	80.81%	71.32%
16 dB ………	92.10%	88.32%	79.77%	70.88%
20 dB ………	88.06%	85.41%	78.43%	70.32%

10 dB, severe S/N degradation is observed, especially for the flat average slope cases. This is primarily due to that the gain in S/N from oversampling is suppressed as a consequence of the high-power regions of two inputs departing from each other. In the case with 10 dB and -10 dB input slopes, the resulting S/N is even worse than that of a system with both identical -10 dB slopes. Compensation of the spectral response of one receiver by the other to yield a better S/N, which is valid for the analog correlator, is found implausible for the 1 bit digital correlator.

8. CONCLUSIONS

In this article, we provide an extensive analysis to the performance of the low-bit, particularly 1 bit, digital correlator in the regime of small correlated signal. In this regime, the add-on digitization noise can leak from the input band to other bands, whereas the correlated signal remains unaffected by digitization. With this understanding, we examine various cases and quantify the S/N degradation due to spectral nonuniformity in the front-end receivers. Related to this, when the input bandwidth is substantially lower than the sampling frequency or when the input spectrum has a logarithmic slope, we find such a situation corresponds to oversampling, which can enhance S/N, and we also quantify the gain of oversampling. However, when the two input spectra for correlation are different from each other, the gain with oversampling can be lost.

This work is a part of the system assessment for the performance of FX digital correlator system in the NTU Array, which is currently near the end of its construction phase. NTU Array is equipped with W band receivers of 35 Ghz instantaneous bandwidth, which is subdivided and downconverted to 4 base bands of 0–8.8 Ghz as the inputs to the digitizers. The FX correlator takes the inputs from 18 Ghz sampling, 1 bit analog-to-digital converters for real-time digital processing. The extreme case of 1 bit digitization addressed here highlights the nature of digitization noise in the small-signal regime. In particular, we address the leakage of digitization noise among different frequency channels that leads to S/N degradation. For a nonflat input spectrum, which is often the case for wide-bandwidth analog input, we find that the loss in S/N is tolerable. Such an understanding provides a foundation to assess the future few-bit sampling wideband digital correlator when the input spectrum is nonideal.

REFERENCES

Bennett, W. R. 1948, Bell Syst. Tech. J., 27, 446

Cole, T. 1968, Austral. J. Phys., 21, 273

Cooper, B. F. C. 1970, Austral. J. Phys., 23, 521

Gary, R. M. 1990, IEEE Trans. Inf. Theory, 36, 1220

Kouwenhoven, M. L. A., & Voute, J. L. L. 2001, A&A, 378, 700

Lamb, J. W. 2001, NRAO, ALMA Memo Series No. 407

Roberts, P. P. 1997, A&A Suppl. Ser., 126, 379

Sripad, A. B., & Snyder, D. L. 1977, IEEE Trans. Acoustics Speech Signal Proc., 25, 442

Thompson, A. R., & Emerson, D. T. 2005, EVLA Memo Series 88

Weinreb, S. 1961, Proc. IEEE, 49, 1099

Widrow, B. 1956, IRE Trans. Circuit Theory, 3, 266

———. 1961, Trans. AIEE Part II: Applications and Industry, 79, 555

Widrow, B., Kollar, I., & Liu, M. C. 1996, IEEE Trans. Instrum. Meas., 45, 353