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Piecewise continuous distribution function method in the theory of wave disturbances of inhomogeneous gas

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Abstract

The system of hydrodynamic-type equations for a stratified gas in gravity field is derived from BGK equation by method of piecewise continuous distribution function. The obtained system of the equations generalizes the Navier–Stokes one at arbitrary Knudsen numbers. The problem of a wave disturbance propagation in a rarefied gas is explored. The verification of the model is made for a limiting case of a homogeneous medium. The phase velocity and attenuation coefficient values are in an agreement with former fluid mechanics theories; the attenuation behavior reproduces experiment and kinetics-based results at more wide range of the Knudsen numbers.

1. Introduction

There is a significant number of problems of gas dynamics at which it is necessary to use a theory beyond the limits of traditional hydrodynamics of Navier–Stokes. The classical fluid mechanics is valid under the condition for the Knudsen number $Kn = l/L \ll 1$, where l is a mean free path, and L is a characteristic scale of inhomogeneity of a problem under consideration. The first work, in which wave perturbations of a gas was investigated from the point of view of more general kinetic approach (Boltzmann

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equation), perhaps, is the paper of Wang Chang and Uhlenbeck [2].

Numerous researches on a sound propagation in a homogeneous gas at arbitrary Knudsen numbers were made, see, e.g., the classic experimental papers [3,4] and, may be the most advanced theoretical, kinetic-based [15]. The investigations have shown, that at arbitrary Knudsen numbers the behavior of a wave differs considerably from one predicted on a basis of hydrodynamical equations of Navier–Stokes. These researches have revealed two essential features: first, propagating perturbations keep wave properties at larger values of Kn, than it could be assumed on the basis of a hydrodynamical description. Secondly, at $Kn \geqslant 1$ such concepts as a wave vector and frequency of a wave become ill-determined.

The case, when the Knudsen number Kn is nonuniform in space or in time is more difficult for investigation and hence need more simplifications in kinetic equations or their model analogues. A constructions of such approaches for analytical solutions based on kinetic equation of Bhatnagar–Gross–Krook (BGK) or of Gross–Jackson [5] in a case of exponentially stratified gas were considered at [18,19,21,22] in connection with general fluid mechanics development. A progress was launched by more deep understanding of perturbation theory (so-called nonsingular perturbations), see, e.g., [17]. Recently the interest to the problem of Kn regime wave propagation has grown again [23–27].

In this Letter we develop the method of a piecewise continuous distribution function launched by ideas of von Karman, mentioned in a pioneering paper of Lees [9] and applied for a gas in gravity field in [21,22]. We consider the example of one-dimensional wave perturbations theory for a gas stratified in gravity field so that the Knudsen number exponentially depends on the (vertical) coordinate and generalize the results of earlier [21,22] to take into account the complete set of nonlinearities. We start with the method review and the generalization at the Section 2, go down to the linearized equations at the Section 3, deriving the dispersion relation at the Section 4 and, then, we study a solution of linear boundary problem to extract the attenuation parameter of the sound. At the Section 5 we pick up all the theoretical curves against the experimental data of Meyer [4] and Greenspan [3] and discuss the results.

2. Piecewise continuous distribution function method

The kinetic equation with the model integral of collisions in BGK form looks like:

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} - g\frac{\partial f}{\partial v_z} = \nu(f_l - f), \tag{1}$$

here f is the distribution function of a gas, t is time, $\vec{v}$ is velocity of a particle of a gas, $\vec{r}$ is coordinate,

$$f_l(\vec{r}, \vec{v}, t) = \frac{n}{\pi^{3/2}v_T^3}\exp\left(-\frac{(\vec{v}-\vec{U})^2}{v_T^2}\right)$$

is the local-equilibrium distribution function, $v_T = \sqrt{2kT/m}$ denotes the average thermal velocity of particles of gas, $\nu = \nu_0\exp(-z/H)$ is the effective frequency of collisions between particles of gas at height z, $H = kT/mg$ is a parameter of the gas stratification. It is supposed, that density of gas n, its average speed $\vec{U} = (u_x, u_y, u_z)$ and temperature T are functions of time and coordinates.

Following the idea of the method of piecewise continuous distribution functions let us search for the solution f of Eq. (2) as a combination of two locally equilibrium distribution functions, each of which gives the contribution in its own area of velocities space:

$$f(t, \vec{r}, \vec{V}) = \begin{cases} f^+ = n^+(\frac{m}{2\pi kT^+})^{3/2}\exp(-\frac{m(\vec{V}-\vec{U}^+)^2}{2kT^+}), \\ \quad v_z \geqslant 0, \\ f^- = n^-(\frac{m}{2\pi kT^-})^{3/2}\exp(-\frac{m(\vec{V}-\vec{U}^-)^2}{2kT^-}), \\ \quad v_z < 0, \end{cases} \tag{2}$$

here $n^\pm = n^\pm(t, z), U^\pm = U^\pm(t, z), T^\pm = T^\pm(t, z)$ are functional parameters of these locally equilibrium distributions functions.

Thus, a set of the parameters determining a state of the perturbed gas is increased twice. The increase of the number of parameters of distribution function (2) results in that the distribution function generally differs from a local-equilibrium one and describes deviations from hydrodynamical regime. In the range of small Knudsen numbers $l \ll L$ we automatically have $n^+ = n^-, U^+ = U^-, T^+ = T^-$ and distribution function (2) tends to local-equilibrium one, reproducing exactly the hydrodynamics of Euler and at the small difference of the functional "up" and "down" parameters—the Navier–Stokes equations. In

the range of big Knudsen numbers the formula (2) gives solutions of collisionless problems. Similar ideas have resulted successfully in a series of investigations. For example, in papers [7,9] a method of piecewise continuous distribution function was used for the description of flat and cylindrical (neutral and plasma) Couette flows. Thus for a flat problem the surface of break in the velocity space was determined by a natural condition $V_z = 0$, we follow the same geometry of the break. The approach was developed for a description of a nonlinear sound in stratified gas [18, 21]. In a cylindrical case $V_r = 0$, where V_z and V_r are, accordingly, vertical and radial components of velocity of particles. Some problem of a flow caused by pulse movement of plane was solved by perturbations [8,10]. Solving a problem of a shock wave structure [1,10] the solution was represented as a combination of two locally equilibrium functions, one of which determines the solution before front of a wave, and another—after. In the problem of condensation/evaporation of drops of a given size [13,16] a surface break was determined by so-called "cone of influence", thus all particles were divided into two types: flying "from a drop" and flying "not from a drop".

The idea of a method of two-fold distribution functions given by (2) is realized as follows. Let us introduce a set of linearly independent eigen functions of the linearized Boltzmann operator. In this Letter we restrict ourselves by the case of one-dimensional disturbances $\vec{U} = (0, 0, U)$, but for the sake of realistic constant values reproducing, velocities of gas particles are three dimensional. Hence the following subset of basic functions (we apply the BGK equation) is used:

$$\varphi_1 = m, \qquad \varphi_2 = mV_z, \qquad \varphi_3 = \frac{1}{2}m|\vec{V} - \vec{U}|^2,$$

$$\varphi_4 = m(V_z - U)^2, \qquad \varphi_5 = \frac{1}{2}m(V_z - U)|\vec{V} - \vec{U}|^2,$$

$$\varphi_6 = \frac{1}{2}m(V_z - U)^3. \tag{3}$$

Let us define a scalar product in velocity space:

$$\langle \varphi_n, f \rangle \equiv \langle \varphi_n \rangle \equiv \int d\vec{v}\, \varphi_n(t, z, \vec{V}) f(t, z, \vec{V}), \tag{4}$$

$$\langle \varphi_1 \rangle = \rho(t, z), \qquad \langle \varphi_2 \rangle = \rho U, \qquad \langle \varphi_3 \rangle = \frac{3}{2}\frac{\rho}{m}kT,$$

$$\langle \varphi_4 \rangle = P_{zz}, \qquad \langle \varphi_5 \rangle = q_z, \qquad \langle \varphi_6 \rangle = \bar{q}_z. \tag{5}$$

Here ρ is mass density, P_{zz} is the diagonal component of the pressure tensor, q_z is a vertical component of a heat flow, $\bar{q}_z$ is a parameter having dimension of the heat flow.

Projecting Eq. (1) on the eigen functions (3) subspaces we obtain the system of differential equations:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z}(\rho U) = 0,$$

$$\frac{\partial}{\partial t}U + U\frac{\partial}{\partial z}U + \frac{1}{\rho}\frac{\partial}{\partial z}P_{zz} + g = 0,$$

$$\frac{3}{2}\frac{k}{m}\frac{\partial}{\partial t}(\rho T) + \frac{3}{2}\frac{k}{m}U\frac{\partial}{\partial z}(\rho T)$$

$$+ \left(\frac{3}{2}\frac{k}{m}\rho T + P_{zz}\right)\frac{\partial}{\partial z}U + \frac{\partial}{\partial z}q_z = 0,$$

$$\frac{\partial}{\partial t}P_{zz} + U\frac{\partial}{\partial z}P_{zz} + 3P_{zz}\frac{\partial}{\partial z}U + 2\frac{\partial}{\partial z}\bar{q}_z$$

$$= -\nu(z)\left(P_{zz} - \frac{\rho}{m}kT\right),$$

$$\frac{\partial}{\partial t}q_z + U\frac{\partial}{\partial z}q_z + 2(q_z + \bar{q}_z)\frac{\partial}{\partial z}U$$

$$- \left(\frac{3}{2}\frac{k}{m}T + \frac{1}{\rho}P_{zz}\right)\frac{\partial}{\partial z}P_{zz} + \frac{\partial}{\partial z}J_1 = -\nu(z)q_z,$$

$$\frac{\partial}{\partial t}\bar{q}_z + U\frac{\partial}{\partial z}\bar{q}_z + 4\bar{q}_z\frac{\partial}{\partial z}U - \frac{3}{2\rho}P_{zz}\frac{\partial}{\partial z}P_{zz} + \frac{\partial}{\partial z}J_2$$

$$= -\nu(z)\bar{q}_z, \tag{6}$$

where

$$J_1 = m\langle(V_z - U)^2(\vec{V} - \vec{U})^2\rangle,$$

$$J_2 = m\langle(V_z - U)^4\rangle. \tag{7}$$

While deriving of the continuity equation we have used the following property of the distribution function:

$$\left\langle \frac{\partial f}{\partial V_z}\right\rangle = 0. \tag{8}$$

It is obtained by integration by parts and taking into account the condition $\lim_{V \to \infty} f = 0$ (see [14]).

The system (6) of the equations according to the derivation scheme is valid at all frequencies of collisions and within the limits of the high frequencies should transform to the hydrodynamic equations. It is a system of hydrodynamical type and generalizes the classical equations of a viscous fluid to arbitrary Kn, up to a free-molecule flow. However, the system (6)

is not closed yet. It is necessary to present values of two integrals J_1 and J_2 as functions of thermodynamic parameters of the system (6). Let us evaluate the integrals (7) and (5) directly, plugging the function (2),

$$\rho = \frac{1}{2}(\rho^+ + \rho^-) + \frac{1}{2\sqrt{\pi}}\left(\frac{\rho^+ U^+}{V_T^+} - \frac{\rho^- U^-}{V_T^-}\right),$$

$$\rho U = \frac{1}{2\sqrt{\pi}}(\rho^+ V_T^+ - \rho^- V_T^-) + \frac{1}{2}(\rho^+ U^+ + \rho^- U^-),$$

$$\frac{3}{2}\frac{\rho}{m}kT = \frac{3}{8}(\rho^+ V_T^{+2} + \rho^- V_T^{-2})$$
$$+ \frac{1}{2\sqrt{\pi}}\big[\rho^+ V_T^+ (2U^+ - U)$$
$$- \rho^- V_T^- (2U^- - U)\big],$$

$$P_{zz} = \frac{1}{4}(\rho^+ V_T^{+2} + \rho^- V_T^{-2})$$
$$+ \frac{1}{\sqrt{\pi}}\big[\rho^+ V_T^+ (U^+ - U) - \rho^- V_T^- (U^- - U)\big],$$

$$q_z = \frac{1}{2\sqrt{\pi}}(\rho^+ V_T^{+3} - \rho^- V_T^{-3})$$
$$+ \frac{5}{8}\big[\rho^+ V_T^{+2}(U^+ - U) + \rho^- V_T^{-2}(U^- - U)\big],$$

$$\bar{q}_z = \frac{1}{4\sqrt{\pi}}(\rho^+ V_T^{+3} - \rho^- V_T^{-3})$$
$$+ \frac{3}{8}\big[\rho^+ V_T^{+2}(U^+ - U) + \rho^- V_T^{-2}(U^- - U)\big],$$
$$\tag{9}$$

where $\rho^\pm = mn^\pm$, $V_T^\pm = \sqrt{2kT^\pm/m}$. Solving the system (9) with respect to the six functional variables $U^\pm, \rho^\pm, V_T^\pm$ and plugging the results into the expressions for $J_{1,2}$ close the general nonlinear system (6). The outlined procedure looks complicated so we apply an expansion with respect to the Mach number.

3. Linearization of the problem

We estimate the functions $U^\pm/V_T^\pm$ as small, that corresponds to small Mach numbers $M = \max |U/v_T|$. We shall base here on an expansion in M, up to the first order. Within the specified approximation the integrals in terms of the functional parameters are

$$J_1 = \frac{5}{16}(\rho^+ V_T^{+4} + \rho^- V_T^{-4}) + \frac{3}{2\sqrt{\pi}}$$
$$\times \big[\rho^+ V_T^{+3}(U^+ - U) - \rho^- V_T^{-3}(U^- - U)\big]$$
$$+ O(M^2),$$

$$J_2 = \frac{3}{16}(\rho^+ V_T^{+4} + \rho^- V_T^{-4}) + \frac{1}{\sqrt{\pi}}$$
$$\times \big[\rho^+ V_T^{+3}(U^+ - U) - \rho^- V_T^{-3}(U^- - U)\big]$$
$$+ O(M^2). \tag{10}$$

For the two-sides distribution function (2) the property (8) looks as

$$\frac{\rho^+}{V_T^+} - \frac{\rho^-}{V_T^-} = 0 + O(M^2). \tag{11}$$

For finding $n^\pm$ and $V_T^\pm$ within the zero order (at $U^\pm = 0$) we solve together four equations: (11) and first three equations of the system (9). Thus we obtain

$$\rho^+ = \rho, \qquad \rho^- = \rho,$$
$$V_T^+ = \sqrt{\frac{2kT}{m}}, \qquad V_T^- = \sqrt{\frac{2kT}{m}}. \tag{12}$$

To evaluate the integrals J_1 and J_2 in the first order we shall substitute $n^\pm$ and $V_T^\pm$ from (12) in the fourth and fifth equations of system (9) that yields expressions for U^+ and U^-

$$U^+ = -\frac{\sqrt{\pi}}{2}\sqrt{\frac{m}{2kT}}\left(P_{zz} - \frac{\rho}{m}kT\right) + U + \frac{2mq_z}{5\rho kT},$$

$$U^- = \frac{\sqrt{\pi}}{2}\sqrt{\frac{m}{2kT}}\left(P_{zz} - \frac{\rho}{m}kT\right) + U + \frac{2mq_z}{5\rho kT}. \tag{13}$$

Taking into account (12) gives the rest functional parameters in the first order

$$\rho^+ = \rho + \frac{m}{2kT}\left(P_{zz} - \frac{\rho}{m}kT\right) + \frac{2\sqrt{\pi}}{5}\left(\frac{m}{2kT}\right)^{3/2} q_z,$$

$$\rho^- = \rho + \frac{m}{2kT}\left(P_{zz} - \frac{\rho}{m}kT\right) - \frac{2\sqrt{\pi}}{5}\left(\frac{m}{2kT}\right)^{3/2} q_z,$$

$$V_T^+ = \sqrt{\frac{2kT}{m}} - \frac{1}{6\rho}\sqrt{\frac{m}{2kT}}\left(P_{zz} - \frac{\rho}{m}kT\right)$$
$$- \frac{mq_z}{5\rho kT}\sqrt{\pi},$$

$$V_T^- = \sqrt{\frac{2kT}{m}} + \frac{1}{6\rho}\sqrt{\frac{m}{2kT}}\left(P_{zz} - \frac{\rho}{m}kT\right)$$
$$+ \frac{mq_z}{5\rho kT}\sqrt{\pi}. \tag{14}$$

Plugging the values of (14) and (13) into (10) gives the values of $J_{1,2}$ at the first order:

$$J_1 = \frac{5\rho}{2}\left(\frac{kT}{m}\right)^2 + \frac{61}{12}\frac{kT}{m}\left(P_{zz} - \frac{\rho}{m}kT\right),$$

$$J_2 = \frac{3\rho}{2}\left(\frac{kT}{m}\right)^2 + \frac{13}{4}\frac{kT}{m}\left(P_{zz} - \frac{\rho}{m}kT\right). \qquad (15)$$

So we have closed the system (6). Let us consider the system in the hydrodynamics limit ($Kn \ll 1$, $\nu \to \infty$). First the Euler's equations are obtained. At the next order of the perturbation theory in the small parameter ν^{-1} we reproduce the Navier–Stokes equations (for the linear case see [21,22]) having the Prandtl number $Pr = 1$. The general theory (and experiments for noble gases) give for Pr the value 2/3. The wrong Prandtl number is the known disadvantage of BGK model, that, however, can be removed by transition to the more exact models of collision integral, for example, of Gross–Jackson ones [5].

Chen and Spiegel [25,26] by studying the ultrasound propagation correct the wrong Prandtl number phenomenological. They follow a common practice to use the linearized system fluid equations and put the empirical Prandtl number $Pr = 2/3$ into the theoretical results. But the Prandtl number for their system is 1 also.

We have proposed a modification of the procedure for deriving fluid mechanics (hydrodynamic-type) equations from the kinetic theory. We did not begin to derive our equations in the customary way from an expansion in mean free path, as it is usually done. Therefore the obtained system of the equations generalizes the Navier–Stokes at arbitrary density (Knudsen numbers).

As we shall see, our method gives reasonable agreement with the experimental data against the results following from NS, and some other types of the fluid dynamic equations as, e.g., [23,25–27].

4. Homogeneous medium limit. Dispersion relation

Let us deliver a verification of the model for a limiting case of homogeneous medium and compare it with the classic experimental data of [3,4]. We linearize the fluid equations and find the dispersion relation, which has the form

$$\frac{18}{125}\tilde{k}^6 + \left(\frac{3}{5}r^2 - \frac{39}{25} - \frac{48}{25}ir\right)\tilde{k}^4$$

$$+ \left(-ir^3 - \frac{24}{5}r^2 + \frac{23}{3}ir + \frac{58}{15}\right)\tilde{k}^2$$

$$+ ir^3 - 1 - 3ir + 3r^2 = 0. \qquad (16)$$

Here the dimensionless wave number $\tilde{k} = kC_0/w$ and the Reynolds number $r = \nu/w$ are introduced, where w is frequency of a wave, k is the (vertical) component of the wave vector and $C_0 = \sqrt{5/6}V_T$ is the adiabatic sound speed of linear wave. The Reynolds number and the Knudsen number are obviously linked:

$$Kn = \frac{\lambda}{\lambda_b} = \frac{w}{\nu}\frac{V_T}{2\pi C_0} = \sqrt{\frac{6}{5}}\frac{1}{2\pi r}.$$

Let $\tilde{k} = \beta + i\alpha$, then

$$n_i = a_i \exp\left(-iw\left(t - \frac{\beta}{C_0}z\right)\right)\exp\left(-w\frac{\alpha}{C_0}z\right)$$

and the real part $\beta = \text{Re}(k)C_0/w = C_0/C$ is the inverse nondimensional phase velocity, α is the factor of attenuation.

We search the basic Fourier component solution of the linearized system as a superposition of three plane waves

$$n_i = A_i^1 \exp(-iwt + ik_1z) + A_i^2 \exp(-iwt + ik_2z)$$

$$+ A_i^3 \exp(-iwt + ik_3z), \qquad (17)$$

where $k_j = \tilde{k}_j w/C_0$, $i, j = 1, 2, 3$ and $\tilde{k}_j$ are solutions of the dispersion equation (16) correspondent to the modes; $\{n_i\} = \{\rho', U', T', P'_{zz}, q'_z, \bar{q}'_z\}$ is the disturbance variables. For example, ρ' is defined through the relation $\rho = \rho_0(1 + \rho')$.

Substituting (17) into the linearized system, one expresses $A_2^j, A_3^j, A_4^j, A_5^j, A_6^j$ in terms of $A_1^j \equiv A^j$. To determine the coefficients A^1, A^2, A^3 we should choose boundary conditions, considering a problem at half-space and the reflection of molecules from a plane as a diffuse one [12]. The boundary condition for the distribution function looks as

$$f(z = 0, \vec{V}, t) = \frac{n}{\pi^{3/2}V_T^3}\exp\left\{-\frac{(\vec{V} - \vec{U}_0 e^{-iwt})^2}{V_T^2}\right\},$$
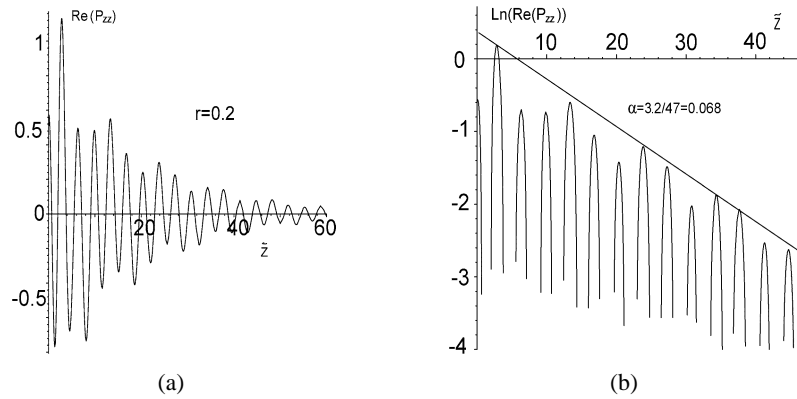
$$V_z > 0.$$

Fig. 1. Finding of attenuation factor.

Here U_0 stands for an amplitude of the hydrodynamic velocity oscillations. For $U_0/V_T \ll 1$ we have:

$$\varphi(z = 0, \vec{V}, t) = \frac{f - f^{(0)}}{f^{(0)}} \sim 2 \frac{U_0 V_Z}{V_T^2} e^{-iwt}, \quad V_z > 0.$$

The correspondent hydrodynamical variables at the boundary has the values

$$\rho'(z = 0, t) = \langle \varphi(z = 0, \vec{V}, t) \rangle$$
$$= \frac{1}{\pi^{3/2} V_T^3} \int d\vec{V} \, \varphi(z = 0, \vec{V}, t) e^{-V^2/V_T^2}$$
$$= \frac{U_0}{\sqrt{\pi}} e^{-iwt},$$
$$U'(z = 0, t) = \left\langle \frac{V_Z}{V_T} \varphi(z = 0, \vec{V}, t) \right\rangle = \frac{U_0}{2} e^{-iwt},$$
$$P'_{zz}(z = 0, t) = \left\langle \frac{V_Z^2}{V_T^2} \varphi(z = 0, \vec{V}, t) \right\rangle = \frac{U_0}{\sqrt{\pi}} e^{-iwt}.$$
$$(18)$$

Substituting the values A_2^j, A_4^j into (17) and comparing right-hand sides of expression (17) and (18) yields in the system of equations in variables A^j. Solving the system of equations we extract the variables A^1, A^2, A^3.

5. Comparison with experiment and results of other evaluations

In experimental acoustics the pressure perturbation amplitude is usually measured. Correspondent combination of the basic variables for the pressure is given

by the formula

$$P'_{zz}(z, t) e^{iwt} = A_4^1 e^{ik_1 z} + A_4^2 e^{ik_2 z} + A_4^3 e^{ik_3 z}. \quad (19)$$

The real part of this expression relates to experiment. In Fig. 1(a) the real part of this expression is represented at $r = 0.2$, where $\tilde{Z} = zw/C_0$ is dimensionless coordinate. The attenuation factor α is determined as a slope ratio of the diagram of the logarithm of the pressure amplitude depending upon distance between the oscillating wall and the receiver. It is illustrated at Fig. 1(b).

The dispersion relation (16) is the binary cubic equation with coefficients parametrized by r. At the vicinity of the limit $r \to 0$ (free molecule flow) we start from the propagation velocity by the formula $C_0/C = 0.54 + 0.15r^2 + o(r^4)$.

Alexeev [23,27] and Chen–Rao–Spiegel [24–26] consider the propagation of the only sound mode. For a finding of the attenuation factor they solve the dispersion relation. Unlike these works we take into account the propagation of three modes. The attenuation factor is determined graphically as shown in the Fig. 1.

In Figs. 2, 3 a comparison of our results of numerical calculation of dimensionless sound speed and attenuation factor depending on r is carried out in a parallel way with the results by other authors. The kinetic theory gives the good agreement with the experimental data at arbitrary *Kn* numbers. The Navier–Stokes prediction is qualitatively wrong at big Knudsen number. Our results for phase speed give the good consistency with the experiments at all Knudsen numbers. However, our results for the attenuation of ultrasound are good (as we can see in experiment) only for the num-
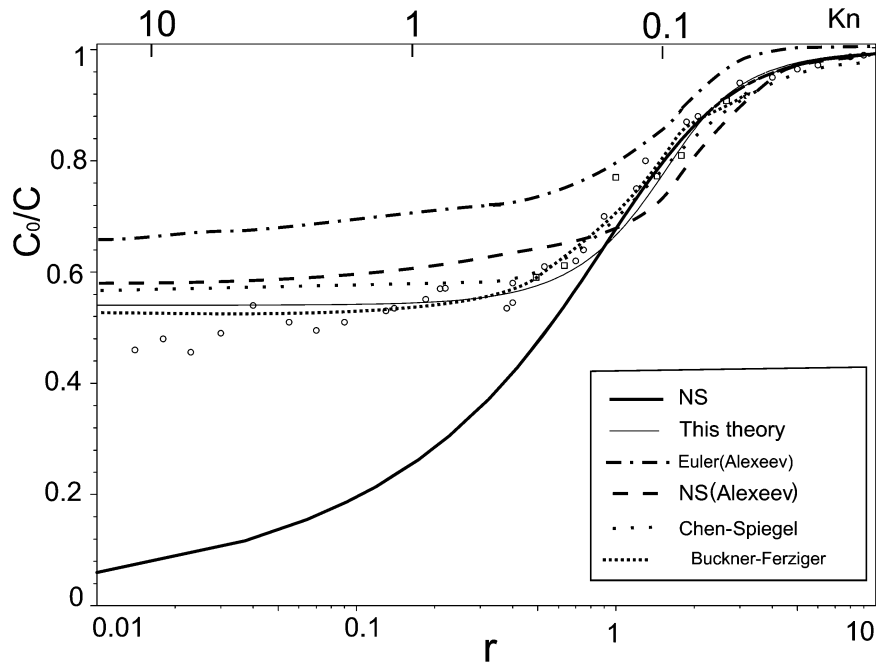
Fig. 2. The inverse nondimensional phase velocity as a function of the inverse Knudsen number. The results of this Letter are compared to Navier–Stokes, the theory of Chen–Spiegel [25,26], the generalized Euler and generalized Navier–Stokes equations of Alexeev [23], the results of Buckner–Ferziger [12] based on the direct solution of the Boltzmann equation (BGK-model) and the experimental data of Meyer–Sessler and Greenspan [3,4].
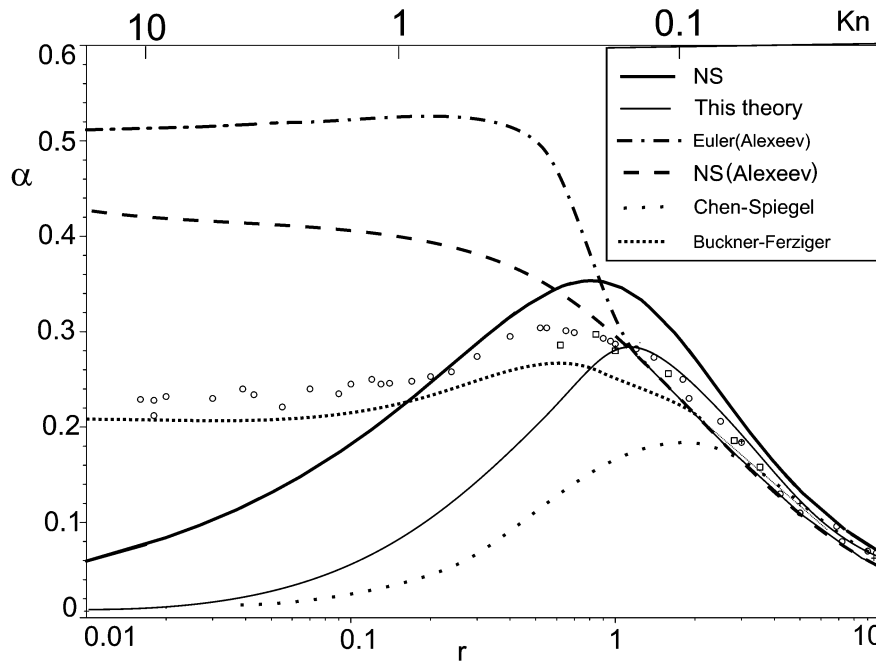

Fig. 3. The attenuation factor as a function of the inverse Knudsen number.

ber r up to order unity. But our results look a bit better than Navier–Stokes, Alexeev [23,27] and Chen–Rao–Spiegel ones [24–26] (the other fluid equations [6,23, 25,26]).

Chen–Rao–Spiegel derive the equations of fluid dynamics from the kinetic theory of a simple gas. As in the works of Hilbert, of Chapman and Enskog [6], they started from an expansion in the Knudsen number. But they do not apply solvability condition in each order, as it is usually done. This improves the equations, by the procedure arisen at Bogoliubov method [17]. Their expressions, unlike the Navier–Stokes equations, for the pressure tensor P_{ij} and heat current $\vec{q}$ are expressed in terms of the fluid fields (ρ, U, T) and their derivatives. That is, their final formulae for P_{ij} and $\vec{q}$ implicitly contain terms of all orders in Kn and so they may hope, that their procedure will produce higher accuracy than Navier–Stokes equations. But there is also a divergence with the experiment for attenuation.

Alexeev derived the generalized Boltzmann equation from the BBGKY-hierarchy of the kinetic equations taking into account three possible levels of scales, connected with the mean time of particle interactions, mean time between collisions and the hydrodynamic time. He obtained the generalized Navier–Stokes and generalized Euler equations with the help of the Chapman–Enskog method. His theory gives a qualitative agreement with the experimental data for attenuation. However it is necessary to solve the system of the differential equations of twice higher order, than the traditional hydrodynamical equations. We can apply our method for the generalized Boltzmann equation of Alexeev [20,27] and we hope, that such "joint" theory would give a better agreement with the experimental data for attenuation at arbitrary Knudsen number.

6. Conclusion

The attenuation of sound at big Knudsen numbers is not "damping" (due to intermolecular collisions), but rather "phase mixing" (due to molecules which left the oscillator at different phases arriving at the receiver at the same time). Solving the Boltzmann equation by the method of the Gross–Jackson [5] revealed a disappearance of discrete modes at some values $r = r_c$ [11,12]. When the number of the moments increases,

r_c decreased. For example, Buckner and Ferziger in the paper [12] have shown, that for $r > 1$ the solution is determined mainly by the discrete sound mode and the dispersion relation may be used in calculating the sound parameters. For $r < 1$, the continuous modes are important. The solution remains "wavelike", but it is no longer a classical plane wave. In fact, the sound parameters depend on a position of the receiver.

Below r_c the solution is represented as superposition of eigen functions from continuous spectrum, therefore the classical understanding of a sound should be changed. The concept of a dispersion relation is not applicable more. It is unclear whether a continuum theory, based on BGK-model, can do much better than this results for attenuation.

The attenuation factor at big Knudsen numbers $Kn > 1$ is modeled by the account of effects of a relaxation in integral of collisions. The model of the Gross–Jackson at given N limits an account of higher relaxation times (fast attenuation) as essentially bases on the condition:

$$\lambda_i = \lambda_{N+1}, \quad i > N + 1.$$

Supreme times of a relaxation are assumed identical. It means, that the inclusion of the supreme eigen functions χ_i, $i \geqslant N + 1$ is necessary, that would allow to move in the range of higher Knudsen numbers.

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