©Freund Publishing House Ltd. London International Journal of Turbo and Jet Engines,14,229-238(1997)

Shock Boundary Layer Interaction Structure

Tony W.H. Sheu

Department of Naval Architecture and Ocean Engineering, National Taiwan University, 73, Chou-Shan Road, Taipei, Taiwan, R.O.C.

Abstract

A model is presented to study the interaction between an oblique shock and a fully-unseparated turbulent boundary layer over the nonadiabatic wall. The method of matched asymptotic expansion, valid in the double limit of free-stream Mach number $M \to 1$ and Reynolds number Re $\to \infty$, is adopted to analyze the nature of nonadiabatic weak interaction. This study addresses the impact of heat transfer on the shock-boundary layer interaction structure. Understanding of the interaction structure will help to provide an adequate grid-point distribution for the numerical study of an oblique shock impinging on the boundary layer over a body of axisymmetric configuration.

Nomenclature

γ	: Specific heat ratio
C_p	: Specific heat at constant pressure
P_r	: Laminar Prandtl number
μ	: Laminar viscosity
Υ_t	: Turbulence viscosity
ε	: Shock strength
Re	: Reynolds number
U_τ	: Shear velocity
X, Y	: Spatial coordinates
x	: Stretched space variable in the x-direction
y^+	: Stretched variable in the y-direction in the external transonic flow region
y	: Stretched variable in the y-direction of the velocity defect flow region
$\hat{y}$	: Stretched variable in the y-direction of the Reynolds stress layer region
$\tilde{y}$	: Stretched variable in the y-direction of the wall layer region.
δ_c	: Boundary layer thickness
δ	: Velocity defect layer thickness
δ_r	: Reynolds stress sublayer thickness
δ_l	: Wall layer thickness
L	: Reference length
$\bar{a}_o^*$	: Reference velocity
P	: Pressure
u	: Velocity in x-direction
v	: Velocity in y-direction
ρ	: Density
T	: Temperature (stagnation)
λ_u, $\lambda^{(n)}$, Υ	: Interaction coefficients
Subscript w	: Wall condition
Subscript e	: Edge of the boundary
*	: Critical condition
^	: Reynolds stress sublayer indicator
~	: Wall layer indicator
'	: Fluctuation indicator
+	: External transonic indicator
(n)	: Indicator of interaction

1. Introduction

Upon impinging on a solid surface, a shock wave will penetrate into the boundary layer which is attached to the body surface. Under such circumstances, both the shock strength and its inclination will undergo change in a complex manner. The reason for such changes is the fluid viscosity and the interaction structure between the shock and the developed boundary layer. The shock penetration will finally terminate in the subsonic region which is just above the body surface. Very often, such shock-boundary layer interaction is observed in the inlet ducts and exhaust nozzles of civilian and military airplanes. A fundamental understanding of how shock waves interact with boundary layers is, thus, important in aircraft design. The interaction structure has been the subject of considerable interest in the past few decades.

The principal interaction arises from the displacement thickness, which causes a significant change in the surface pressure. Also, fluid viscosity may cause the already established pressure jump to smear in the boundary layer. This smearing of the pressure may persist and extend through several boundary layer thicknesses along the primary flow direction. The compression disturbances produced on the supersonic stream side will propagate along the Mach line and gradually reach the subsonic part of the boundary layer. Usually, the shock wave is very thin, on the order of a few mean free path thicknesses. The reduction of the kinetic energy in association with the gas flow passing over the shock will be converted into pressure force, thus compressing the gas and increasing the temperature and entropy. Since the shock thickness is too thin to spread the large pressure change, the pressure responds to an increase ahead of the impingement point through the upstream influence. The established compression family of waves will interact among themselves and weaken the incoming shock. This upstream propagation mechanism is believed to play an important role in the shock wave and turbulent boundary layer interaction structure. The incoming shock partly reflects and partly penetrates such that new disturbances are produced and propagate along another family of Mach lines in the supersonic region.

Numerous efforts have been devoted in the past four decades to achieving a better understanding of the shock-boundary layer interaction. Much of the previous work has focused on the laminar and adiabatic cases. This study refers back to the work of Ackeret *et al.* /1/, who investigated the influences of the high-speed compression shock on the development of the boundary layer. Substantial advances in understanding the interaction structure between the shock and boundary layer have been achieved by such mathematicians as Howarth /2/ and Tsien and Finston /3/ in the past half-century, and much attention has been devoted to the laminar and adiabatic cases. A wide variety of interaction models has been proposed, among which the two-layer model of Lighthill /4/ and the triple deck model of Brilliant Stewartson /5/ are often referred to. Theoretical investigation into the nature of the interaction can be mainly divided into the integral /6,7/ and the asymptotic perturbation /8-15/ methods. In this study we prefer the asymptotic perturbation approach since use of this approach can provide useful flow details at different length scales.

2. Derivation of Working Equations

We aim to apply the asymptotic and perturbation method to analysis of the interaction of an externally generated oblique shock with the steady turbulent flow over a flat plate. For this study, the oblique shock is assumed to be weak, so that the boundary layer is considered to be attached to the wall surface. In addition, the interaction region under investigation is assumed to be distant from the leading edge, thus avoiding the difficulty of dealing with the so-called Goldstein singularity. Both the molecular Prandtl number, Pr, and the compressibility parameter, $(\gamma-1)M_\infty^2$, have values of order one. The resulting turbulent structure is, consequently, not effected by the Markovin hypothesis /16/.

In the present turbulent flow analysis, the scales of turbulence fluctuations are determined on the basis of experimental observations given by Kistler /17/. He found that the turbulent fluctuations in the velocity, temperature, density, and fluid viscosity are on the order of the friction velocity in the case of non-hypersonic flow (M_∞ < 5). The fluctuation pressure, then, is on the order of the square of the friction velocity. In this study, we neglect fluctuations of transport properties, such as μ', k', c'_p and p'_r, for simplicity. As far as turbulent dissipation and other double fluctuation terms are concerned, their values are negligibly small /18/. An exception is the turbulent heat transfer $<v'T'>$, which is approximated using the Boussinesq's eddy viscosity hypothesis.

With the above assumptions, we normalize the resulting conservation equations. This dimensionless procedure is desired from the application viewpoint. The referred characteristic quantities involve L for length, $\bar{a}_{\infty}^{*}$ for velocity, $\bar{\rho}_{\infty}^{*}\bar{a}_{\infty}^{*2}/\gamma$ for pressure, $\bar{a}_{\infty}^{*2}/\gamma\bar{R}$ for temperature, and $\bar{\rho}_{\infty}^{*}$ for density, $\bar{\mu}_{\infty}^{*}$ for fluid viscosity, and $\bar{u}_{\tau}$ for turbulent velocity. The remaining dimensional turbulence quantities employed here are summarized as follows:

$$T' = \bar{T}' \cdot \bar{a}_{\infty}^{*} / \bar{T}_{\infty}^{*}\bar{u}_{\tau},$$
$$\rho' = \bar{\rho}'\bar{a}_{\infty}^{*} / \bar{\rho}_{\infty}^{*}\bar{u}_{\tau},$$
$$p' = \bar{p}'\bar{a}_{\infty}^{*2} / \bar{p}_{\infty}^{*}\bar{u}_{\tau}^{2}.$$

Based on the premise that the turbulence is modeled by the eddy viscosity model, the nondimensional governing equations, up to an accuracy order of $o\left(\bar{u}_{\tau}^{2}\right)$, are those given below:

$$\frac{\partial}{\partial x}(\rho u + u_\tau^2 < \rho' u' >) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$(\rho u + u_\tau^2 < \rho' u' >)\frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \tag{2}$$
$$-\frac{1}{\gamma}\frac{\partial p}{\partial x} - u_\tau^2 \frac{\partial}{\partial x}[(\rho < u'u' > + u < \rho' u' >) +$$
$$\frac{\partial}{\partial y}(\rho < u'v' > + v < \rho' u' >)] +$$
$$\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) + O(u_\tau^3),$$

$$(\rho u + u_\tau^2 < \rho' u' >)\frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = \tag{3}$$
$$-\frac{1}{\gamma}\frac{\partial p}{\partial y} - u_\tau^2 [\frac{\partial}{\partial x}(\rho < u'v' > + u < \rho' v' >) +$$
$$\frac{\partial}{\partial y}(\rho < v'v' > + v < \rho' v' >) -$$
$$\rho u \frac{\partial}{\partial x}(< \rho' v' > / \rho) - \rho v \frac{\partial}{\partial y}(< \rho' v' > / \rho)] +$$
$$\frac{1}{\mathrm{Re}}[\frac{\partial}{\partial y}(\mu(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x})) + \frac{\partial}{\partial x}(\mu \frac{\partial u}{\partial y})]$$
$$+ O(u_\tau^3),$$

$$(\rho u + u_\tau^2 < \rho' u' >)\frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \tag{4}$$
$$\frac{1}{\mathrm{Re}\, P_r}\frac{\partial^2 T}{\partial y^2} + (\gamma - 1)\mu(\frac{\partial u}{\partial y})^2 + \frac{\gamma - 1}{\gamma} u \frac{\partial p}{\partial x} -$$
$$u_\tau^2 \frac{\partial}{\partial y}(\rho < v'T' >) - (\gamma - 1)\rho u_\tau^2 \frac{\partial u}{\partial y} < u'v' >$$
$$+ O(u_\tau^3),$$

where

$$p = \rho T + u_\tau^2 < \rho' T' > + o(u_\tau^3). \tag{5}$$

3. Asymptotic and Perturbation Study

Analytic analysis of the working set of nonlinear partial differential equations to elucidate the interactive flow structure is hardly possible. For this reason, we will consider a simplified set of coupled equations. The approach taken here is to adopt the asymptotic and perturbation method. The basis is as follows. We first decompose the physical domain of interest into several domains, each of which has physical importance. In between two adjacent domains, appropriate matching conditions are prescribed to blend them. The primary subdomains to be investigated include the transonic external flow region, undisturbed turbulent boundary layer and interaction region comprised of the velocity defect, blending, and the wall layers. Asymptotic perturbation method is, in principle, an art. The quality of the computed solution is drastically dependent on the choice of the expansion parameter and of the method for perturbing the dependent variables. In each layer, we expand dependent variables as functions of legitimative perturbation quantities, which should be physically supported through well-known phenomena. As for independent variables, they are stretched in the interaction region to achieve a match of solutions at the interface. In the following, we will proceed with step-by-step analysis in each layer of the physical domain.

3.1. External transonic flow region

Given that heat transfer prevails in regions confined to the thermal boundary layer, the incoming dependent variables in the external region can be approximated as follows:

$$u_\infty^e = 1 + \varepsilon + \cdots, \tag{6a}$$

$$T_\infty^e = 1 - (\gamma - 1)\varepsilon + \cdots, \tag{6b}$$

$$\rho_\infty^e = 1 - \frac{2}{\gamma + 2}\varepsilon + \cdots, \tag{6c}$$

$$P_\infty^e = 1 - \frac{\gamma(\gamma + 1)}{\gamma + 2}\varepsilon + \cdots. \tag{6d}$$

In this region, the conservation of energy equation is simplified as

$$T + \frac{\gamma - 1}{2}u^2 = \frac{1}{2}(\gamma + 1). \tag{6e}$$

In the presence of compression disturbances resulting from the incoming shock, the strength of which is on the order of ε, dependent variables in this region can be expressed using the following expansions:

$$u = 1 + \varepsilon u_1^+(x, y^+) + \cdots, \tag{7a}$$

$$v = \upsilon_1^+ + \varepsilon v_1^+(x, y^+) + \cdots, \tag{7b}$$

$$\rho = 1 + \varepsilon \rho_1^+(x, y^+) + \cdots, \tag{7c}$$

$$P = 1 + \varepsilon P_1^+(x, y^+) + \cdots. \tag{7d}$$

As to independent variables, they are stretched by

$$x = \frac{X - 1}{\Delta}, \tag{8a}$$

$$y^+ = \frac{Y}{\delta e}. \tag{8b}$$

By substituting equations (7),(8) into working equations (1)-(5), one can derive the transonic equation, irrotationality condition, and pressure-velocity relation, respectively, as follows for the disturbances:

$$\frac{\gamma + 1}{2}(u_1^{+2})_{,xx} - (u_1^+)_{,y^+y^+} = 0, \tag{9a}$$

$$(u_1^+)_{,y^+} - (v_1^+)_{,x} = 0, \tag{9b}$$

$$P_1^+ + \gamma u_1^+ = 0. \tag{9c}$$

These equations are obtained together with the attendant relations as

$$\Delta = \varepsilon^{1/2}\delta e, \tag{10a}$$

$$\upsilon_1^+ = \varepsilon^{3/2}. \tag{10b}$$

For this study, the flow is considered to be supersonic everywhere in the external transonic flow region. This permits the application of the method of characteristics to determine the change in the physical properties across the shock. All the disturbances propagate along the left- and right-running characteristics, C^+ and C^-, respectively. Exceptions to this are the entropy and the vorticity. In regions ahead of the shock, simple waves with straight characteristic lines having constant properties are present. As a result, the fluctuation velocities u_{1u}^+, v_{1u}^+ along the right-running characteristic C^- can be determined according to the expression given below:

$$\frac{2}{3}(\gamma + 1)^{1/2}(u_{1u}^+)^{3/2} + v_{1u}^+ = \frac{2}{3}(\gamma + 1)^{1/2}. \tag{11}$$

Downstream of the shock fluctuation velocities, u_{1d}^+ and v_{1d}^+ are, on the other hand, determined along the line of the left-running characteristic C^+ by

$$\frac{2}{3}(\gamma + 1)^{1/2}(u_{1d}^+)^{3/2} + v_{1d}^+ = \frac{2}{3}(\gamma + 1)^{1/2}\Lambda, \tag{12}$$

where $\Lambda(<1)$ is called the shock strength parameter.

Across the oblique shock, the discontinuity condition is governed by the Prandtl polar equation given below:

$$u_{1d}^+ + u_{1u}^+ = \frac{2}{\gamma + 1}(\beta^+)^2, \tag{13}$$

$$v_{1d}^+ - v_{1u}^+ = \frac{2}{\gamma + 1}(\beta^+)^3 - 2u_{1u}^+\beta^+, \tag{14}$$

where the shock angle β^+ with respect to the incoming flow is defined by

$$\beta^+ = (v_{1u}^+ - v_{1d}^+)/(u_{1u}^+ - u_{1d}^+). \tag{15}$$

The solutions of the upstream fluctuation velocities u_{1u}^+, v_{1u}^+ and the downstream fluctuation velocities u_{1d}^+, v_{1d}^+ are determined by equations (11)-(15).

3.2. Undisturbed turbulent boundary layer

Turbulent flow is characterized by the existence of velocity defect and inner wall layers. In the velocity defect layer, the convective terms in the x-momentum equation are taken to be on the order of the y-derivative of the Reynolds shear stresses. In the wall layer, the viscous and Reynolds shear stresses dominate other quantities and are balanced with each other.

For the sake of applying a matching condition along the flow direction, in between undisturbed and interaction regions, we stretch independent variables in the velocity defect layer by $x = \frac{X-1}{\Delta}$, $y = \frac{Y}{\delta}$. We adopt the non-dimensional "Law of the wake", as proposed by Coles /19/, to take the compressibility effect into consideration so as to expand the velocity components as follows:

$$u = 1 + \varepsilon + \frac{u_\tau}{\kappa}(\ln y - 2\pi) + \cdots, \tag{16}$$

$$v = \delta_i(x)\, u_\tau(x)\, v_{01}(y) + x\Delta\delta u_\tau v_{11}(y) + \cdots, \tag{17}$$

where the shear velocity u_τ is defined by

$$u_\tau = \left(\frac{\tau_w}{\rho_e}\right)^{1/2}. \tag{18}$$

In equation (16), π and κ are known as wake parameters. Their values are taken as two different constants.

As for the independent variables in the wall layer, they are also stretched by $x = \frac{X-1}{\Delta}, \tilde{y} = \frac{Y}{\tilde{\delta}}$. This is useful for achieving the matching condition at the interface of the undisturbed and interaction regions. The "Law of the wall", proposed by Coles /19/, is employed for the velocity component u:

$$u = u_\tau (T_w / T_e)^{1/2} (\kappa^{-1} \ln \tilde{y} + 5). \tag{19}$$

The y-component velocity is obtained by performing a Taylor series expansion:

$$v = \delta_i u_\tau \tilde{v}_{01}(\tilde{y}) + x\Delta\delta_i u_\tau \tilde{v}_{01}(\tilde{y}) + \cdots,$$

where

$$\tilde{\delta} = \upsilon_w / (\tau_w / \rho_w)^{1/2}. \tag{20}$$

By examining equations (16, 19), the velocity u in the velocity defect layer fails to match that in the wall layer as $y \to \infty$. For compressible flows, this difficulty may be avoided by employing a velocity expression which holds in the region of $\tilde{\delta} \ll y \ll \delta$. The velocity profile for the compressible turbulent flow over the flat plate, subject to the heat transfer, can be written as

$$u = \frac{u_\infty}{2A^2}\{(B^2 + 4A^2)^{1/2} \sin[\text{constant} \cdot A + \frac{A}{u_\infty \kappa} u_\tau \sqrt{\frac{T_w}{T_\infty}} \ln \tilde{y}] + B\}, \tag{21}$$

where

$$A^2 = \frac{\gamma - 1}{2} M_\infty^2 \frac{T_\infty}{T_w}, \tag{22}$$

$$B = \left(\frac{\gamma - 1}{2} M_\infty^2 + 1\right)\frac{T_\infty}{T_w} - 1, \tag{23}$$

and

$$M_\infty^2 = (1+\varepsilon)^2 / [1 - \frac{\gamma - 1}{2}(\varepsilon^2 + 2\varepsilon)]. \tag{24}$$

To derive equation (3), the modified Prandtl mixing length theory for a variable density is used together with the Crocco-Busemann relation for both temperature and velocity.

4. Interaction Region

The interaction region inside the turbulent boundary layer consists of the velocity defect outer layer, Reynolds stress sublayer, and wall layer. They will be discussed separately in the following sections.

4.1. Velocity defect layer

To obtain a legitimate way to perturb dependent variables in this layer, it is instructive to depict the interaction effect and provide a way to match solutions with those at the undisturbed outer turbulent layer. The dependent variables are expressed as the sum of a series of undisturbed quantities and their interaction counterparts by

$$u = 1 + \varepsilon + u_\tau u_{01}(y) + x\Delta u_\tau u_{11}(y) + \cdots \qquad (25a)$$
$$+ \lambda_1 u_1(x,y) + \lambda_2 u_2(x,y) + \cdots +$$
$$\lambda_1^{(1)} u_1^{(1)}(x,y) + \lambda_2^{(1)} u_2^{(1)}(x,y) + \cdots,$$

$$v = \delta u_\tau v_{01}(y) + x\Delta\delta u_\tau v_{11}(y) + \cdots \qquad (25b)$$
$$+ \Upsilon_1 v_1(x,y) + \Upsilon_2 v_2(x,y) + \cdots,$$

$$P = P_e + \cdots + \lambda_1 P_1(x,y) + \lambda_2 P_2(x,y) \qquad (25c)$$
$$+ \cdots \lambda_1^{(1)} P_1^{(1)}(x,y) + \cdots,$$

$$\rho = \rho_e + \cdots + \lambda_1 \rho_1(x,y) + \lambda_2 \rho_2(x,y) + \qquad (25d)$$
$$\cdots + \lambda_1^{(1)} \rho_1^{(1)}(x,y) + \cdots,$$

$$T = T_e + u_\tau \tilde{T}_{01}(y) + \cdots + \lambda_1 T_1(x,y) + \qquad (25e)$$
$$\cdots + \lambda_1^{(1)} T_1^{(1)}(x,y) + \cdots.$$

In the above equations, all the independent variables are stretched by $x = \dfrac{X-1}{\Delta}$, $y = \dfrac{Y}{\delta}$. Revealed by these equations is the interaction effect, which is expressed in terms of the coefficients λ_1, $\lambda_2,\ldots,\lambda_1^{(1)}$, $\lambda_2^{(1)}$ and Υ_1, $\Upsilon_2\ldots$. Note that $\upsilon_1 >> \Delta\delta u_\tau$ holds in equation (25).

Substituting the stretched independent variables and the dependent variables into the X-momentum equation (2), one can find that the Reynolds and shear stresses serve as higher order terms. Consequently, the nature of the flow in the layer is, in essence, inviscid. In this layer, the interaction parameters are related by

$$\lambda_1 = \varepsilon, \qquad (26)$$
$$\Upsilon_1 = \varepsilon^{3/2}, \qquad (27)$$
$$\lambda_1^{(1)} = u_\tau \hat{\lambda}_1. \qquad (28)$$

As for the leading interaction variables, they are given by

$$u_1 = -P_1/\gamma, \qquad (29)$$

$$\rho_1 = P_1/\gamma, \qquad (30)$$

$$v_1(x) = \frac{\gamma+1}{\gamma}\frac{\partial P_1}{\partial x}\int_0^y (\alpha\cdot\ln y + \beta_0)\,dy + f(x). \qquad (31)$$

In these equations, $P_1(x) = y + P_1^+(x,0)$, which is calculated by matching solutions between this layer and the external region. From the energy equation in its lowest order form, one can derive

$$T_1 + (\gamma-1)u_1 = 0 \qquad (32)$$

since

$$T_1 = u_1 = 0 \text{ as } x \to -\infty. \qquad (33)$$

Since we consider the viscosity as the second order contribution to the energy equation, the interaction length, $\Delta = \mathrm{Re}\delta^3$, results. P_1, a function of x only, is derived from the Y-momentum equation. Given the above statements, one can have $\delta = \Delta^2$ since the convection term plays the role of the second order term in the Y-momentum equation. In the velocity defect layer, we have

$$u_1^{(1)} = \frac{1-\gamma}{\gamma-2}\int v_{1,y}\,dx + \frac{1-2\gamma}{\gamma(\gamma-2)}u_{01}(y)P_1, \qquad (34)$$

$$P_1^{(1)} = \gamma P_1 u_{01} - \frac{\gamma(1-\gamma)}{\gamma-2}\int v_{1,y}\,dx - \frac{1-2\gamma}{\gamma-2}u_{01}(y)P_1, \qquad (35)$$

$$T_1^{(1)} = (1-\gamma)\left[\frac{1-\gamma}{\gamma-2}\int v_{1,y}\,dx - \frac{3(1-\gamma)}{\gamma(\gamma-2)}u_{01}(y)P_1\right], \qquad (36)$$

$$\rho_1^{(1)} = \frac{\gamma-1}{\gamma-2}\int v_{1,y}\,dx + \frac{2\gamma^3 - 7\gamma + 3}{\gamma(\gamma-2)}u_{01}(y)P_1. \qquad (37)$$

The remaining relations derived from this analysis are

$$\Delta = \varepsilon^{-1/2}u_\tau^2 = \varepsilon^{1/2}\delta_e = \mathrm{Re}\,\delta^3, \qquad (38)$$

and

$$\rho_{01} = -T_{01} = (\gamma-1)u_{01}. \qquad (39)$$

4.2. Reynolds stress sublayer

In the velocity defect layer, the magnitude of the inertial force has the same order of magnitude as does that of the streamwise pressure gradient. However, the Reynolds and viscous stresses are dominant in the wall layer; hence, an intermediate region must be inserted to link the transport phenomena between the velocity defect and the wall layers. In this layer, inertial, pressure gradient and Reynolds stress terms have the same order of magnitude. The transverse momentum transfer is transported by the turbulence rather than by the molecular mechanism.

In this layer, the dependent variables are expanded by

$$u = 1 + \varepsilon + u_\tau \hat{u}_{01}(\hat{y}) + x\Delta u_\tau \hat{u}_{11}(\hat{y}) + \cdots \tag{40}$$

$$+ \hat{\lambda}_1 \hat{u}_1 + \hat{\lambda}_2 \hat{u}_2 + \cdots + \hat{\lambda}_1^{(1)} \hat{u}_1^{(1)} + \cdots,$$

$$v = \delta_s u_\tau \hat{v}_{01} + x\Delta\delta_s u_\tau \hat{v}_{11}(\hat{y}) + \cdots + \hat{\Upsilon}_1 \hat{v}_1 + \cdots, \tag{41}$$

$$P = P_e + \cdots + \hat{\lambda}_1 \hat{P}_1 + \cdots \hat{\lambda}_1^{(1)} \hat{P}_1^{(1)} + \cdots, \tag{42}$$

$$T = T_e + \cdots + \hat{\lambda}_1 \hat{T}_1 + \cdots + \hat{\lambda}_1^{(1)} \hat{T}_1^{(1)} + \cdots. \tag{43}$$

The independent variables are stretched by $x = \dfrac{X-1}{\Delta}$, $y = \dfrac{Y}{\delta_e}$. For this study, the eddy viscosity model

$$-\rho < u'v' >= \rho \Upsilon_t u_{,Y} \tag{44}$$

is chosen as the closure of the working Reynolds average transport equations. The turbulent viscosity, Υ_t, in equation (44) is expressed by $\Upsilon_t = \dfrac{\delta_s}{\delta}\hat{\Upsilon}_t$. Substituting the above equation into the x-momentum equation, one has

$$\delta_s = u_\tau \Delta, \tag{45}$$

and

$$\hat{u}_1 + \hat{P}_1 / \gamma = 0. \tag{46}$$

Under these circumstances, the Reynolds stresses can remain, up to their lowest order, in the interaction equations. From the continuity equation, one can derive

$$\hat{u}_1 + \hat{\rho}_1 = 0 \tag{47}$$

since

$$\hat{u}_1 = \hat{\rho}_1 = 0 \text{ as } x \to -\infty. \tag{48}$$

The relations between the interaction parameters in this layer are given by

$$\hat{\Upsilon}_1 = u_\tau \delta_s \hat{\lambda}_1 / \Delta, \tag{49}$$

$$\lambda_1^{(1)} = u_\tau \hat{\lambda}_1, \tag{50}$$

$$\hat{\lambda}_1 = \varepsilon. \tag{51}$$

From the energy equation, the interaction temperature $\hat{T}_1$ can be related to $\hat{u}_1$ by

$$\hat{T}_1 = (1-\gamma)\hat{u}_1. \tag{52}$$

The pressure gradient along the normal direction, $\hat{y}$, remains as the higher order term in the Y-momentum equation. The remaining functional relation valid in this layer is:

$$\hat{v}_1(x) = (\gamma + 1)\hat{u}_{1,x} \int_0^{\hat{y}} \hat{u}_{01}\, d\hat{y} - \gamma \hat{\Upsilon}_t \hat{u}_{,\hat{y}} + \hat{f}(x), \tag{53}$$

where

$$\hat{u}_{01} = \alpha \cdot \ln \hat{y} + \alpha \cdot \ln(\frac{\delta_s}{\delta}) + \beta_0.$$

It is worth noting that $\hat{u}$ satisfies

$$\hat{u}_{,x} = C(\hat{y}\hat{u}_{,\hat{y}})_{,\hat{y}}, \tag{54}$$

and

$$\hat{u}(-\infty) = 0. \tag{55}$$

As for the non-interaction dependent variables, $\hat{\rho}_{01}, \hat{T}_{01}$, they are expressed as functions of $\hat{u}_{01}$:

$$\hat{\rho}_{01} = -\hat{T}_{01} = (\gamma - 1)\hat{u}_{01}. \quad (56)$$

The functional forms for the second lowest interaction dependent variables are

$$\hat{u}_1^{(1)}(x) = (\gamma + 1)u_{1,x}\int_0^{\hat{y}}\left(\frac{1}{\hat{\Upsilon}_t}\int_0^{\hat{y}}\hat{u}_{01}\right)d\hat{y} - \gamma\hat{u} + \int_0^{\hat{y}}\hat{f}(x)/\hat{\Upsilon}_t\,d\hat{y}, \quad (57)$$

$$\hat{T}_1^{(1)} = (1-\gamma)\hat{u}_1^{(1)} - (\gamma - 1)\hat{u}_1\hat{u}_{01}, \quad (58)$$

$$\hat{\rho}_1^{(1)} = \gamma\hat{u} + (2\gamma^2 - 2\gamma - 1)\hat{u}_1\hat{u}_{01} - \hat{u}_1^{(1)}, \quad (59)$$

$$\hat{P}_1^{(1)} = \gamma\hat{u} + (\gamma^2 - 2\gamma)\hat{u}_1\hat{u}_{01} - \gamma\hat{u}_1^{(1)}. \quad (60)$$

4.3. Wall layer

As the wall is approached, the Reynolds shear stresses decrease. As a result, the viscous shear stress becomes dominant over other terms. The momentum transfer is mainly due to the molecular rather than the turbulent means. By performing expansion of the dependent variables, one can obtain

$$u = u_\tau\tilde{u}_{01}(\tilde{y}) + x\Delta u_\tau\tilde{u}_{11}(\tilde{y}) + \cdots + \tilde{\lambda}_1\tilde{u}_1^{(1)}(x,\tilde{y}) + \cdots, \quad (61)$$

$$v = \delta_i u_\tau\tilde{v}_{01}(\tilde{y}) + x\Delta\delta_i u_\tau\tilde{v}_{11}(\tilde{y}) + \cdots + \tilde{\Upsilon}_1\tilde{v}_1^{(1)}(x,\tilde{y}) + \cdots, \quad (62)$$

$$P = P_e + \cdots + \tilde{\lambda}_1\tilde{P}^{(1)}(x,\tilde{y}) + \cdots, \quad (63)$$

$$\rho = \tilde{\rho}_{00}(\tilde{y}) + x\Delta\tilde{\rho}_{11}(\tilde{y}) + \cdots + \tilde{\lambda}_1^{(1)}\tilde{\rho}^{(1)}(x,\tilde{y}) + \cdots, \quad (64)$$

$$T = \tilde{T}_{00}(\tilde{y}) + x\Delta u_\tau\tilde{T}_{11}(\tilde{y}) + \cdots + \tilde{\lambda}_1\tilde{T}^{(1)}(x,\tilde{y}) + \cdots. \quad (65)$$

In the above, the independent variables are stretched by $x = \frac{X-1}{\Delta}, \tilde{y} = \frac{Y}{\delta_i}$. The interaction parameters in the above equations are obtained as

$$\mathrm{Re}\,\delta_i u_\tau = O(1), \quad (66)$$

$$\tilde{\Upsilon}_1 = \delta_i\tilde{\lambda}_1/\Delta, \quad (67)$$

$$\delta\delta_i = \mathrm{Re}^{-1}. \quad (68)$$

The dependent interaction variables are expressed in functional forms as follows, where the constants are determined by matching conditions in the wall layer with those in the Reynolds sublayer:

$$\tilde{v}_1^{(1)} = \int_0^{\tilde{y}}\left(\exp\left(-\int_0^{\tilde{y}}\frac{\tilde{\mu}_{00,\tilde{y}}}{\tilde{\mu}_{00}}\right)d\tilde{y} + F_1(x)\tilde{y} + F_2(x),\right. \quad (69)$$

$$\tilde{\rho}_1^{(1)} = -\int_0^{\tilde{y}}((\tilde{\rho}_{00}\tilde{v}_1^{(1)})_{,\tilde{y}} - \tilde{\rho}_{00}\tilde{u}_{1,x}^{(1)})/\tilde{u}_{01}dx, \quad (70)$$

$$\tilde{u}_1^{(1)} = \frac{\tilde{f}_1^{(1)}}{\rho_w} - \tilde{u}_{01} - \frac{1}{\rho_w}\int_0^{\tilde{y}}(\tilde{\rho}_1\tilde{\Upsilon}_t + \tilde{\mu}_1)(\tilde{u}_{01,\tilde{y}})^2\,d\tilde{y}, \quad (71)$$

$$\tilde{P}_1^{(1)} = \gamma\int_0^{\tilde{y}}x((\tilde{\rho}_1 + \tilde{\mu}_1)\tilde{u}_{11,y})_{,\tilde{y}}\,dx + \ell_1(\tilde{y}), \quad (72)$$

$$\tilde{T}_1^{(1)} = (\tilde{P}_1 - \tilde{T}_{00}\tilde{\rho}_1)/\tilde{\rho}_{00}. \quad (73)$$

In the above equations, $\tilde{v}_1^{(1)}$ is determined from the equation given below:

$$-\frac{4\tilde{u}_{01}\tilde{\mu}_{00}}{3\tilde{\Upsilon}_t\tilde{u}_{01,\tilde{y}}}\tilde{v}_{1,\tilde{y}\tilde{y}}^{(1)} + \left(\tilde{\rho}_{00} - \frac{4\tilde{u}_{01}\tilde{\mu}_{00,\tilde{y}}}{3\tilde{\Upsilon}_t\tilde{u}_{01,\tilde{y}}}\right)\tilde{v}_{,\tilde{y}}^{(1)} + \tilde{\rho}_{00,\tilde{y}}\tilde{v}_1^{(1)} + \left(\tilde{\rho}_{00}\tilde{u}_{1,x}^{(1)} - \frac{\tilde{u}_{01}}{\tilde{u}_{01,\tilde{y}}}\tilde{\rho}_{00}\right)\tilde{u}_{1,x\tilde{y}}^{(1)} = 0. \quad (74)$$

From the relation $\delta\delta_i = O(\mathrm{Re}^{-1})$ and $\delta = O(\mathrm{Re}^{-2/5})$, one can obtain

$$\delta_i = O(\mathrm{Re}^{-3/5}). \quad (75)$$

From the relations $\mathrm{Re}\delta_i u_\tau = O(1)$, $\Delta = \mathrm{Re}\delta^3$, $\delta = O(\mathrm{Re}^{-2/5})$ and $\delta_s = u_\tau$, Δ, one can derive $\delta_s = O(\mathrm{Re}^{-3/5})$.

5. Concluding Remarks

We have presented a mathematical model used to explore in detail the heat transfer and the interaction flow structure between an oblique shock wave and the turbulence boundary layer over the axisymmetric configuration. Asymptotic and perturbation analysis has formed the basis for models of shock-flow interaction in turbulent regions comprising triple layers.

Asymptotic and perturbation analysis has been conducted in the transonic external layer, velocity defect layer, and wall layer in the downstream and upstream undisturbed regions, and in the velocity defect layer, Reynolds sublayer, and the wall layer in the interaction region. In this study, different length scales in the boundary layer have been obtained and are compared with those obtained under adiabatic conditions. We summarize their length scales in the following table:

Apparent from this study is that heat transfer has a considerable impact on the interaction structure between the shock and the turbulent boundary layer. The diffusion of heat causes the velocity defect layer to behave more like a laminar case ($\delta / \Delta \ll 1$). Conclusions drawn from the results of the present fundamental study provide us with useful knowledge about appropriate clustering grids in the turbulent boundary layers.

	Non-adiabatic case	Adiabatic case
Velocity defect layer thickness δ	$O(\mathrm{Re}^{-2/5})$	$\frac{1}{\ln(\mathrm{Re})}$ or $\Delta^{1/2}\varepsilon^{1/4}$
Reynolds stress sublayer δ_s	$O(\mathrm{Re}^{-3/5})$	$\delta^3\varepsilon^{-1/2}$
Wall layer thickness δ_l	$O(\mathrm{Re}^{-3/5})$	$\frac{\ln(\mathrm{Re})}{\mathrm{Re}}$ or $\varepsilon\tilde{\gamma}_l / \Delta$
Δ / δ	$\gg 1$	$\ll 1$

References

1. J. Ackeret, F. Feldmann and N. Rott. Investigation of compression shocks and boundary layers in gases moving at high speeds, NACA TM 1113 (1947).
2. L. Howarth. The propagation of steady disturbances in a supersonic stream bounded on one side by a parallel stream, *Proc. Camb. Philos. Soc.*, 44, 380-390 (1948).
3. H.S. Tsien and M. Finston. Interaction between parallel streams of subsonic and supersonic velocities, *J. Aero. Science*, **16**, 515-528 (1949).
4. M.J. Lighthill. On boundary layers and upstream influence: II. Supersonic flows without separation, *Proc. Roy. Soc. Ser. A*, **217**, 478-507 (1953).
5. H.M. Brilliant and T.C. Adamson. Shock wave boundary layer interactions in laminar transonic flow, *AIAA J.*, 323-329 (1974).
6. W.B. Bush and F.E. Fendell. Asymptotic analysis of turbulent channel and boundary layer flow, *J. Fluid Mech.*, **56**, 657-681 (1972).
7. J.E. Green. Interactions between shock waves and turbulent boundary layers, *Proc. Aerosp. Sci.*, **11**, 235-340 (1970).
8. L. Lees. Interaction between the laminar boundary layer over a plane surface and the incident oblique shock wave, *Princeton Univ. Aeronaut. Eng. Lab.*, Rep. No. 143 (1949).
9. T.C. Adamson and A. Feo. Interaction between a shock wave and a turbulent boundary layer in transonic flow, *SIAMJ. Appl. Math.*, **29** (1), 103-138 (1980).
10. T.C. Adamson and A.F. Messiter. Analysis of two dimensional interactions between shock waves and boundary layers, *Ann. Rev. Fluid Mech.*, 103-138 (1980).
11. N. Afzal. A higher order theory for compressible turbulent boundary layers at moderately large Reynolds number, *J. Fluid Mech.*, **57**, 1-25 (1973).

12. G.L. Mellor. The large Reynolds number asymptotic theory of turbulent boundary layer, *Int. J. Engrg. Sci.*, **10**, 850-873 (1972).
13. A.F. Messiter, M.S. Liou and T.C. Adamson. Interaction between a normal shock wave and a turbulent boundary layer at high transonic speeds, *ZAMP*, **31**, 204-246 (1980).
14. R.E. Melnik and B. Grossman. Analysis of the interaction of a weak normal shock wave with a turbulent boundary layer, AIAA Paper, No. 598, 32 (1974).
15. H. Tennekes. Outline of a second-order theory of turbulent pipe flow, *AIAA J.*, **6**, 1735-1740 (1968).
16. M.V. Markovin. *Mechanism of Turbulence*, A. Farre (Ed.), Gordon and Breach, 1964.
17. A.L. Kistler. Fluctuation measurement in a supersonic turbulent boundary layer, *Phys. Fluids*, **2**, 3 (1959).
18. E.R. Van Driest. Turbulent boundary layer in compressible fluids, *J. Aero Sci.*, **18**, 145-160 (1951).
19. D.E. Coles. The law of the wake in the turbulent boundary layer, *J. Fluid. Mech.*, **1**, 191-226 (1956).