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Research Notes

Performance Study on Upwind Finite Element Models for a Convection Dominated Problem

Wen-Jey Liang,* Wen-Hann Sheu,**
Cheng-Hway Su,*** Ling-Jie Chung*** and
Nien-Bang Chiu***

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ABSTRACT

A study is conducted on the performance accuracy and stability of six upwind finite element models. Standard Galerkin, adaptive Galerkin, modified weighted residual, reduced integration, quadratic upwind Petrov-Galerkin and cubic upwind Petrov-Galerkin finite element models are investigated. Computed solutions for the convection-diffusion test problem show that the reduced integration model has the best performance.

INTRODUCTION

Engineering problems in the range of high Peclet number are frequently encountered. Analytic solutions for such problems are rarely available, the majority of such analyses being numerical. In the context of numerical simulation, oscillatory solutions are found if a center-based discretization scheme is used [5]. Such numerical oscillations can be suppressed simply by taking the direction of transport velocity into account during the discretization of convective terms [6]. The resulting upwind-based scheme can remove a large amount of numerical oscillation since it can enhance the positive influence coefficient matrix.

As far as multi-dimensional problems are concerned, computed error from the upwind model has been recognized along the direction normal to the local velocity direction [3,5-7]. These false diffusion errors stem mainly from the local one-dimensional discretization on convective terms. The quality of the computed solution becomes worse as the angle between the flow direction and grid system increases. The improvement on this subject is therefore related to the successful designing a flow-oriented model.

The present study addresses the evaluation of computed accuracy and stability using different upwind models for a one-dimensional problem. This assessment study may provide useful knowledge for conducting a multi-dimensional analysis using the operator splitting technique.

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*Professor, Institute of Mechanical Engineering, National Taiwan University, and Institute of Physics, Academia Sicnica, Taipei, Taiwan, R.O.C.

**Associate Professor, Institute of Naval Architecture and Ocean Engineering, National Taiwan University,

***Graduate student, Institute of Mechanical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.

BASIC FORMULATION

Consider the following one-dimensional heat conduction problem as the model equation:

$$-kT_{,xx} + uT_{,x} - 1 = 0 \qquad (1)$$

$$T_{(x=0)} = T_{(x=1)} = 0 \qquad (2)$$

which has the following analytic solution:

$$T(x) = \{x - [(1-\exp(ux/k))/(1-\exp(u/k))]\}/u. \qquad (3)$$

The discretized equation using the Galerkin finite element model can be expressed as follows:

$$\sum_e A_{ij}^e T_i = \sum_e F_j^e \quad (i = 1, 2, \ j = 1, 2) \qquad (4)$$

where

$$\sum_e A_{ij}^e = \int_{\Omega_e} (k\phi_{i,x}\phi_{j,x} + u\phi_{i,x}\phi_j)\, dx$$

$$F_j^e = \int_{\Omega_e} \phi_j\, dx .$$

The discretized Eq. (4) at a point j can be derived to be

$$-k \frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2} + u \frac{T_{j+1} - T_{j-1}}{2\Delta x} = 1 . \qquad (5)$$

The resulting modified equation for (4) is

$$-kT_{,xx} + uT_{,x} - 1 = \frac{(\Delta x)^2}{12}\left(k\frac{\partial^4 T}{\partial x^4} - 2u\frac{\partial^3 T}{\partial x^3}\right) + \frac{(\Delta x)^4}{360}\left(k\frac{\partial^6 T}{\partial x^6} - 3u\frac{\partial^5 T}{\partial x^5}\right) + O((\Delta x)^5) . \qquad (6)$$

The computed solutions at different Peclet numbers, defined by $P_e = u\Delta x/k$, are shown in Fig. 1. All the calculations were performed on a uniform discretized domain with $\Delta x = 0.05$ and $k = 1$. Typical oscillatory solutions, as shown in Fig. 2, appear as $P_e > 2$. This leads to numerical instability and clearly indicates that the center-based discretization model is no longer valid for any convection dominated problem.

The remedy for suppressing such oscillations may be accomplished by modifying the previous fixed grid formulation to an adaptive model such that the maximum Peclet number is less than 2. The presently employed adaptive model belongs to r type (or mesh movement adaptive type) which redistributes the nodal points in an adaptive manner to keep P_e as uniform as possible. The computed results in Fig. 3 show that the monotonicity pro-

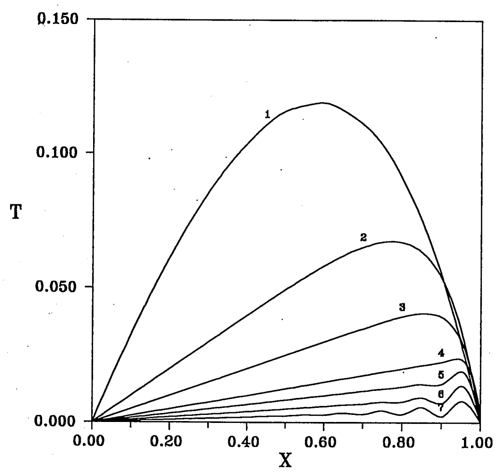

Fig. 1. Computed linear Galerkin solutions. 1. Pe = 0.1, u = 2; 2. Pe = 0.5, u = 10; 3. Pe = 1.0, u = 20; 4. Pe = 2.0, u = 40; 5. Pe = 3.0, u = 60; 6. Pe = 5.0, u = 100; 7. Pe = 10, u = 200.

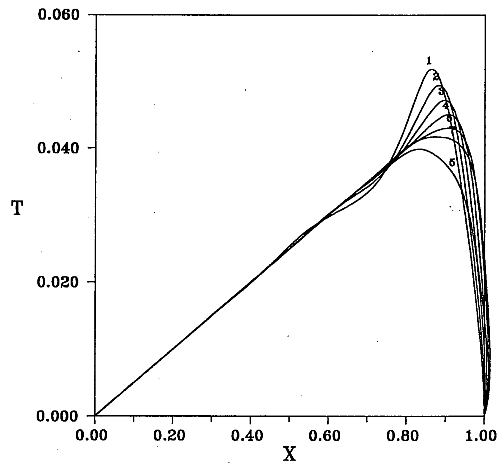

Fig. 2. Computed linear Galerkin solutions around Pe = 2. 1. Pe = 2.85; 2. Pe = 2.5; 3. Pe = 2.22; 4. Pe = 2; 5. exact; 6. Pe = 1.82; 7. Pe = 1.6.

perty can be preserved quite well up to P_e = 3.5, indicating that the adaptive arrangement aids the solutions in becoming oscillation-free even where one uses a center-based formulation. It is unfortunate that such adaptivity still has its limitiation as shown in Fig. 2 as P_e increases to 10. The analysis should be upgraded to the upwind context for achieving an oscillation-free solution. Upwinding schemes can be established in a variety of ways. The major approaches advanced in current literature will be in-

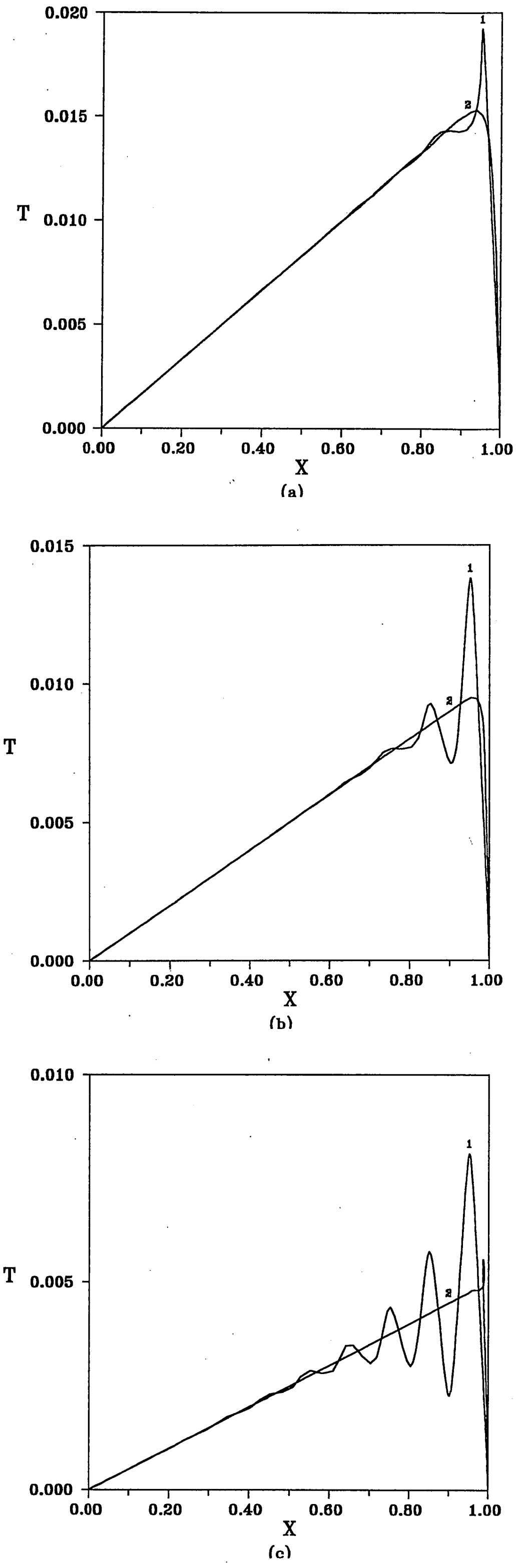

Fig. 3. Computed temperatures. (a) Pe = 3, u = 60; (b) Pe = 5, u = 100; (c) Pe = 10, u = 200. 1. represents standard Galerkin; 2. represents adaptive Galerkin.

vestigated here for conducting the assessment study.

MODIFIED WEIGHTED RESIDUAL MODEL (MWR)

The upwind finite element model was first proposed by Christie et al. [1] who no longer imposed the equal weight on both sides of a point j of interest. The weighting functions used in the present weighted residual formulation are given in a biased manner that larger weight can be placed on the upwind side. The employed weighting function is:

$$w_j(x) = \begin{cases} \phi_j(x) + \dfrac{\alpha}{h_j} N\left(\dfrac{x-x_{j-1}}{h_j}\right); & x_{j-1} \le x \le x_j \\ \phi_j(x) - \dfrac{\alpha}{h_{j+1}} N\left(\dfrac{x_{j+1}-x}{h_{j+1}}\right); & x_j \le x \le x_{j+1} \end{cases} \tag{7}$$

where $h_j = x_j - x_{j-1}$, $h_{j+1} = x_{j+1} - x_j$, and ϕ_j is a linear shape function. α in Eq. (7) is used to control the degree of upwinding. The optimal α in terms of accuracy, can be derived to be

$$\alpha = \coth\frac{P_e}{2} - \frac{2}{P_e} \tag{8}$$

using the discretized analytic solution. The corresponding algebraic equation for this model is

$$-k\left[1 + \left(\frac{P_e}{2}\coth\frac{P_e}{2} - 1\right)\right]\frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2} + u\,\frac{T_{j+1} - T_{j-1}}{2\Delta x} - 1 = 0\,. \tag{9}$$

One can easily observe the difference between Eqs. (5) and (9): the extra term ($\frac{P_e}{2}\coth\frac{P_e}{2} - 1$) in Eq. (9) is responsible for the upwinding effect. This addition enlarges the diagonal terms such that the matrix stability is enhanced and the numerical stability is improved. As seen in Fig. 4, the difficulty could still arise when P_e is large.

REDUCED INTEGRATION MODEL

The upwind effect can be incorporated into the formulation by treating the convection terms differently from the diffusion term [2]. The weighting function is chosen to be the shape function but the convection terms in the Galerkin model are modified. The resulting coefficient matrix becomes

$$A_{ij}^e = \int_{\Omega_e} (k\phi_{i,x}\,\phi_{j,x})\,dx + (x_j - x_{j-1})u\phi_i(x^k)\phi_{j,x}(x^k)$$

where

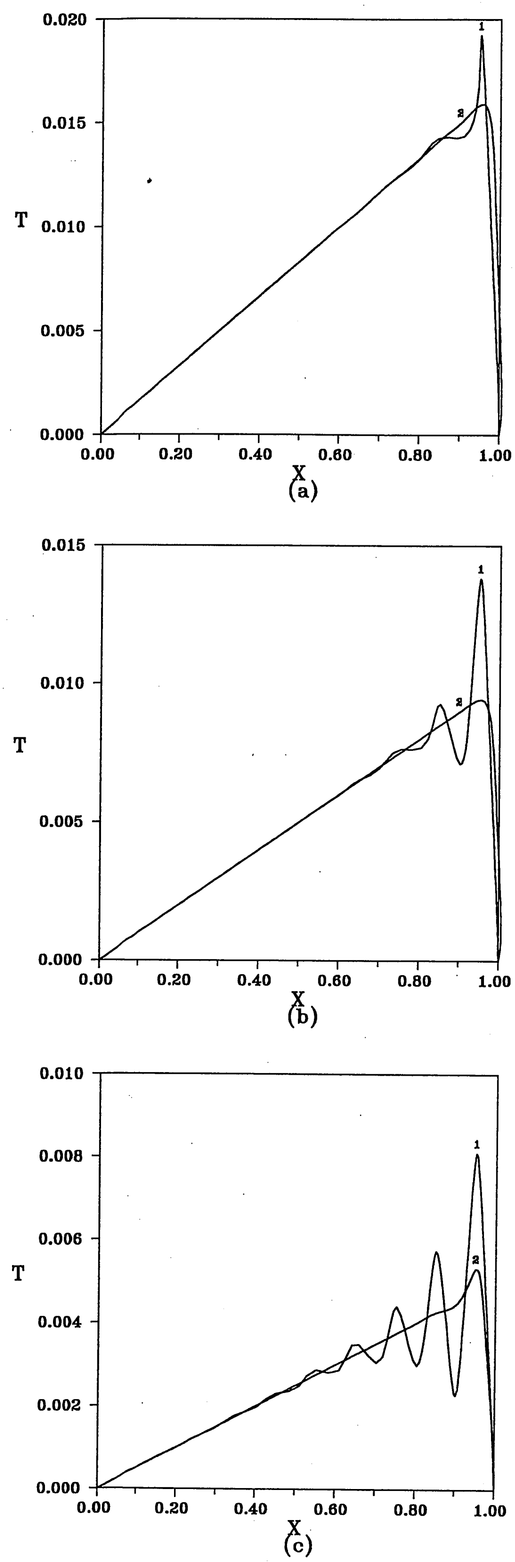

Fig. 4. Computed temperatures by modified weighted residual model. (a) Pe=3, u=60; (b) Pe=5, u=100; (c) Pe=10, u=200. (1 indicates standard Galerkin model, 2 indicates M.W.R. model.)

$$x^k = \frac{1}{2}(x_j + x_{j-1}) + \alpha_{j-1/2}\frac{x_j - x_{j-1}}{2}$$

$$\alpha_{j-1/2} = \coth\frac{P_{e,j-1/2}}{2} - \frac{2}{P_{e,j-1/2}}$$

$$P_{e,j-1/2} = \frac{u(x_j - x_{j-1})}{k} .$$

The resulting algebraic equation can be derived to be

$$-(k + u\frac{\alpha\Delta x}{2})\ \frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2}$$

$$+ \frac{u}{2\Delta x}(T_{j+1} - T_{j-1}) - 1 = 0 . \quad (10)$$

If one chooses α to be the optimal value, Eq. (10) is then equal to the following exact discretized equation and a high quality solution can be obtained:

$$-k\frac{P_e}{2}\left[\frac{1+\exp(P_e)}{1-\exp(P_e)}\right]\frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2} +$$

$$\frac{u}{2\Delta x}(T_{j+1} - T_{j-1}) - 1 = 0 \quad (11)$$

or

$$a_W T_{j-1} + a_P T_j + a_E T_{j+1} = \textit{source term ...} \quad (12)$$

where

$$a_W = K + 1$$

$$a_P = -2K$$

$$a_E = K - 1$$

$$K = \frac{1+\exp(P_e)}{1-\exp(P_e)} \le -1 .$$

It is noted that if one shifts the conventional Gaussian integration point used in Eq. (4) to x^k, one can obtain the same algebraic equation as Eq. (10).

The above two models are formulated on the basis of a linear interpolation function. Higher-order elements, namely quadratic and cubic elements, will be also investigated in the context of upwind models.

QUADRATIC UPWIND PETROV – GALERKIN (QUPG) MODEL

The Galerkin finite element model is a degenerated version of the Petrov-Galerkin family which allows the use of different shape and weighting functions. The weighting (test) functions are chosen to be linear but the shape (basis) functions are chosen to be quadratic functions [4] as follows:

$$\begin{cases} N_1(\xi) = \frac{1}{4}(3+\xi)(1-\xi) \\ N_2(\xi) = \frac{1}{8}(1+\xi)(3+\xi)\ ; \quad \xi \in (-1,1) . \\ N_3(\xi) = -\frac{1}{8}(1+\xi)(1-\xi) \end{cases}$$

The resulting algebraic equation at a point j is

$$\frac{u}{12\Delta x}(T_{j-2} - 9T_{j-1} + 3T_j + 5T_{j+1}) -$$

$$\frac{k}{(\Delta x)^2}(T_{j-1} - 2T_j + T_{j+1}) - 1 = 0 .$$

The accuracy was upgraded to fourth-order since the corresponding modified equation becomes

$$u\frac{\partial T}{\partial x} - k\frac{\partial^2 T}{\partial x^2} - 1 = -\frac{(\Delta x)^4}{720}(21u\frac{\partial^5 T}{\partial x^5} - 2k\frac{\partial^6 T}{\partial x^6})$$

$$+ O((\Delta x)^5) .$$

CUBIC UPWIND PETROV – GALERKIN (CUPG) MODEL

The test function is still kept linear but the order of the basis function is increased to be cubic [4] as follows:

$$\begin{cases} N_1(\xi) = \frac{1}{16}(5+\xi)(3+\xi)(1-\xi) \\ N_2(\xi) = \frac{1}{48}(1+\xi)(3+\xi)(5+\xi)\ ; \quad \xi \in (-1,1) \\ N_3(\xi) = -\frac{1}{16}(1-\xi)(1+\xi)(5+\xi) \\ N_4(\xi) = \frac{1}{48}(1-\xi)(1+\xi)(3+\xi) \end{cases}$$

The resulting algebraic equation at a point j is derived to be

$$\frac{u}{24\Delta x}(-T_{j-3} + 6T_{j-2} - 24T_{j-1} + 10T_j + 9T_{j+1}) +$$

$$\frac{k}{(\Delta x)^2}(-T_{j-1} + 2T_j - T_{j+1}) - 1 = 0 .$$

The corresponding modified equation can be derived to be

$$u\frac{\partial T}{\partial x} - k\frac{\partial^2 T}{\partial x^2} - 1 = -\frac{(\Delta x)^3}{12}u\frac{\partial^4 T}{\partial x^4} - \frac{(\Delta x)^4}{720}$$

$$(9u\frac{\partial^5 T}{\partial x^5} - 2k\frac{\partial^6 T}{\partial x^6}) + O((\Delta x)^5) .$$

COMPUTED RESULTS

The computed results of the modified weighted residual model for the present investigated problem at P_e = 3, 5, 10 were presented in Fig. 4. These results are not oscillatory as P_e is greater than 2. It indicates that the biased weighting function can enhance stability. As P_e number increases to 10, a kink is shown near the right hand side. That means the M.W.R. model might be unable to resolve the high gradient region in the case of the highly convective dominated situation shown in Fig. 4c.

As far as the reduced integration model is concerned, $a_W \leq 0$, $a_E \leq -2$, while $a_P \geq 2$ in Eq. (12). This situation provides a perfect numerical stability. Unfortunately, this model fails as $P_e = 0$ since the value of K does not exist. The computed solutions of the reduced integration model at $P_e = 3, 5, 10$ are presented in Fig. 5. The results are not oscillatory at all no matter how high the Peclet number may be. It indicates that the construction of a upwind model can be made via the appropriate relocation of the Gaussian integration point along the flow direction. The comparison with the exact solution shown in Fig. 6 indicates that this model is almost perfect as far as the one-dimensional analysis is concerned.

The computed high-order solutions are plotted in Figs. 7 and 8 respectively for QUPG and CUPG. The performance of QUPG is good considering the whole range of investigated Peclet numbers. The computed results of CUPG, on the other hand, becomes worse as the Peclet number increases to 5.

In summary, the comparison among the investigated models is made on the basis of computed error ϵ defined by

$$\epsilon = \frac{|\ \textit{computed solution} - \textit{exact solution}\ |}{\textit{exact solution}}$$

As seen in Fig. 9, the reduced integration model has the best performance in accuracy as well as stability.

CONCLUSIONS

An assessment of the standard Galerkin, adaptive Galerkin, modified weighted residual, reduced integration, QUPG, and CUPG finite element models was made in solving the one-dimensional convection-diffusion heat transfer problem. The best performance of the investigated schemes was found is the reduced integration model. No matter how high the Peclet number is, accuracy and stability are nearly perfect. The main reason for its success may be the optimal choice of the upwinding parameter $[1+\exp(P_e)]\ /\ [1-\exp(P_e)]$ which is always less than -1. It results in good numerical stability.

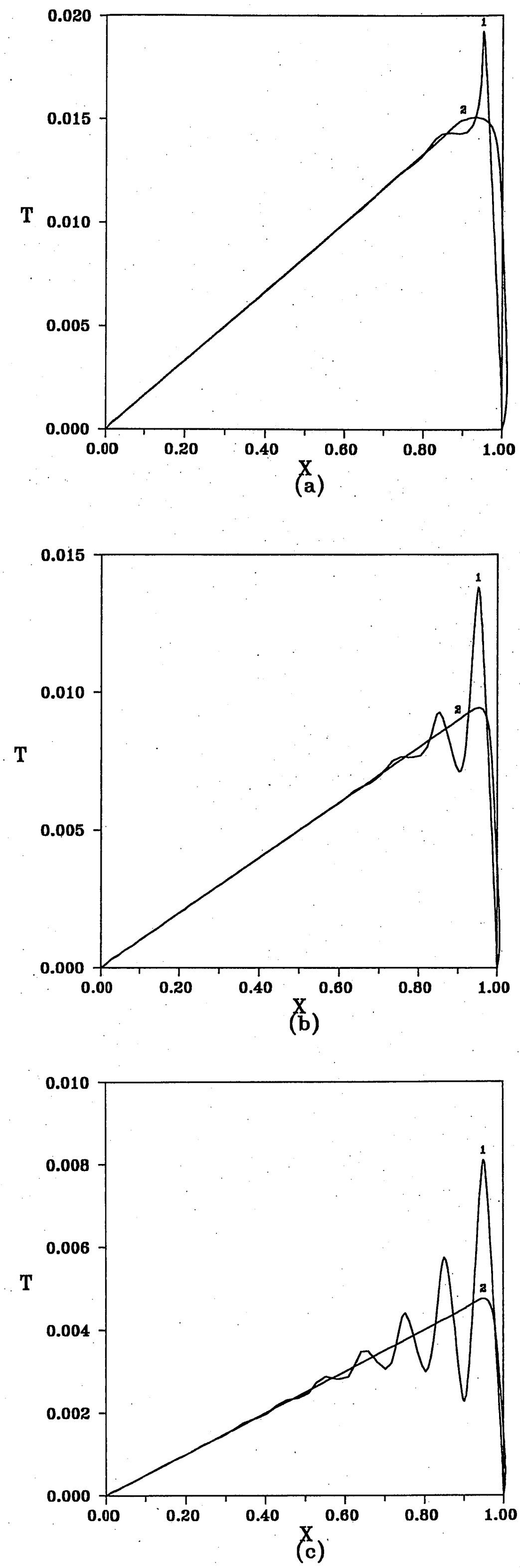

Fig. 5. Computed temperatures by reduced integration model. (a) Pe = 3, u = 60; (b) Pe = 5, u = 100; (c) Pe = 10, u = 200. (1 indicates conventional Galerkin, 2 indicates reduced integration model)

REFERENCE

1. Christie, I., D.F. Griffiths, D.F., A.R. Mitchell, and O.C.

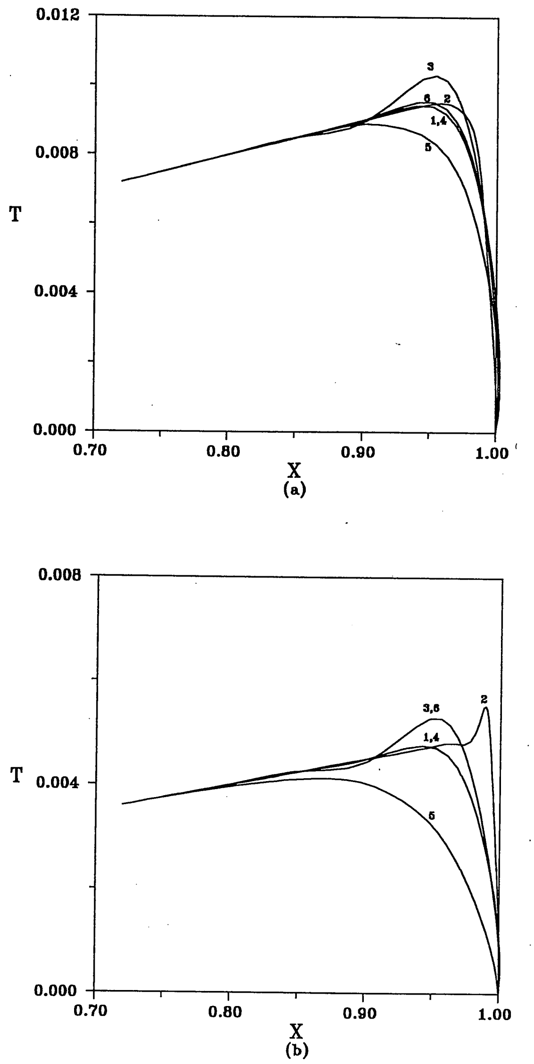

Fig. 6. Comparison study for Pe = 10. 1. exact; 2. adaptive Galerkin; 3. M.W.R.; 4. reduced integration model; 5. QUPG; 6. CUPG. (a) Pe = 5, (b) Pe = 10.

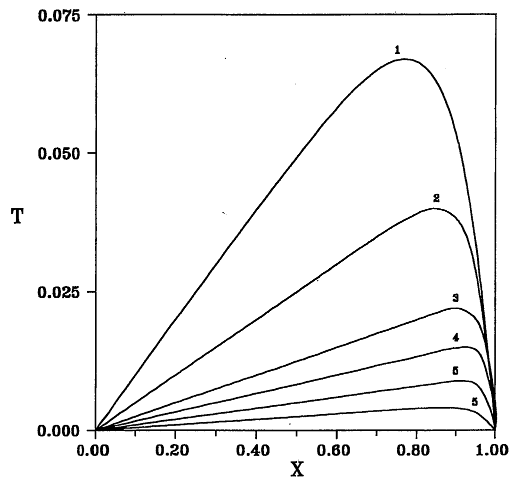

Fig. 7. Computed temperatures by QUPG model. 1. Pe = 0.5, u = 20; 2. Pe = 1, u = 20; 3. Pe = 2, u = 40; 4. Pe = 3, u = 60; 5. Pe = 5, u = 100; 6. Pe = 10, u = 200.

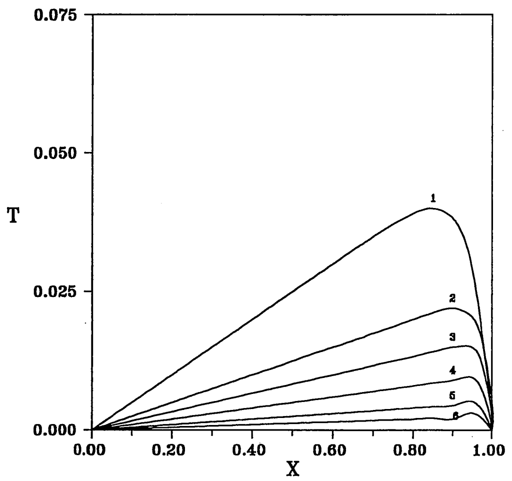

Fig. 8. Computed temperatures by CUPG model. 1. Pe = 1, u = 20; 2. Pe = 2, u = 40; 3. Pe = 3, u = 60; 4. Pe = 5, u = 100; 5. Pe = 10, u = 200; 6. Pe = 20, u = 400.

Zienckiewicz, Finite element methods for second order differential equations with significant first derivatives, Int. J. Num. Meth. Engrg., Vol. 10, pp. 1389-1396, 1976.

2. Hughes, T.J.R., A Simple Finite Element Scheme for Developing Upwind Finite Elements, Int. J. Num. Meth. Engrg., Vol. 12, pp. 1359-1365, 1978.
3. Leonard, B.P., A Stable and Accurate Convection Modelling Procedure Based on Quadratic Upstream Interpolation, J. Comput. Meth. Appl. Mech. Engrg. Vol. 19, pp. 58-59, 1979.
4. Steffler, P.M., Upwind Basis Finite Elements for Convection-Dominated Problems, Int. J. Num. Meths. in Fluid, Vol. 9, pp. 385-403, 1989.

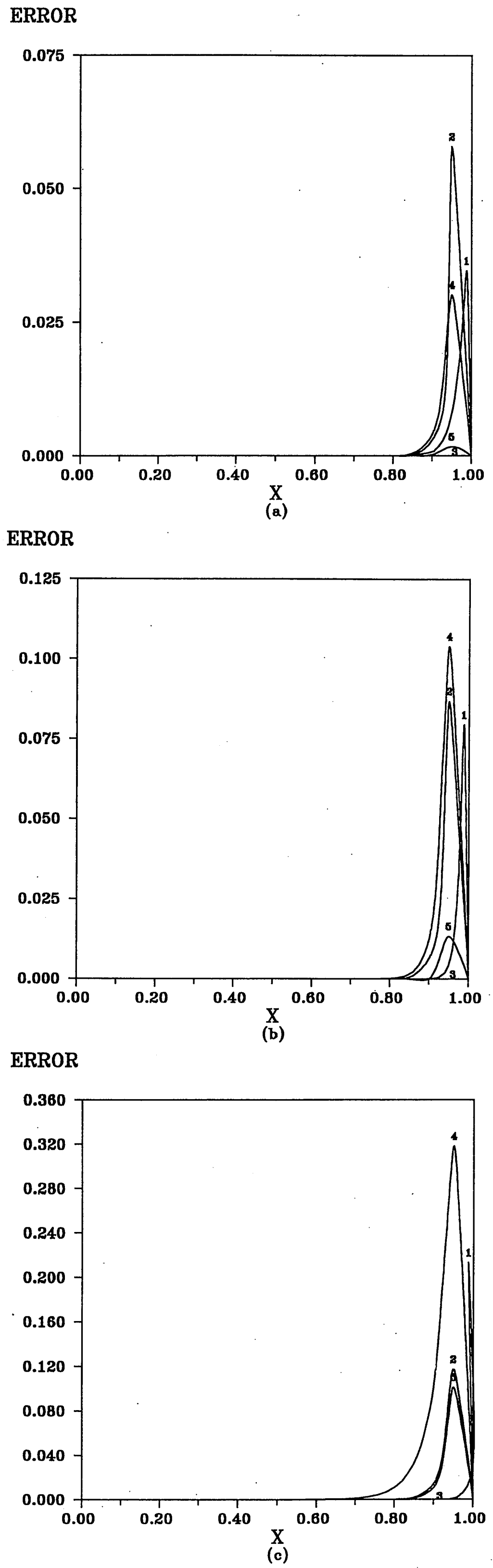

Fig. 9. Error analysis for the investigated models. (a) Pe = 3, u = 60; (b) Pe = 5, u = 100; (c) Pe = 10, u = 200. 1. adaptive Galerkin; 2. M.W.R. 3. reduced integration model; 4. QUPG; 5. CUPG.

5. Tony Sheu, W.H., S.M. Lee, B.J.Y. Chiou and K. Ou Yang, An Assessment of Various Schemes on Incompressible Navier-Stokes Flows, 13th Nat. Conf. on Ther & Appl. Mech., Dec. 15, pp. 1045-1054, 1989.
6. Tony Sheu, W.H., S.M. Lee, K. Ou Yang and J.Y. Bob Chiou, A Non-oscillating Solution Technique for Skew Upwind and QUICK-Family Scheme, Computational Mechanics, Vol. 8, pp. 365-382, 1991.
7. Raithby, G.D., Skew-Upwind Differencing Schemes for Problems Involving Fluid Flow, J. Comput. Meths. Appl. Mech. Engrg. Vol. 9, pp. 153-164, 1976.

分析－主要流場的上風有限元素方法的評估

梁文傑
國立臺灣大學機械工程系所及中央研究院物理所
許文翰
國立臺灣大學造船及海洋工程系所
蘇振輝　鍾靈傑　邱念邦
國立臺灣大學機械工程系所

摘　要

本文就不同的上風有限元素方法評估其準確性及穩定性。所分析的上風有限元素方法包括Standard Galerkin, adaptive Galerkin, modified weighted residual, reduced integration, quadratic upwind Petrov-Galerkin，及Cubic upwinding Petrov-Galerkin方法。本文針對一維的Convective-Diffusive問題測試，得知reduced integration方法具有最好的表現。