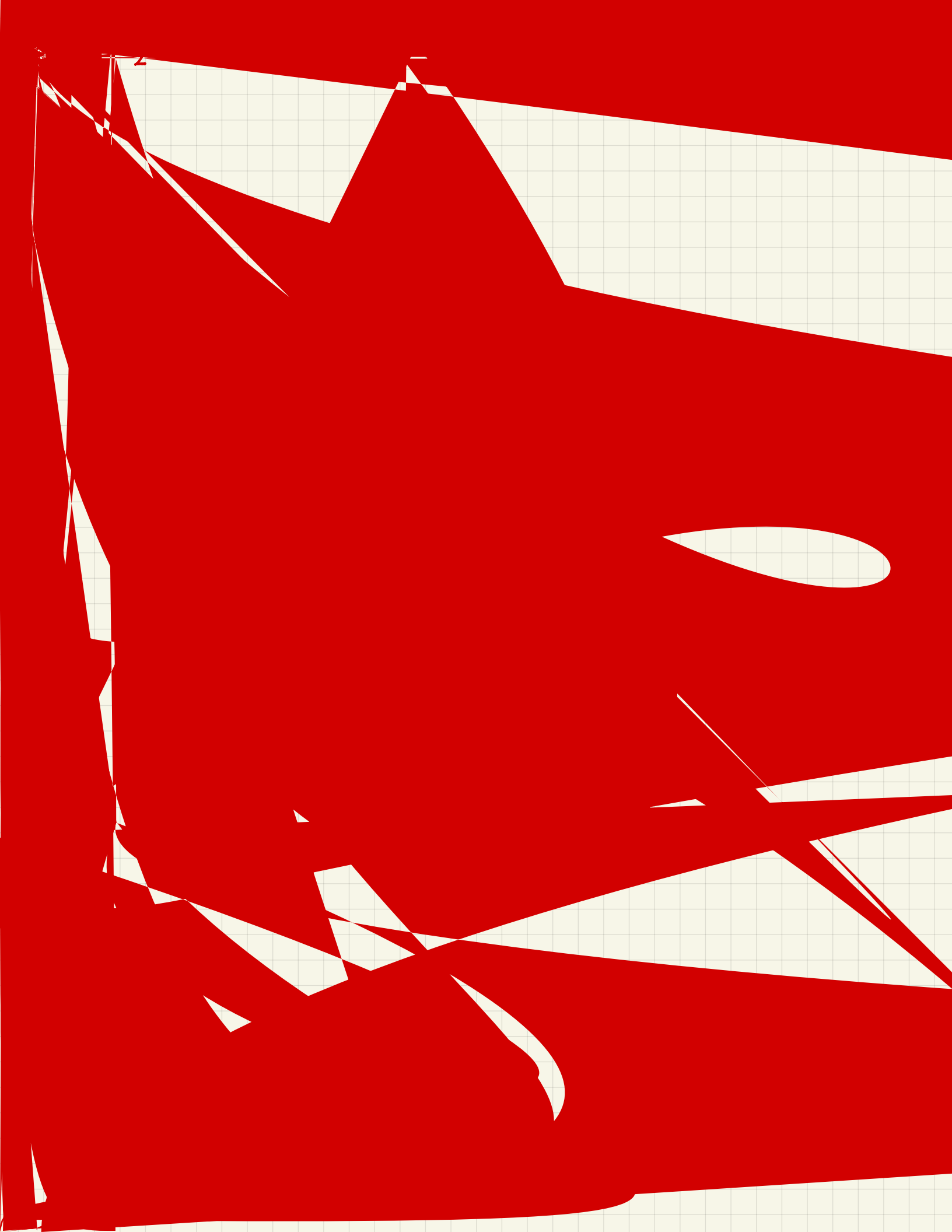


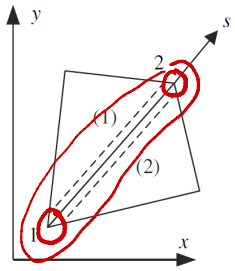
Finite Element Method

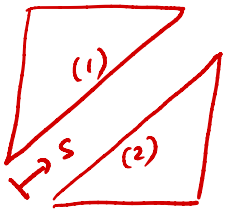
Q: $u_x^e(x,y)$

α_i^e

A: $u_x^e(x,y) = \alpha_0^e + \alpha_1^e x + \alpha_2^e y + \alpha_3^e x^2 + \alpha_4^e xy + \alpha_5^e y^2$

C^0





$u_x^{(1)}(x,y) = \alpha_0^{(1)} + \alpha_1^{(1)} x + \alpha_2^{(1)} y$

$u_x^{(2)}(x,y) = \alpha_0^{(2)} + \alpha_1^{(2)} x + \alpha_2^{(2)} y$

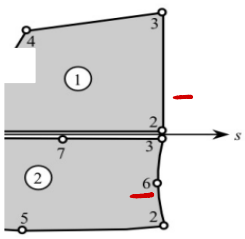
C^0 C^0

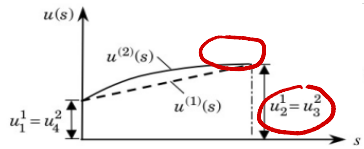
C^0 **at every point on the interfaces between the elements (not just at the nodes)**

$u_x^{(1)}(s) = u_x^{(2)}(s)$

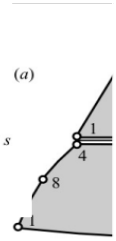
$^{(1)}(\ ) = {}^{(2)}(\ )$

C^0





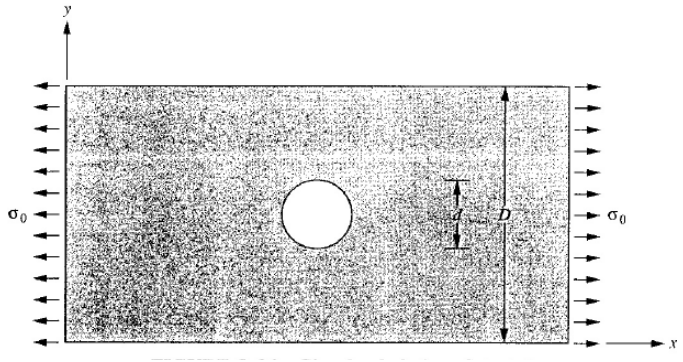
Constraint c

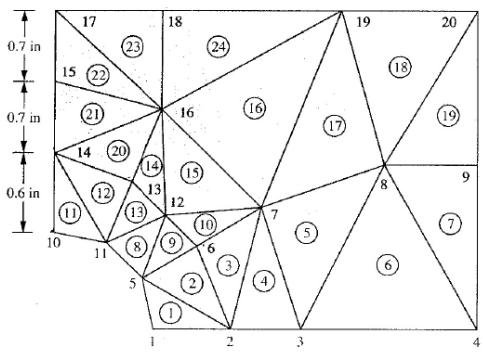


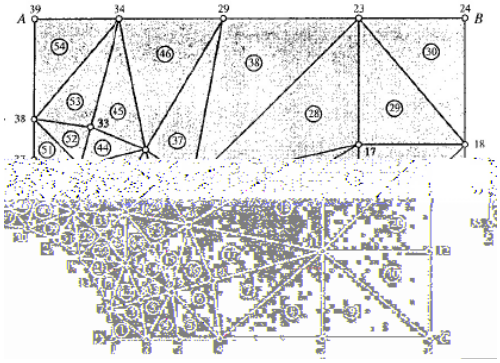
three-node triangular element

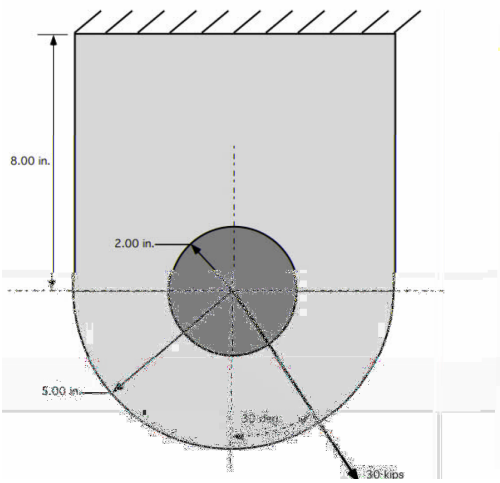
3.7 Three-Node Triangular Element (T3)

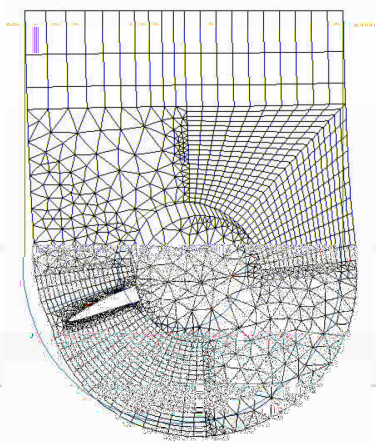
almost any geometry with triangular elements







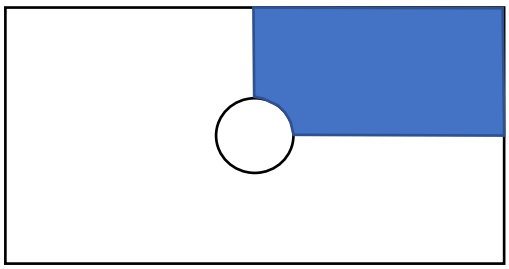


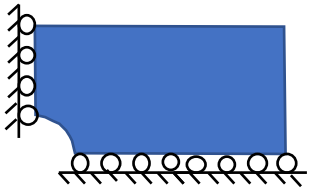


Finite Element Method

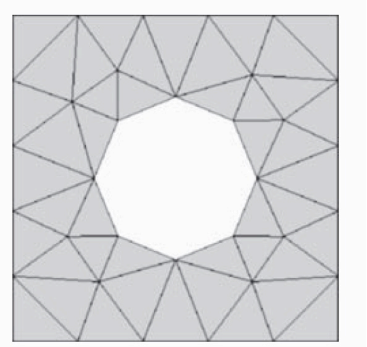
Remarks

$\frac{1}{4}$



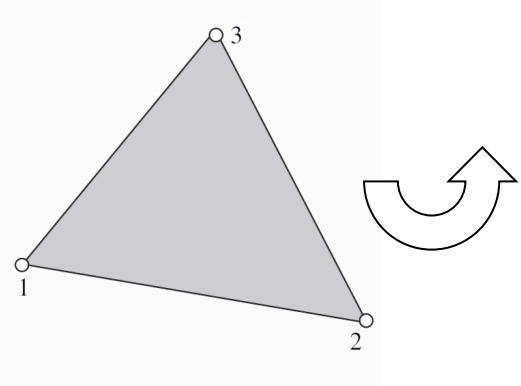


disadvantage **(T3) is that it is a relatively inaccurate element**

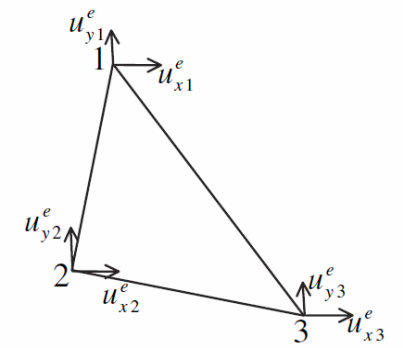


should NOT be very acute.

3.7.1 Shape Functions: T3



the nodes need to be numbered counterclockwise.



Remark:

u_x u_y

u_x

u_x u_y

x y

$$(\ , \ ) = \alpha_0 + \alpha_1 \quad + \alpha_2$$

$$(\ , \ ) = [1 \qquad] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= \mathbf{p}(\ , \ )\boldsymbol{\alpha}$$

the number of unknown coefficients
the number of nodes **coefficients** α_i^e

---

unknown coefficients α_0^e α_1^e α_2^e **unknown nodal values**

$$_1(\ , \ ) = \alpha_0 + \alpha_1 \ _1 + \alpha_2 \ _1$$

$$u_{x2}^e(x,y) = \ !_0^e + \ !_1^e x_2 + \ !_2^e y_2$$

$$u_{x3}^e(x,y) = \ _0^e + \ _1^e x_3 + \ _2^e y_3$$

$$\begin{bmatrix} u_{x1}^e \\ u_{x2}^e \\ u_{x3}^e \end{bmatrix} = \begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix} \quad \Rightarrow \quad \mathbf{d}_x^e = \mathbf{M}^e \boldsymbol{\alpha}^e$$

Finite Element Method

$$\boldsymbol{\alpha}^e = \left(\mathbf{M}^e\right)^{-1}\mathbf{d}_x^e$$

$$(\ ,\ ) = \mathbf{N}(\ ,\ )\mathbf{d} \qquad \text{where} \quad \mathbf{N}(\ ,\ ) = \mathbf{p}(\ ,\ )\left(\mathbf{M}\right)^{-1}$$

$$\mathbf{N}^e(x,y) = \begin{bmatrix} N_1^e(x,y) & N_2^e(x,y) & N_3^e(x,y) \end{bmatrix}$$ **element shape function**

$\mathbf{M}^e$

$$\left(\mathbf{M}^e\right)^{-1} = \frac{1}{2A^e}\begin{bmatrix} x_2^e y_3^e - x_3^e y_2^e & x_3^e y_1^e - x_1^e y_3^e & x_1^e y_2^e - x_2^e y_1^e \\ y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

A^e e

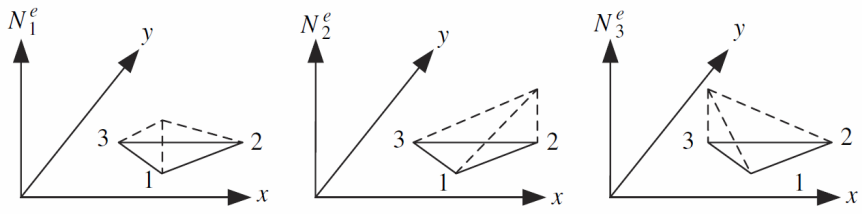
$\mathbf{M}^e$

$$2A^e = \det\left(\mathbf{M}^e\right) = (x_2^e y_3^e - x_3^e y_2^e) + (x_3^e y_1^e - x_1^e y_3^e) + (x_1^e y_2^e - x_2^e y_1^e)$$

$$N_1^e(x,y) = \frac{1}{2A^e}((x_2^e y_3^e - x_3^e y_2^e) + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y)$$

$$N_2^e(x,y) = \frac{1}{2A^e}((x_3^e y_1^e - x_1^e y_3^e) + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y)$$

$$N_3^e(x,y) = \frac{1}{2A^e}((x_1^e y_2^e - x_2^e y_1^e) + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y)$$



$$N_I^e(x_J^e, y_J^e) = \delta_{IJ}$$

$$N_1^e(x,y) = {}_0^e + {}_1^e x + {}_2^e y$$

3 conditions $N_1^e(x_1^e, y_1^e) = 1 \quad N_1^e(x_2^e, y_2^e) = N_1^e(x_3^e, y_3^e) = 0$

$$\Rightarrow \begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix} \begin{bmatrix} \beta_0^e \\ \beta_1^e \\ \beta_2^e \end{bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad \beta_0^e = \frac{1}{2A^e} a_1 \qquad \beta_1^e = \frac{1}{2A^e} b_1 \qquad \beta_2^e = \frac{1}{2A^e} c_1$$

$$a_1 = x_2^e y_3^e \quad x_3^e y_2^e$$

$$b_1 = y_2^e - y_3^e$$

$$c_1 = x_3^e - x_2^e$$

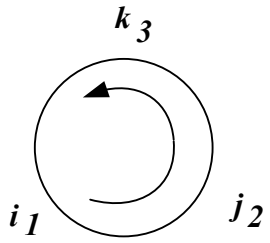
$$A^e = \qquad = \frac{1}{2} \begin{vmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix} = \frac{1}{2} \det\left(\mathbf{M}^e\right)$$

$$N_2^e(x,y) = \frac{1}{2A^e}(a_2 + b_2 x + c_2 y) \qquad N_3^e(x,y) = \frac{1}{2A^e}(a_3 + b_3 x + c_3 y)$$

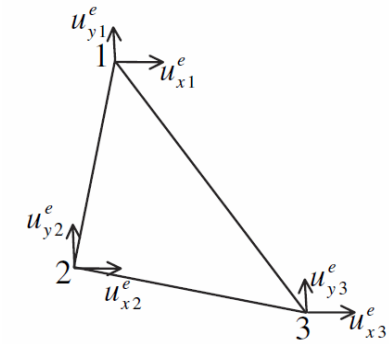
$$\begin{cases} a_2 = x_3^e y_1^e - x_1^e y_3^e \\ b_2 = y_3^e - y_1^e \\ c_2 = x_1^e - x_3^e \end{cases} \qquad \begin{cases} a_3 = x_1^e y_2^e - x_2^e y_1^e \\ b_3 = y_1^e - y_2^e \\ c_3 = x_2^e - x_1^e \end{cases}$$

$$N_i^e(x,y)=\frac{1}{2A^e}(a_i+b_ix+c_iy)$$

$$\begin{cases} a_i = x_j^e y_k^e - x_k^e y_j^e \\ b_i = y_j^e - y_k^e \\ c_i = x_k^e - x_j^e \end{cases}$$



3.7.2 Element Matrices: T3



$\mathbf{N}^e$

$$\mathbf{u}^e(x,y)=\mathbf{N}^e(x,y)\mathbf{d}^e$$

$\mathbf{B}^e$

$$\boldsymbol{\varepsilon}^e=\nabla_S\mathbf{u}^e=\nabla_S\mathbf{N}^e\mathbf{d}^e=\mathbf{B}^e\mathbf{d}^e$$

$$\mathbf{B}^e = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix}$$

$\mathbf{K}^e$ $\mathbf{f}^e$ $\mathbf{N}^e$ $\mathbf{B}^e$

$\mathbf{K}^e$ $\mathbf{N}^e$ $\mathbf{B}^e$

$$\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega$$

t

$$\mathbf{K}^e = tA^e\mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e$$

Q: $\mathbf{K}^e$

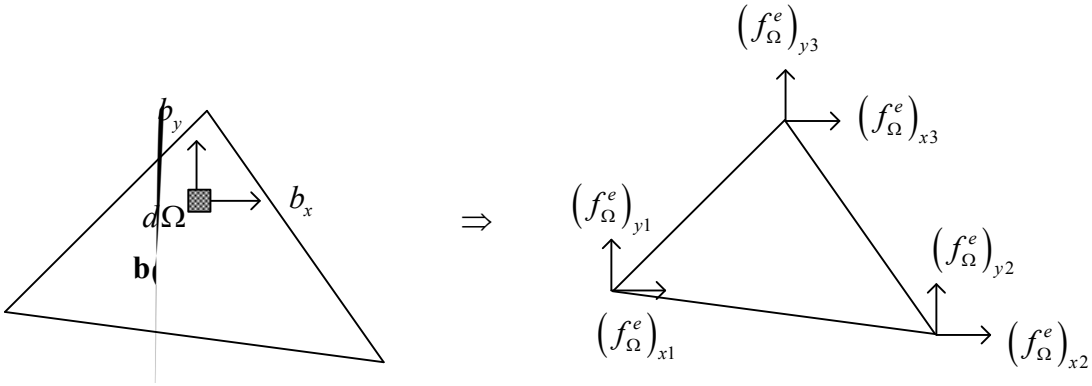
A:

$\mathbf{f}^e$ $\mathbf{N}^e$ $\mathbf{B}^e$

t

$$\mathbf{f}^e = \mathbf{f}_\Omega^e + \mathbf{f}_\Gamma^e = t\int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega + t\int_{\Gamma_t^e} \mathbf{N}^{eT}\bar{\mathbf{t}}\, d\Gamma$$

3.7.2.1 $\mathbf{f}^e$ due to body forces ($\mathbf{f}_\Omega^e$)



$$\mathbf{f}_{\Omega}^{e} = t \int_{\Omega^e} \mathbf{N}^{eT} \mathbf{b} \, d\Omega$$

_____ $\bar{\mathbf{b}}$

$$\bar{\mathbf{b}} = \begin{bmatrix} \bar{b}_x \\ \bar{b}_y \end{bmatrix} \qquad \bar{b}_x \qquad \bar{b}_y$$

$$\mathbf{f}^{e} = t \int_{e} \mathbf{N}^{e} \, d \qquad \bar{\mathbf{b}}^{T}$$

Q:

A:

Finite Element Method

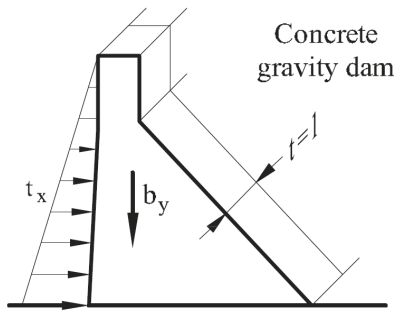
Q: $\int_{\Omega^e} N_1^e \, d\Omega$

A:

$\bar{\mathbf{b}}$

$$\begin{bmatrix} \left(f_\Omega^e\right)_{x1} \\ \left(f_\Omega^e\right)_{y1} \\ \left(f_\Omega^e\right)_{x2} \\ \left(f_\Omega^e\right)_{y2} \\ \left(f_\Omega^e\right)_{x3} \\ \left(f_\Omega^e\right)_{y3} \end{bmatrix} = \frac{tA^e}{3} \begin{bmatrix} \bar{b}_x \\ \bar{b}_y \\ \bar{b}_x \\ \bar{b}_y \\ \bar{b}_x \\ \bar{b}_y \end{bmatrix}$$

Example:



A:

$$\mathbf{b}(x,y) = \mathbf{N}^e(x,y)\mathbf{b}^e$$

$\mathbf{b}^e$

$$\mathbf{b}^e = \begin{bmatrix} b^e_{x1} \\ b^e_{y1} \\ b^e_{x2} \\ b^e_{y2} \\ b^e_{x3} \\ b^e_{y3} \end{bmatrix}$$

f = N N b

$$\int_{\Omega^e} \left(N_1^e\right)^a \left(N_2^e\right)^b \left(N_3^e\right)^c \, d\Omega = \frac{a!\; b!\; c!}{(a+b+c+2)!} 2A^e$$

$$\mathbf{f}^e_\Omega = \frac{tA^e}{12} \begin{bmatrix} 2b_{x1} + b_{x2} + b_{x3} \\ 2b_{y1} + b_{y2} + b_{y3} \\ b_{x1} + 2b_{x2} + b_{x3} \\ b_{y1} + 2b_{y2} + b_{y3} \\ b_{x1} + b_{x2} + 2b_{x3} \\ b_{y1} + b_{y2} + 2b_{y3} \end{bmatrix}$$

Remark $b_{x1} = b_{x2} = b_{x3} = \bar{b}_x$ $b_{y1} = b_{y2} = b_{y3} = \bar{b}_y$