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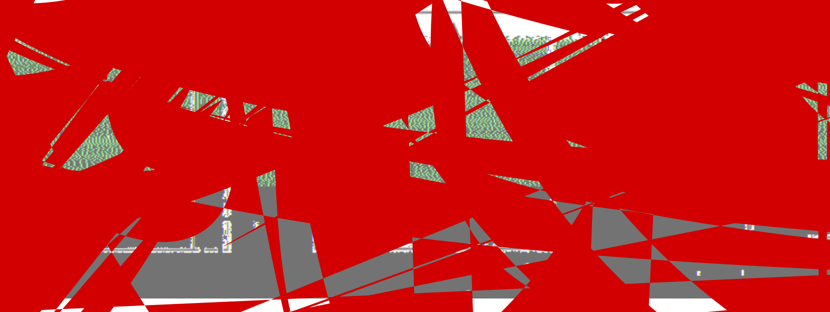

Remark:

Two forms of the shear strain appear commonly in finite element software: the engineering shear strain γ_{xy} given above and the tensor shear strain $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$.

In finite element methods, the strains are usually arranged in a column matrix:

$$\text{(3.10)} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

And strain-displacement equations can be expressed by:

from (3.9)

$$\text{(3.11)} \qquad \boldsymbol{\varepsilon} = \boldsymbol{\nabla}_S \mathbf{u} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

where $\boldsymbol{\nabla}_S$ is **a symmetric gradient matrix operator**. ↑ small del

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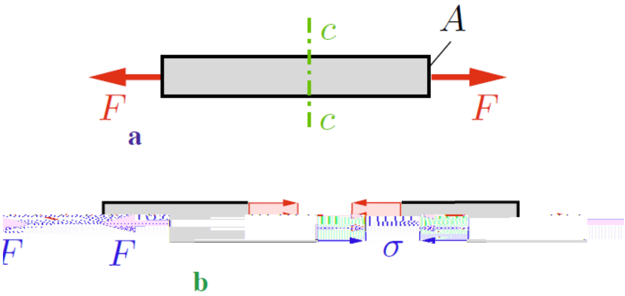
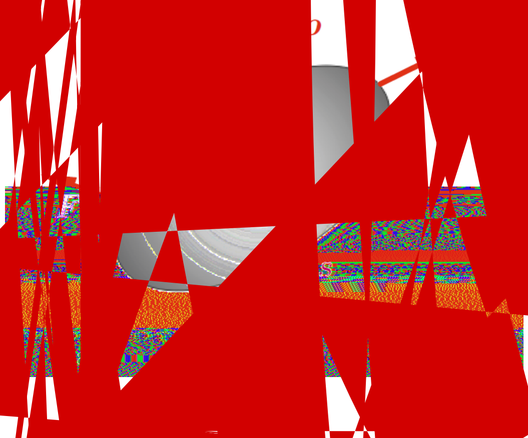
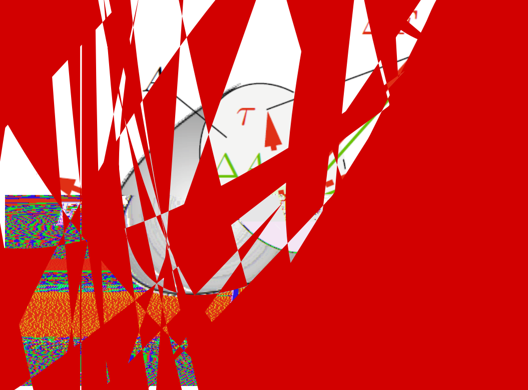
Internal Force, Stress Vector and Stress

The external load causes internal forces. In 1D, the internal forces can be visualized by an imaginary cut of the bar. Stresses are constant over the cross section



Let us now extend the idea to 2D and 3D. For this purpose let us consider a body which is loaded arbitrarily, e.g. by point forces $\vec{F_i}$ and distributed forces $\vec{p}$ as shown in the Fig (a)





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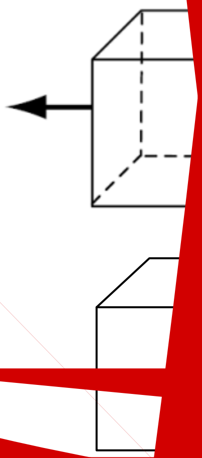

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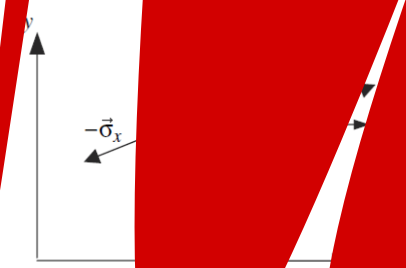


Now cons… t squa… nt P, stresses in 2D correspond to the **stress vectors** $\vec{\sigma}_x$ and $\vec{\sigma}$…

(3.13) $\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$

(3.14) $\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}$

Stresses are thus mathematical description of internal forces. The stress state in a two-dimensional body is described by two normal stresses σ_{xx} and σ_{yy} and shear stresses σ_{xy} and σ_{yx}. From moment equilibrium in a unit square, it can be shown that $\sigma_{xy} = \sigma_{yx}$. The proof is skipped and taken for granted (you can find the proof in virtually any Continuum Mechanics textbook).

The stress state at any point P in a 2D body is thus **uniquely** determined by σ_{xx}, σ_{yy}, σ_{xy}.

In finite element methods, the stresses are usually arranged in a column matrix similar to strains:

(3.15) $$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

Occasionally, it is convenient to arrange stress components in a 2×2 symmetric matrix $\boldsymbol{\tau}$ as

(3.16) $$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

Returning to (3.12) and multiplying $\vec{i}$ and $\vec{j}$ in (3.13) and (3.14) yields:

(3.17) $$\begin{aligned} t_x &= \sigma_{xx} n_x + \sigma_{xy} n_y = \vec{\sigma}_x \cdot \vec{n} \\ t_y &= \sigma_{xy} n_x + \sigma_{yy} n_y = \vec{\sigma}_y \cdot \vec{n} \end{aligned}$$

Or in matrix form

***

(3.18) $$\mathbf{t} = \boldsymbol{\tau}\mathbf{n} \quad \boxed{\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}}$$

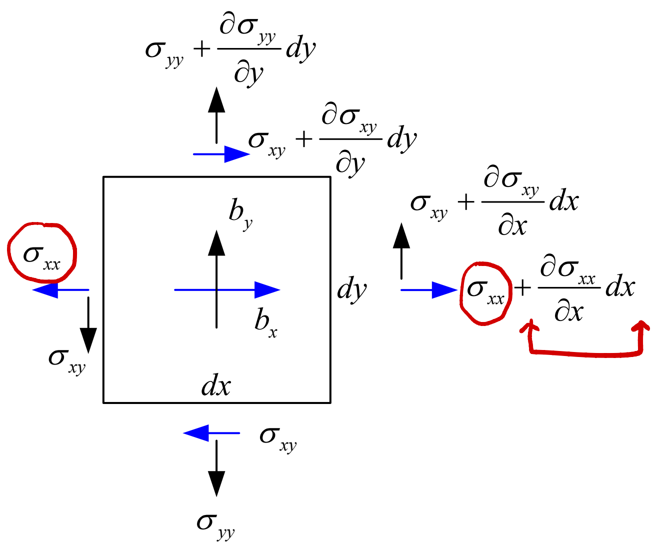
Equation (3.18) is the famous Cauchy's formula: the formal relationship between stress, normal and stress vector derived by Augustin Cauchy. The Cauchy's formula said:

> Knowing the components σ_{xx}, σ_{yy}, σ_{xy}, we can write down at once the stress vector $\vec{t}$ acting on any surface with unit outer normal vector $\vec{n}$.

Remark: Cauchy in 1822 realized that there is no inherent difference between external forces acting on the physical surfaces of a body and internal forces acting across virtual surfaces within the body. This realization may appear to be a very simple, almost trivial, observation. However, it cleared up the confusion resulting from nearly 100 years of failed and partly failed attempts to understand internal forces that preceded Cauchy and paved the way for the

Equilibrium Equations

Figure below shows stresses that act on a differential element of a unit thickness in 2D.



Q: What is the static equilibrium of forces in the x direction?

A:

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dy + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y} dy\right) dx$$

$$- (\sigma_{xx}\, dy) - (\sigma_{xy})\, dx + b_x\, dx dy = 0$$

$$= \frac{\partial \sigma_{xx}}{\partial x} dx dy + \frac{\partial \sigma_{xy}}{\partial y} dy dx + b_x\, dx dy = 0$$

$=$

We can do the same things for the y-direction. Thus the differential equations of equilibrium for a 2D problem are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \tag{3.19}$$

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can also express the e ... ons ... we c ... tr nsp ... ymmetric gradient o ... .11) d ... for ... re , w at ... atrix form of equili

2×3

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \end{bmatrix}$$

$$\underset{\sim}{\nabla}_s^T \underset{\sim}{\sigma} \quad \underset{\sim}{0}$$

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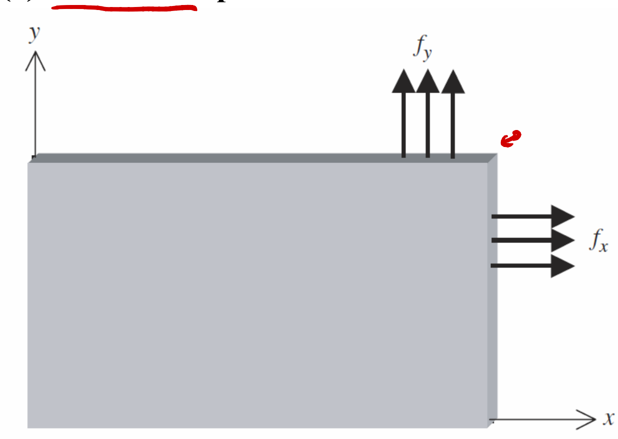
Hooke's law. It is always a symmetric, positive-definite matrix. In two-dimensional problems, the matrix **D** depends on whether one assumes a plane stress or a plane strain condition. These assumptions determine how the model is simplified from a three-dimensional physical body to a two-dimensional model.

(a) Plane stress problem



The plate is very thin ($t = l_z << l_x \approx l_y$) and subjected to loadings in the x-y plane (in plane loading).

$$\Rightarrow \sigma_{zz} = 0, \;\; \sigma_{zx} = 0, \;\; \sigma_{zy} = 0 \;\; \text{(throughout the plate)}$$

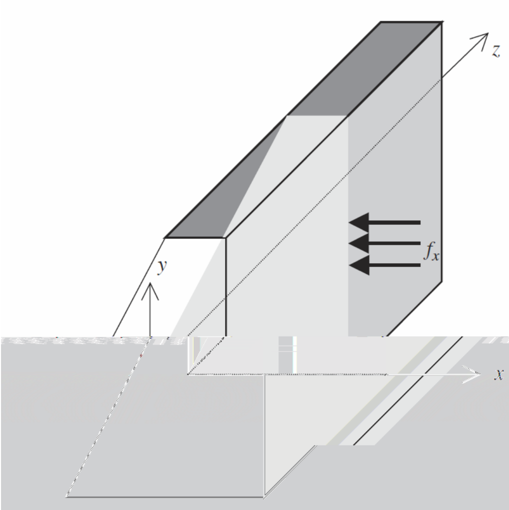
$$\sigma_{zz} = 0 \;\; \Rightarrow \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_x + \sigma_y) \neq 0 \;\; \text{in general}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

linear, isotropic materials, we only need two independent materi

$$\begin{bmatrix} \sigma \\ \sigma \\ \sigma \end{bmatrix} = \frac{}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon \\ \varepsilon \\ \gamma \end{bmatrix}$$

(b) Plane strain problem



Loading is a function of only x and y.

$$l_z >> l_x \approx l_y$$

$$w = u_z = 0 \Rightarrow \varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$$

$$\Rightarrow \sigma_{zz} = \nu(\sigma_x + \sigma_y) \neq 0 \quad \text{in general}$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Remark: The matrix **D** in plane stress and plane strain problems can be obtained from the generalized Hooke's law for 3D isotropic materials.

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

where $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$ $\boldsymbol{\varepsilon}$ $\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$ and notice that the shear modulus that relates shear stress and shear strain is $G = \dfrac{E}{+\nu}$. Consult Timoshenko, S. and Goodier, J. *Theory of Elasticity*, 3$^{\text{rd}}$ Edition, 1970 or DL Logan, *A First Course in Finite Element Method, 4th Edition*, 2011 (Appendix C, available from the course website) for further elaboration.

3.3 Linear Elasticity: Strong Form

Recall (3.19), the differential equations of equilibrium for a 2D problem are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

The Governing differential equations can be expressed in vector form by recalling the stress vectors $\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$ and $\vec{\sigma}_y = \sigma_{xy}\vec{i} + \sigma_{yy}\vec{j}$ (from (3.13) and (3.14)) and the definition of divergence (3.6):

(3.22) $$\begin{aligned} \vec{\nabla}\cdot\vec{\sigma} \;+\; &\quad = 0 \\ \vec{\nabla}\cdot\vec{\sigma} \;+\; &\quad = 0 \end{aligned}$$

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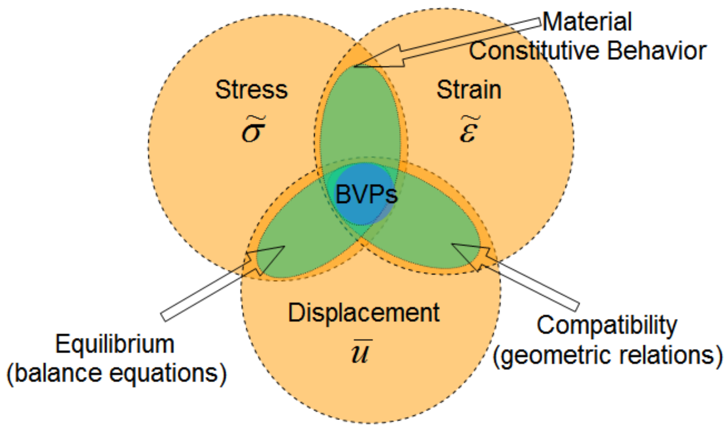
Summary

The key ingredients of linear elasticity are stresses, strains, and displacements. Kinematics defines the compatibility (geometric relations between displacements and strains). Constitutive model defines the material constitutive behavior and relate stresses with strains. Finally, equilibrium establishes the Governing differential equations and we often solve unknown displacements in a boundary value problem (BVP).



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3.4 Linear Elasticity: Weak Form

We will follow the **basic three steps** as for the 1D problem to obtain the weak form. Before we can proceed, we shall introduce the Green's theorem: the 2D and 3D counterpart of the integration by parts in 1D.

Mathematic Primer

Green's theorem: the 2D and 3D counterpart of the integration by parts in 1D. Consider **ANY** scalar function $\phi(x,y)$ (for example, u_x and u_y are scalar functions and $\vec{u}$ is a vector function). The essence of Green's theorem is to relate **an integral of a gradient of a scalar function over an area (volume) to a contour (surface) integral** by:

$$\boxed{(3.25) \qquad \int_{\Omega} \vec{\nabla}\phi \, d\Omega = \int_{\Gamma} \phi \vec{n} \, d\Gamma}$$

The proof is skipped and taken for granted (you can find the proof in virtually any Engineering Mathematics textbook).

Q: We can furthermore expand the Green's theorem in 2D into two equations, one for the x-direction and the other for the y-direction. What are them? (hint: $\vec{\nabla} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y})$)

A:

(3.26a)

(3.26b)

We can use (3.26a) and (3.26b) to develop the divergence theorem which relates **an area (volume) integral of the divergence of a vector field to the contour (surface) integral of a vector field**:

$$\boxed{(3.27) \qquad \int_\Omega \vec{\nabla} \cdot \vec{u}\; d\Omega = \int_\Gamma \vec{u} \cdot \vec{n}\; d\Gamma}$$

By replacing the scalar function $\phi(x, y)$ in (3.26a) with u_x and (3.26b) with u_y:

$$\int_\Omega \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) d\Omega = \int_\Gamma (u_x n_x + u_y n_y)\; d\Gamma$$

which is the divergence theorem given in (3.27).

Remark: Divergence theorem is an important result for the mathematics of engineering, in particular in electrostatics and fluid dynamics. Consult wiki for further elaboration: http://en.wikipedia.org/wiki/Divergence_theorem.

Finally we are now ready to show the 2D and 3D counterpart of integration by parts:

$$\boxed{(3.28) \qquad \int_\Omega w\, \vec{\nabla} \cdot \vec{u}\; d\Omega = \int_\Gamma w\, \vec{u} \cdot \vec{n}\; d\Gamma - \int_\Omega \vec{\nabla} w \cdot \vec{u}\; d\Omega}$$

We shall first recall the identify similar to the chain rule for taking the **derivative of a product** in 1D $\frac{d}{dx}(wf) = w\frac{df}{dx} + f\frac{dw}{dx}$. **The power of the gradient operator $\vec{\nabla}$ is shown by the**

following product rule in 2D and 3D:

(3.29) $\vec{\nabla}\cdot(w\vec{u}) = \vec{\nabla}w\cdot\vec{u} + w\vec{\nabla}\cdot\vec{u}$

(Derivation Exercise) Verify the identify (3.29) in 2D (hint: you will need to expand it)

Answer:

Integrating (3.29) over the domain Ω yields:

(3.30) $\int_{\Omega}\vec{\nabla}\cdot(w\vec{u})\ d\Omega = \int_{\Omega}\vec{\nabla}w\cdot\vec{u}\ d\Omega + \int_{\Omega}w\vec{\nabla}\cdot\vec{u}\ d\Omega$

Q: What happens if we apply the divergence theorem to the term on the LHS in (3.30)?

A:

(3.31)

Substituting (3.31) into (3.30), we then have:

$$\int_{\Gamma} w\,\vec{u}\ \cdot\ \vec{n}\ d\Gamma = \int_{\Omega} w\ \vec{\nabla}\cdot\vec{u}\ d\Omega + \int_{\Omega}\vec{\nabla}w\ \cdot\ \vec{u}\ d\Omega$$

which yield the Green's formula (3.28), the counterpart of integration by parts in 2D and 3D.

<u>Weak Form for Linear Elasticity</u>

We are now ready to derive the **weak form** for linear elasticity problems following the **three-step** procedure presented in 1D.

<u>**Step 1**</u>

The first step is to multiply the differential equations (3.24) by the corresponding weigh functions and integrate over the domain.

(3.32a) $\int_\Omega w_x \ \vec{\nabla} \cdot \vec{\sigma}_x \ d\Omega + \int_\Omega w_x b_x \ d\Omega = 0 \quad \forall w_x \in U_0$

(3.32b) $\int_\Omega w_y \ \vec{\nabla} \cdot \vec{\sigma}_y \ d\Omega + \int_\Omega w_y b_y \ d\Omega = 0 \quad \forall w_y \in U_0$

where U_0 is the set of smooth functions that vanish on the essential boundaries (displacement boundaries) $U_0 = \left\{ w_i(x,y) \middle| w_i = 0 \text{ on } \Gamma_u \right\}$ and

$\mathbf{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ (matrix form) $\qquad \vec{w} = w_x \vec{i} + w_y \vec{j}$ (vector form)

<u>Step 2</u>

In the second step, we distribute the differentiation among $\vec{\ }$ and $\vec{\ }$ equally. To achieve this, we do the Green's formula (3.28), the counterpart of integration by parts in 2D and 3D.

Q: What happens if we apply Green's formula (3.28) to the first terms in (3.32a) and (3.32b)?
A:
(3.33a)

(3.33b)

<u>Step 3</u>

Impose essential boundary conditions: adding (3.33a) and (3.33b) and recalling the weight functions vanish on Γ_u yields:

(3.34a) $\int_\Omega (\vec{\nabla} w_x \ \cdot \vec{\sigma}_x + \vec{\nabla} w_y \ \cdot \vec{\sigma}_y) \ d\Omega = \int_{\Gamma_t} (w_x \vec{\sigma}_x \cdot \vec{n} + w_y \vec{\sigma}_y \cdot \vec{n}) \ d\Gamma + \int_\Omega (w_x b_x + w_y b_y) \ d\Omega$

Or writing the RHS in the vector form gives:

(3.34b) $$\int_{\Omega}(\vec{\nabla}w_x \cdot\vec{\sigma}_x+\vec{\nabla}w_y \cdot\vec{\sigma}_y)\,d\Omega=\int_{\Gamma_t}\vec{w}\cdot\vec{\bar{t}}\;d\Gamma+\int_{\Omega}\vec{w}\cdot\vec{b}\;d\Omega$$

Q: What is $\vec{\nabla}w_x \cdot\vec{\sigma}_x$ and $\vec{\nabla}w_y \cdot\vec{\sigma}_y$ if we carry out the scalar product?

A:

Thus, the integrand on the LHS of (3.34b) can be written in matrix form

(3.35) $$\begin{aligned}\vec{\nabla}w_x \cdot\vec{\sigma}_x+\vec{\nabla}w_y \cdot\vec{\sigma}_y &= \begin{bmatrix}\frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial y} & \frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}\\ &=(\nabla_S\mathbf{w})^T\boldsymbol{\sigma}\\ &=(\nabla_S\mathbf{w})^T\mathbf{D}\nabla_S\mathbf{u}\end{aligned}$$

in which $\boldsymbol{\nabla}_S$ is **the symmetric gradient matrix operator** defined in (3.11) when we related strain with displacement in 2D and $\boldsymbol{\nabla}_S=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}$.

Thus, the weak form in 2D can be written as (in matrix form):

(3.36)

Find $\mathbf{u} \in U$ such that

$$\int_{\Omega} (\nabla_S \mathbf{w})^T \mathbf{D} \nabla_S \mathbf{u} \; d\Omega = \int_{\Gamma_t} \mathbf{w}^T \bar{\mathbf{t}} \; d\Gamma + \int_{\Omega} \mathbf{w}^T \mathbf{b} \; d\Omega \qquad \forall \mathbf{w} \in U_0$$

where $U = \left\{ \mathbf{u} \middle| \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_u \right\}$ and $U_0 = \left\{ \mathbf{w} \middle| \mathbf{w} = 0 \text{ on } \Gamma_u \right\}$.

Expressing the weak form (3.36) from an integral over the entire domain Ω to a sum of integrals over element domains, we have the elementwise weak form:

(3.37) $$\sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} (\nabla_S \mathbf{w}^e)^T \mathbf{D}^e \left(\nabla_S \mathbf{u}^e \right) d\Omega - \int_{\Omega^e} \mathbf{w}^{eT} \mathbf{b} \; d\Omega - \int_{\Gamma_t^e} \mathbf{w}^{eT} \bar{\mathbf{t}} \; d\Gamma \right\} = 0$$

Remark:

Recall the weak form in 1D (Eq. (1.11))

Find $u(x)$ among the smooth functions that satisfy $u = \bar{u}$ on Γ_u such that

$$\int_{\Omega} \left(\frac{dw}{dx} \right)^T AE \frac{du}{dx} dx - \int_{\Omega} w^T b \; dx - (w^T A \bar{t}) \Big|_{\Gamma_t} = 0 \qquad w = 0 \text{ on } \Gamma_u$$

And elementwise weak form in 1D:

(2.31) $$\sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} \left(\frac{dw^e}{dx} \right)^T A^e E^e \left(\frac{du^e}{dx} \right) dx - \int_{\Omega^e} w^{eT} b \; dx - (w^{eT} A^e \bar{t})_{\Gamma_t^e} \right\} = 0$$

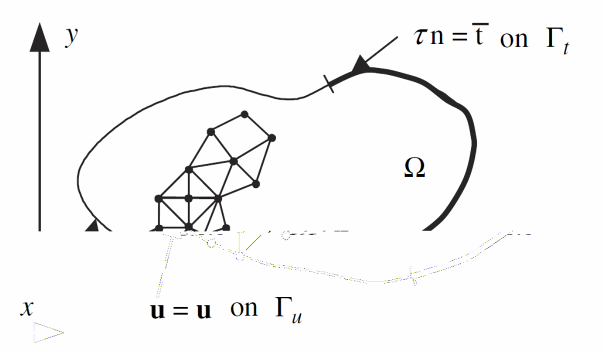
You can immediately observe the similarity between one-dimensional and multi-dimensional cases. Thus, it is not surprised that the FE formulations (and implementations) will share a lot of similarities.

3.5 Linear Elasticity: Finite Element Formulation from Weak Form

<u>Element Matrices</u>

We will derive finite element equations from the weak form in general for 2D stress analysis.

Consider a domain Ω with a boundary Γ discretized with 2D elements (triangles or quadrilaterals) shown below:



There are **two degrees-of-freedom per node** corresponding to the two components of the global displacements, so the **global nodal displacement matrix** is:

$$\mathbf{d} = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & \cdots & u_{xn_{np}} & u_{yn_{np}} \end{bmatrix}^T \tag{3.38}$$

where n_{np} is the number of nodes in the finite element mesh.

In addition, the finite element approximation of the trial solution and weight function on **each element** can be expressed as:

$$\begin{aligned} &\mathbf{u}\ (\quad) = \mathbf{N}\ (\quad)\mathbf{d} \\ &\mathbf{w}\ \ (\quad) = \mathbf{w}\ \ \mathbf{N}\ (\quad) \end{aligned} \tag{3.39}$$

where the element shape function matrix $\mathbf{N}^e$ in (3.39) is given as:

$$\mathbf{N}^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \cdots & N_{n_{en}}^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \cdots & 0 & N_{n_{en}}^e \end{bmatrix} \tag{3.40}$$

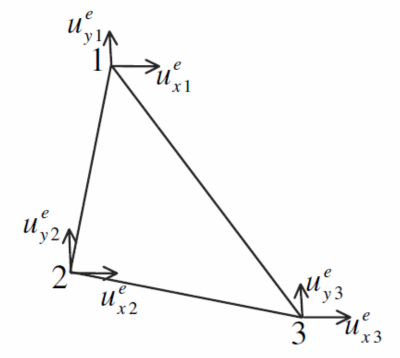
where n_{en} is number of nodes for an element.

The element nodal displacement matrix $\mathbf{d}^e$ in (3.39) is:

(3.41) $$\mathbf{d}^e = \begin{bmatrix} u_{x1}^e & u_{y1}^e & u_{x2}^e & u_{y2}^e & \cdots & u_{xn_{en}}^e & u_{yn_{en}}^e \end{bmatrix}^T$$

For example, in a linear triangular element, $n_{en} = 3$ and the column matrix $\mathbf{d}^e$ consists of six terms:

$$\mathbf{d}^e = \begin{bmatrix} u_{x1}^e \\ u_{y1}^e \\ u_{x2}^e \\ u_{y2}^e \\ u_{x3}^e \\ u_{y3}^e \end{bmatrix}$$



Similarly, the matrix for the element nodal values of weight functions $\hat{\mathbf{w}}^e$ in (3.39) is:

(3.42) $$\hat{\mathbf{w}}^e = \begin{bmatrix} \hat{w}_{x1}^e & \hat{w}_{y1}^e & \hat{w}_{x2}^e & \hat{w}_{y2}^e & \cdots & \hat{w}_{xn_{en}}^e & \hat{w}_{yn_{en}}^e \end{bmatrix}^T$$

Q: How to find element stiffness matrix and element external force matrix?
A:

Applying the symmetric gradient operator to $\mathbf{N}^e$ gives $\mathbf{B}^e$:

(3.43) $$\boldsymbol{\varepsilon}^e = \nabla_S \mathbf{u}^e = \nabla_S \mathbf{N}^e \mathbf{d}^e = \mathbf{B}^e \mathbf{d}^e$$

The strain-displacement matrix $\mathbf{B}^e$ is:

(3.44)

Similarly, the derivatives of weight functions are:

(3.45)

Substituting (3.44) and (3.45) into the elementwise weak form (3.37) and recalling and yields:

(3.46)

is the portion of corresponding to nodes that are **NOT** because of the higher dimension. The procedure for developing element matrices is, <u>however</u>, very similar to that for the 1D elements. These steps can be summarized in the following **three-step procedure for developing element matrices**:

1. Construction of shape functions matrix $\mathbf{N}^e$.
2. Formulation of the strain-displacement matrix $\mathbf{B}^e$.
3. Calculation of $\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega$ and $\mathbf{f}^e = \mathbf{f}_\Omega^e + \mathbf{f}_\Gamma^e = \int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega + \int_{\Gamma_t^e} \mathbf{N}^{eT}\bar{\mathbf{t}}\, d\Gamma$ using $\mathbf{N}^e$ and $\mathbf{B}^e$.

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The weak form Eq. (3.46) can then be written as:

(3.49) $$\hat{\mathbf{w}}^T \left\{ \left(\sum_{e=1}^{n_{el}} \mathbf{L}^{eT}\mathbf{K}^e\mathbf{L}^e\mathbf{d} \right) - \left(\sum_{e=1}^{n_{el}} \mathbf{L}^{eT}\mathbf{f}^e \right) \right\} = 0 \qquad \forall \hat{\mathbf{w}}_F$$

Or

(3.50) $$\hat{\mathbf{w}}^T \left(\mathbf{Kd} \quad \mathbf{f} \right) = 0 \qquad \hat{\mathbf{w}}_F$$

We **exactly** reproduce Eq. (2.34) in 1D!!! The rest of finite element formulation and arguments follow exactly the same as we have in 1D (**see Section 2.3**).