National Taiwan University | **Finite Element Method**
Department of Civil Engineering | Instructor: C.-S. CHEN

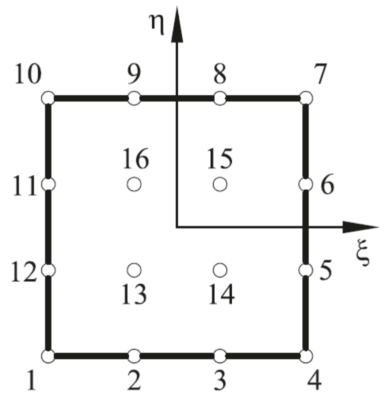
Homework 10, 05/30/2019 Due: 06/05/2019

A4 professional format and you should document your codes, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8 (the solution will be posted usually within a week))

Total 90% + 20% Bonus

1. **(30%)** Consider a cubic Lagrange element shown below. Find the shape functions for nodes 5, 10, 14 and plot these shape functions in MATLAB.



Sol:

$$N_5 = \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\eta+1)(\eta-\frac{1}{3})(\eta-1)}{(1+1)(1+\frac{1}{3})(1-\frac{1}{3})(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})(-\frac{1}{3}-1)}$$

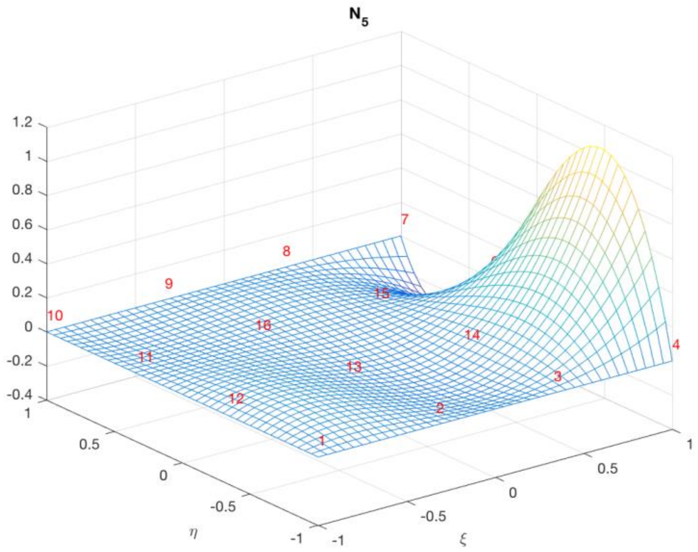
$$= \frac{243}{256}(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\eta+1)(\eta-\frac{1}{3})(\eta-1)$$

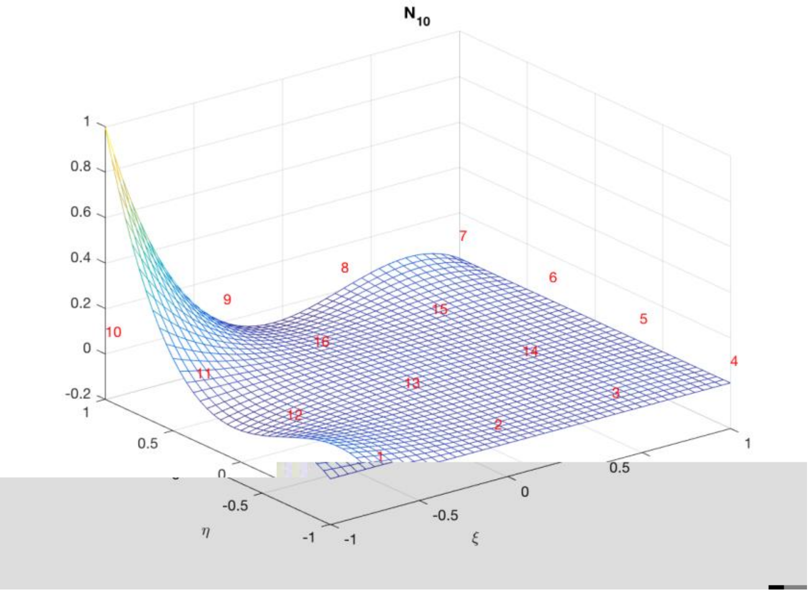
$$N_{10} = \frac{(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1)(\eta+1)(\eta+\frac{1}{3})(\eta-\frac{1}{3})}{(-1+\frac{1}{3})(-1-\frac{1}{3})(-1-1)(1+1)(1+\frac{1}{3})(1-\frac{1}{3})}$$

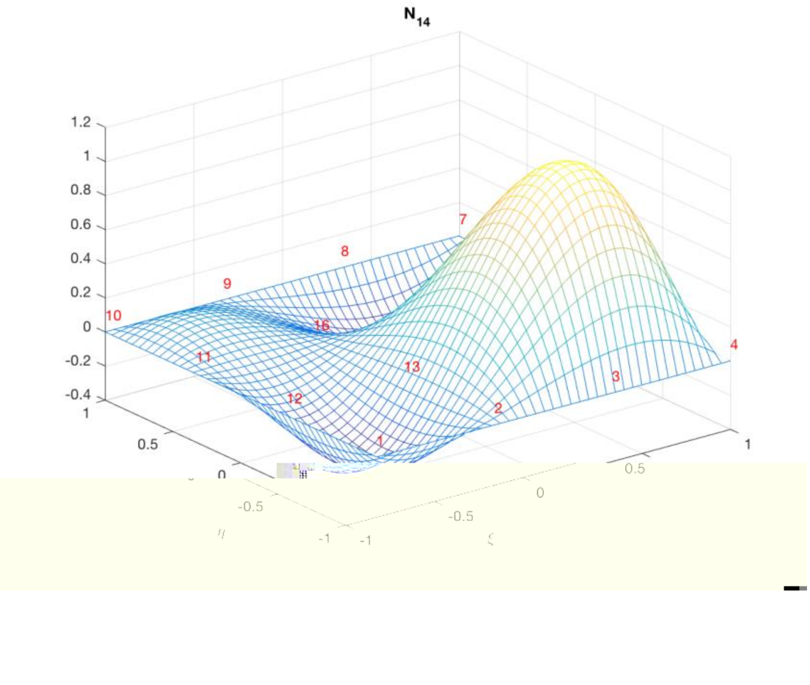
$$= \frac{-81}{256}(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1)(\eta+1)(\eta+\frac{1}{3})(\eta-\frac{1}{3})$$

$$N_{14} = \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)}{(\frac{1}{3}+1)(\frac{1}{3}+\frac{1}{3})(\frac{1}{3}-1)(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})(-\frac{1}{3}-1)}$$

$$= \frac{-729}{256}(\xi+1)(\xi+\frac{1}{3})(\xi-1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)$$


N5
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
1
0.5
0
-0.5
-1
η
-1
-0.5
0
0.5
1
ξ
1 2 3 4 7 8 9 10 11 12 13 14 15 16

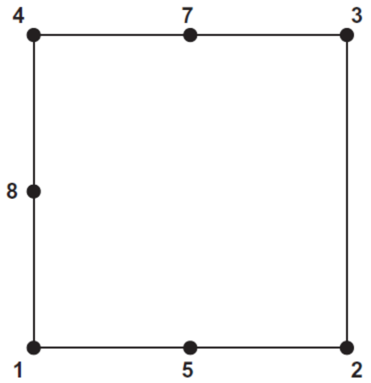

N10
1
0.8
0.6
0.4
0.2
0
-0.2
1
0.5
0
-0.5
-1
η
-1
-0.5
0
0.5
1
ξ
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16


N14
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
1
0.5
0
-0.5
-1
η
-1
-0.5
0
0.5
1
ξ
1 2 3 4 7 8 9 10 11 12 13 16

```
Xi=-1:0.05:1;
Eta=-1:0.05:1;
[xi,eta]=ndgrid(Xi,Eta);
N5=((xi+1).*(xi+1/3).*(xi-1/3))./((1+1).*(1+1/3)*(1-1/3)) ...
    .*((eta+1).*(eta-1/3).*(eta-1))./((-1/3+1).*(-1/3-1/3).*(-1/3-1));
N10=((xi+1/3).*(xi-1/3).*(xi-1))./((-1+1/3).*(-1-1/3)*(-1-1)) ...
    .*((eta+1).*(eta+1/3).*(eta-1/3))./((1+1).*(1+1/3).*(1-1/3));
N14=((xi+1).*(xi+1/3).*(xi-1))./((1/3+1).*(1/3+1/3)*(1/3-1)) ...
    .*((eta+1).*(eta-1/3).*(eta-1))./((-1/3+1).*(-1/3-1/3).*(-1/3-1));

p_xi = -1:2/3:1;
p_eta= -1:2/3:1;
'1''1' point = {'1' '2' '3' '4';
```


'1'

2. **(30%)** Derive the shape functions for a transition element shown below using the hierarchical method.



Sol:

$$N_5 = \frac{1}{2}(\xi+1)(\xi-1)(\eta-1)$$

$$N_7 = \frac{-1}{2}(\xi+1)(\xi-1)(\eta+1)$$

$$N_8 = \frac{1}{2}(\xi-1)(\eta+1)(\eta-1)$$

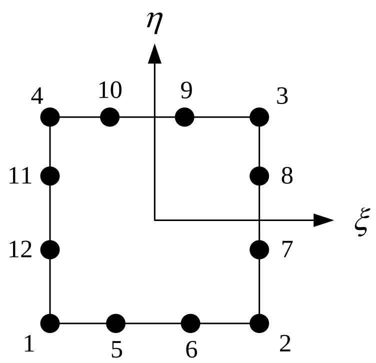
$$N_1 = N_1^L - \frac{1}{2}N_5 - \frac{1}{2}N_8 = \frac{-1}{4}(\xi-1)(\eta-1)(\xi+\eta+1)$$

$$N_2 = N_2^L - \frac{1}{2}N_5 = \frac{-1}{4}\xi(\xi+1)(\eta-1)$$

$$N_3 = N_3^L - \frac{1}{2}N_7 = \frac{1}{4}\xi(\xi+1)(\eta+1)$$

$$N_4 = N_4^L - \frac{1}{2}N_7 - \frac{1}{2}N_8 = \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1)$$

3. **(30%)** Derive the shape functions for node 2 and node 12 of a cubic Serendipity type element and plot these shape functions in MATLAB.



Sol:

(1)

$$N_{12}=\frac{(\xi-1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)}{(-1-1)(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})(-\frac{1}{3}-1)}=\frac{-27}{32}(\xi-1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)$$

(2)

$$N_2^L=\frac{-1}{4}(\xi+1)(\eta-1)$$

$$N_5=\frac{(\xi+1)(\xi-\frac{1}{3})(\xi-1)(\eta-1)}{(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})(-\frac{1}{3}-1)(-1-1)}=\frac{-27}{32}(\xi+1)(\xi-\frac{1}{3})(\xi-1)(\eta-1)$$

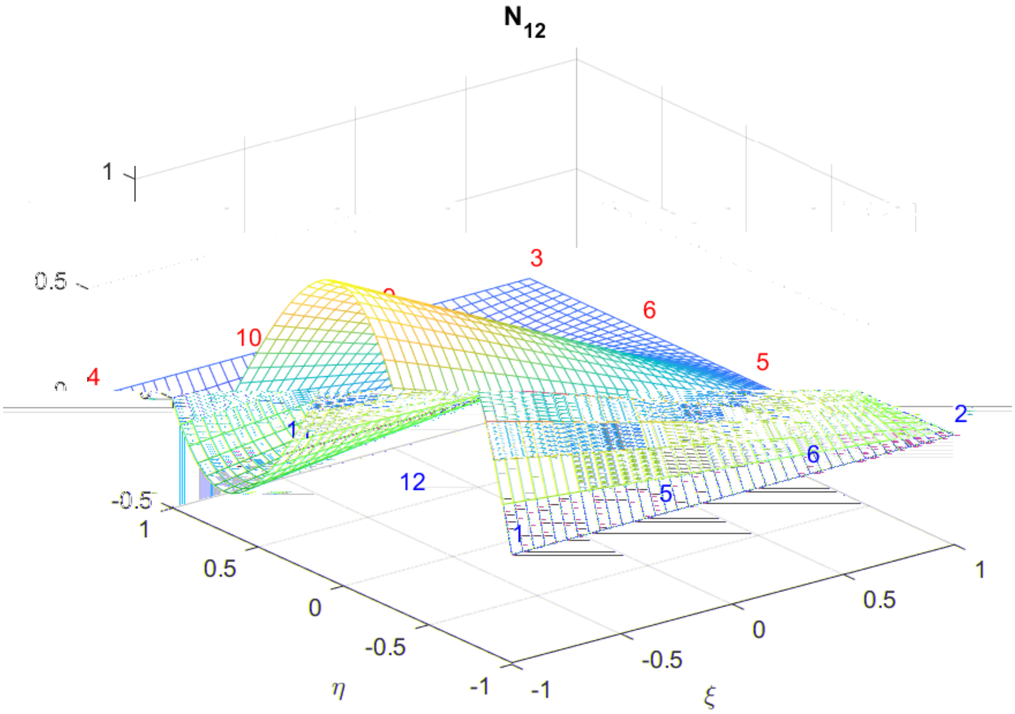
$$N_6=\frac{(\xi+1)(\xi+\frac{1}{3})(\xi-1)(\eta-1)}{(\frac{1}{3}+1)(\frac{1}{3}+\frac{1}{3})(\frac{1}{3}-1)(-1-1)}=\frac{27}{32}(\xi+1)(\xi+\frac{1}{3})(\xi-1)(\eta-1)$$

$$N_7=\frac{(\xi+1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)}{(1+1)(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})(-\frac{1}{3}-1)}=\frac{27}{32}(\xi+1)(\eta+1)(\eta-\frac{1}{3})(\eta-1)$$

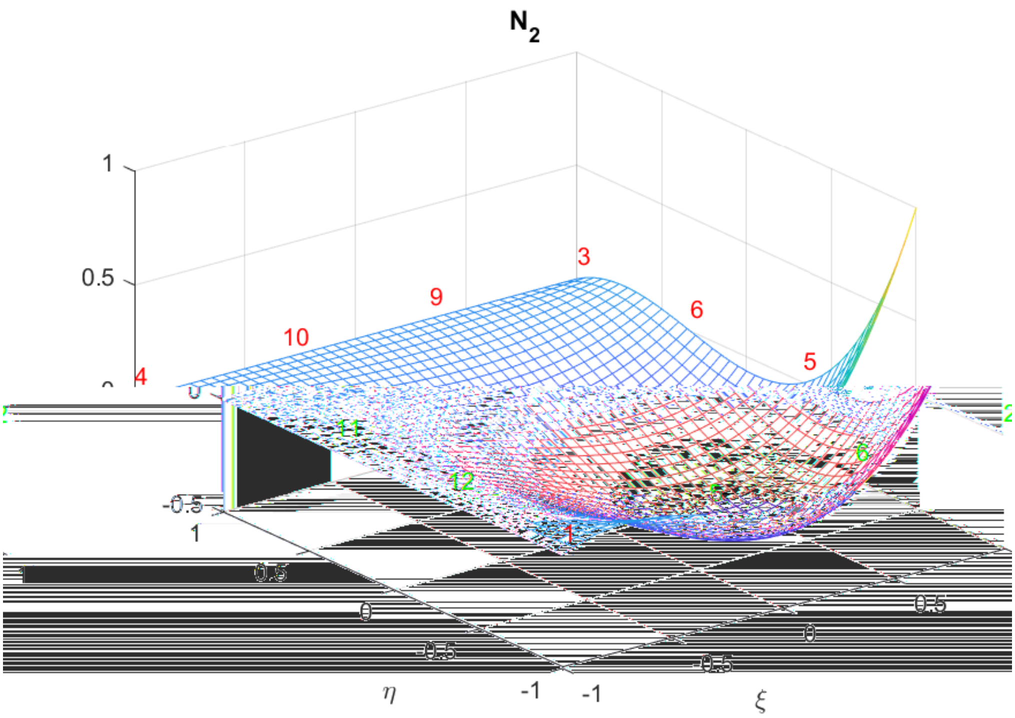
$$N_8=\frac{(\xi+1)(\eta+1)(\eta+\frac{1}{3})(\eta-1)}{(1+1)(\frac{1}{3}+1)(\frac{1}{3}+\frac{1}{3})(\frac{1}{3}-1)}=\frac{-27}{32}(\xi+1)(\eta+1)(\eta+\frac{1}{3})(\eta-1)$$

$$N_2=N_2^L-\frac{1}{3}N_5-\frac{2}{3}N_6-\frac{2}{3}N_7-\frac{1}{3}N_8$$

$$=\frac{1}{32}(1+\xi)(1-\eta)(9(\xi^2+\eta^2)-10)$$



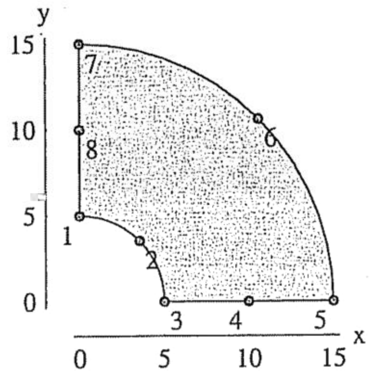
N12
η
ξ



N2
η
ξ

```
Xi=-1:0.05:1;
Eta=-1:0.05:1;
[xi,eta]=ndgrid(Xi,Eta);
N2=((xi+1).*(eta-1))./-4;

N7=(xi+1)./(1+1) ...
    .*((eta+1).*(eta-1).*(eta-1/3))./((-1/3+1).*(-1/3-1).*(-1/3-1/3));
N8=(xi+1)./(1+1) ...
    .*((eta+1).*(eta-1).*(eta+1/3))./((1/3+1).*(1/3-1).*(1/3+1/3));
N5=((xi-1).*(xi+1).*(xi-1/3))./((-1/3-1).*(-1/3+1)*(-1/3-1/3)) ...
    .*(eta-1)./(-1-1);
N6=((xi-1).*(xi+1/3).*(xi+1))./((1/3-1).*(1/3+1/3)*(1/3+1)) ...
    .*(eta-1)./(-1-1);
N2p=N2-(N8+N5)/3-(N7+N6)*2/3;

N12=(xi-1)./(-1-1) ...
    .*((eta+1).*(eta-1/3).*(eta-1))./((-1/3+1).*(-1/3-1/3).*(-1/3-1));

p_xi = -1:2/3:1;
p_eta= -1:2/3:1;
point = {'1' '5' '6' '2'; '12' '' '' '5'; ...
        '11' '' '' '6';'4' '10' '9' '3'};
% draw fig
figure(1);
mesh(xi,eta,N2p);
for i=1:4
for j=1:4
text(p_xi(i),p_eta(j),0.1,point(j,i),'color','r')
end
end
xlabel('\xi')
ylabel('\eta')
title('N_{2}')
figure(2);
mesh(xi,eta,N12);
for i=1:4
for j=1:4
text(p_xi(i),p_eta(j),0.1,point(j,i),'color','r')
end
end
xlabel('\xi')
ylabel('\eta')
title('N_{12}')

saveas(figure(1),'N2','png')
saveas(figure(2),'N12','png')
```

4. **(Bonus, 20%)** Consider the following Q8 element in the Cartesian coordinate. Calculate the strain-displacement matrix $\mathbf{B}^e$ for the element at $\xi = \eta = 0$.



Sol:

```
numberElements=1;
numberNodes=8;
syms xi eta;
elementNodes=[1 2 3 4 5 6 7 8];
nodeCoordinates=[0 5; 5*cosd(45) 5*sind(45);5 0; 10 0; 15
0;15*cosd(45) ...
15*sind(45); 0 15 ;0 10];
% GDof: global number of degrees of freedom
GDof=2*numberNodes;
for e=1:numberElements
numNodePerElement = length(elementNodes(e,:));
numEDOF = 2*numNodePerElement;
% shape functions and derivatives
shape=1/4*[ (1-xi)*(1-eta)*(-xi-eta-1) 2*(1-xi^2)*(1-eta) ...
(1+xi)*(1-eta)*(xi-eta-1) 2*(1-eta^2)*(1+xi) (1+xi)*(1+eta)*(xi+eta-
1) ...
2*(1-xi^2)*(1+eta) (1-xi)*(1+eta)*(-xi+eta-1) 2*(1-eta^2)*(1-xi)];
for i=1:numberNodes
naturalDerivatives(1,i)=diff(shape(i),xi);
naturalDerivatives(2,i)=diff(shape(i),eta);
end
% Jacobian matrix, inverse of Jacobian,
Jacob=naturalDerivatives*nodeCoordinates;
invJacobian=inv(Jacob);
XYderivatives=invJacobian*naturalDerivatives;
% % B matrix
%B=zeros(3,numEDOF);
B(1,1:2:numEDOF) = XYderivatives(1,:);
B(2,2:2:numEDOF) = XYderivatives(2,:);
B(3,1:2:numEDOF) = XYderivatives(2,:);
B(3,2:2:numEDOF) = XYderivatives(1,:);
end
B=double(subs(B,{xi,eta},{0,0}));
```

$$B = \begin{bmatrix} 0.0000 & 0.0000 & -0.0707 & 0.0000 & 0.0000 & 0.0000 & 0.0500 & 0.0000 & 0.0000 & 0.0000 & 0.0707 & 0.0000 & 0.0000 & 0.0000 & -0.0500 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0707 & 0.0000 & 0.0000 & 0.0000 & -0.0500 & 0.0000 & 0.0000 & 0.0000 & 0.0707 & 0.0000 & 0.0000 & 0.0000 & 0.0500 \\ 0.0000 & 0.0000 & -0.0707 & -0.0707 & 0.0000 & 0.0000 & -0.0500 & 0.0500 & 0.0000 & 0.0000 & 0.0707 & 0.0707 & 0.0000 & 0.0000 & 0.0500 & -0.0500 \end{bmatrix}$$