National Taiwan University **Finite Element Method**
Department of Civil Engineering Instructor: C.-S. CHEN

Homework 7, 05/03/2019 Due: 05/08/2019

A4 professional format, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8

(the solution will be posted usually within a week))

Total 80%

1. **(30%)** Derive the component K_{11} for the T3 element stiffness matrix $\mathbf{K}^e$ for plane stress problems.

   Ans：

$$\mathbf{K}^e = \int_{\Omega^e} B^{eT} D B^e d\Omega \ (10\%)$$

$$= \int_{\Omega^e} \frac{1}{(2A^e)^2} \begin{bmatrix} y_2^e - y_3^e & 0 & x_3^e - x_2^e \\ 0 & x_3^e - x_2^e & y_2^e - y_3^e \\ y_3^e - y_1^p & 0 & x_1^e - x_3^e \\ 0 & x_1^e - x_3^e & y_3^e - y_1^e \\ y_1^e - y_2^e & 0 & x_2^e - x_1^e \\ 0 & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix} d\Omega \ (10\%)$$

$$\Rightarrow K_{11} = \int_{\Omega^e} \frac{E}{(2A^e)^2 (1-\nu^2)} \left[(y_2^e - y_3^e)^2 + \frac{1-\nu}{2} (x_3^e - x_2^e)^2 \right] d\Omega = \frac{Et}{4A^e (1-\nu^2)} \left[(y_2^e - y_3^e)^2 + \frac{1-\nu}{2} (x_3^e - x_2^e)^2 \right] \ (10\%)$$

$$where\ A^e = \frac{1}{2} \left[(x_2^e y_3^e - x_3^e y_2^e) + (x_3^e y_1^e - x_1^e y_3^e) + (x_1^e y_2^e - x_2^e y_1^e) \right]$$

2. **(30%)** Considering the body forces varying linearly within the T3 element, show that the element body force matrix can be expressed as:

$$\mathbf{f}_\Omega^e = \frac{tA^e}{12} \begin{bmatrix} 2b_{x1} + b_{x2} + b_{x3} \\ 2b_{y1} + b_{y2} + b_{y3} \\ b_{x1} + 2b_{x2} + b_{x3} \\ b_{y1} + 2b_{y2} + b_{y3} \\ b_{x1} + b_{x2} + 2b_{x3} \\ b_{y1} + b_{y2} + 2b_{y3} \end{bmatrix}$$

   where $b_{x1}^e, b_{y1}^e, b_{x2}^e, b_{y2}^e, b_{x3}^e, b_{y3}^e$ are the element nodal body forces.

   Ans：

$$\mathbf{f}_\Omega^e = t(\int_{\Omega^e} \mathbf{N}^{eT}\mathbf{N}^e d\Omega)\mathbf{b}^e \quad (10\%)$$

$$= t\int_{\Omega^e} \begin{bmatrix} N_1^{e2} & 0 & N_1^e N_2^e & 0 & N_1^e N_3^e & 0 \\ & N_1^{e2} & 0 & N_1^e N_2^e & 0 & N_1^e N_3^e \\ & & N_2^{e2} & 0 & N_2^e N_3^e & 0 \\ & & & N_2^{e2} & 0 & N_2^e N_3^e \\ & & & & N_3^{e2} & 0 \\ sym. & & & & & N_3^{e2} \end{bmatrix} d\Omega \begin{bmatrix} b_{x1}^e \\ b_{y1}^e \\ b_{x2}^e \\ b_{y2}^e \\ b_{x3}^e \\ b_{y3}^e \end{bmatrix} \quad (10\%)$$

$$\int_{\Omega^e} (N_1^e)^a (N_2^e)^b (N_3^e)^c d\Omega = \frac{a!b!c!}{(a+b+c+2)!} 2A^e \quad (10\%)$$

$$\mathbf{f}_\Omega^e = \frac{tA^e}{12} \begin{bmatrix} 2b_{x1}^e + b_{x2}^e + b_{x3}^e \\ 2b_{y1}^e + b_{y2}^e + b_{y3}^e \\ b_{x1}^e + 2b_{x2}^e + b_{x3}^e \\ b_{y1}^e + 2b_{y2}^e + b_{y3}^e \\ b_{x1}^e + b_{x2}^e + 2b_{x3}^e \\ b_{y1}^e + b_{y2}^e + 2b_{y3}^e \end{bmatrix} \quad (10\%)$$

3. (20%) Consider a plane strain element as shown in the figure below.

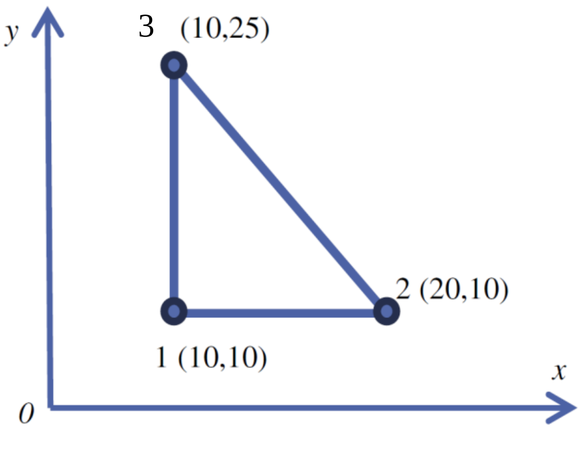
(a) Calculate the shape functions for the element.

(b) The nodal displacements are given below:

$u_{x1} = 0.005$ mm $\quad u_{x2} = 0.0$ mm $\quad u_{x3} = 0.005$ mm

$u_{y1} = 0.002$ mm $\quad u_{y2} = 0.0$ mm $\quad u_{y3} = 0.0$ mm

Let the elastic modulus $E = 70$ GPa and the Poisson ratio $\nu = 0.3$. Calculate the stresses.



Ans：

(a) (10%)

$$N_1 = \frac{1}{2A^e}[(x_2^e y_3^e - x_3^e y_2^e) + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y]$$

$$= \frac{1}{150}(400 - 15x - 10y)$$

$$N_2 = \frac{1}{2A^e}[(x_3^e y_1^e - x_1^e y_3^e) + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y]$$

$$= \frac{1}{150}(-150 + 15x)$$

$$N_3 = \frac{1}{2A^e}[(x_1^e y_2^e - x_2^e y_1^e) + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y]$$

$$= \frac{1}{150}(-100 + 10y)$$

(b) (10%)

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} = \frac{1}{150}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -15 & 0 & 15 & 10 & 0 \end{bmatrix}\left(\frac{1}{mm}\right) \quad (3\%)$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} = \frac{70000}{1.3\times 0.4}\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}\left(\frac{N}{mm^2}\right) \quad (2\%)$$

$$\mathbf{d}^e = \begin{bmatrix} 0.005 \\ 0.002 \\ 0 \\ 0 \\ 0.005 \\ 0 \end{bmatrix}, \boldsymbol{\sigma} = \mathbf{D}\mathbf{B}^e\mathbf{d}^e = \begin{bmatrix} -52.5 \\ -32.76 \\ -5.38 \end{bmatrix}\left(\frac{N}{mm^2}\right) \quad \text{(5\%: 3\% for value, 2\% for unit)}$$