Homework 5, 04/18/2019 Due: 04/24/2019

A4 professional format, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8 (the solution will be posted usually within a week))

Total 60%

1. **(30%)** The plane stress and plane strain constitutive matrix $\mathbf{D}$ can be deduced from the isotropic Hooke's law in 3D. Use the 3D constitutive matrix given in the note and derive the constitutive matrix $\mathbf{D}$ for the plane stress and plane strain problems.

<sol>

From the class note, we have the constitutive matrix

$$\mathbf{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

For the plane stress problem: $\left(l_z \ll l_x \approx l_y\right)$

$$\sigma_{zz}=\sigma_{zx}=\sigma_{zy}=0,\ \varepsilon_{zz}=\frac{-\nu}{E}\left(\sigma_{xx}+\sigma_{yy}\right)\neq 0$$

Then, from Hooke's law:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{bmatrix}=\begin{bmatrix}\sigma_{x}\\ \sigma_{y}\\ 0\\ \sigma_{xy}\\ 0\\ 0\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{x}\\ \varepsilon_{y}\\ \varepsilon_{z}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}$$

$$\Rightarrow \sigma_z=0=\frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x+\nu\varepsilon_y+\left(1-\nu\right)\varepsilon_z\right]\Rightarrow \nu\varepsilon_x+\nu\varepsilon_y+\left(1-\nu\right)\varepsilon_z=0$$

$$\Rightarrow \varepsilon_z=-\frac{\nu}{\left(1-\nu\right)}\left(\varepsilon_x+\varepsilon_y\right) \qquad (1)$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}\left[\left(1-\nu\right)\varepsilon_x+\nu\varepsilon_y+\nu\varepsilon_z\right] \qquad (2)$$

$$\sigma_y=\frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x+\left(1-\nu\right)\varepsilon_y+\nu\varepsilon_z\right] \qquad (3)$$

By substituting (1) into (2) and (3), we get

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y - \frac{\nu^2}{1-\nu}(\varepsilon_x+\varepsilon_y)\right] = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \tag{4}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\varepsilon_y - \frac{\nu^2}{1-\nu}(\varepsilon_x+\varepsilon_y)\right] = \frac{E}{1-\nu^2}\left(\nu\varepsilon_x + \varepsilon_y\right) \tag{5}$$

Similarly, the following equation could be derived from Hooke's law

$$\sigma_{xy} = \frac{E}{(1+\nu)(1-2\nu)} \times \frac{1-2\nu}{2}\gamma_{xy} = \frac{E}{1-\nu^2}\left(\frac{1-\nu}{2}\gamma_{xy}\right) \tag{6}$$

Therefore, from (4), (5) and (6), we have

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \Rightarrow \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$$

For the plane strain problem: $\left(l_z \gg l_x \approx l_y\right)$

$$\varepsilon_z = \varepsilon_{zx} = \varepsilon_{zy} = 0,\ \sigma_{zz} = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\left(\varepsilon_x+\varepsilon_y\right)\right] \neq 0$$

Then, from Hooke's law:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{bmatrix} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ 0\\ \gamma_{xy}\\ 0\\ 0\end{bmatrix}$$

We have

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right] \tag{7}$$

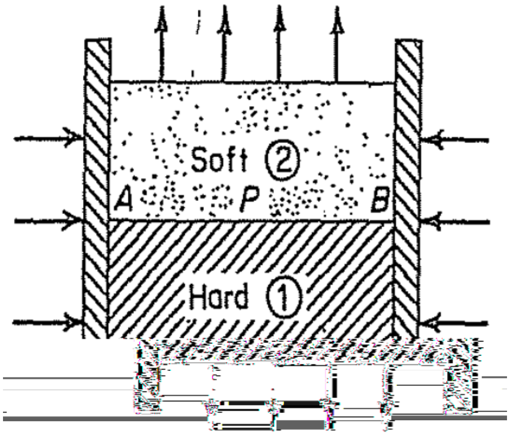
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\varepsilon_y\right] \tag{8}$$

$$\sigma_{xy} = \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{2}\gamma_{xy}\right) \tag{9}$$

From (7), (8) and (9), we get

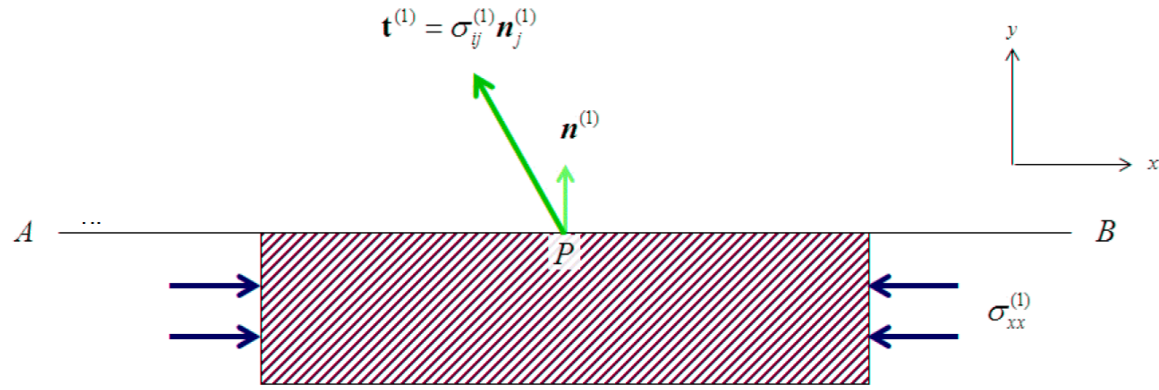
$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \Rightarrow \mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

2. **(30%)** Consider a thin body composed of a hard material (high elastic modulus) joined to a soft material (low elastic modulus), as shown in the Figure below. Let the body be compressed between two walls. Both the soft material and the hard material will be stressed. At a point P on the interface AB, show that $\sigma_{xx}^{(1)} \neq \sigma_{xx}^{(2)}$, $\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)}$ and $\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}$.

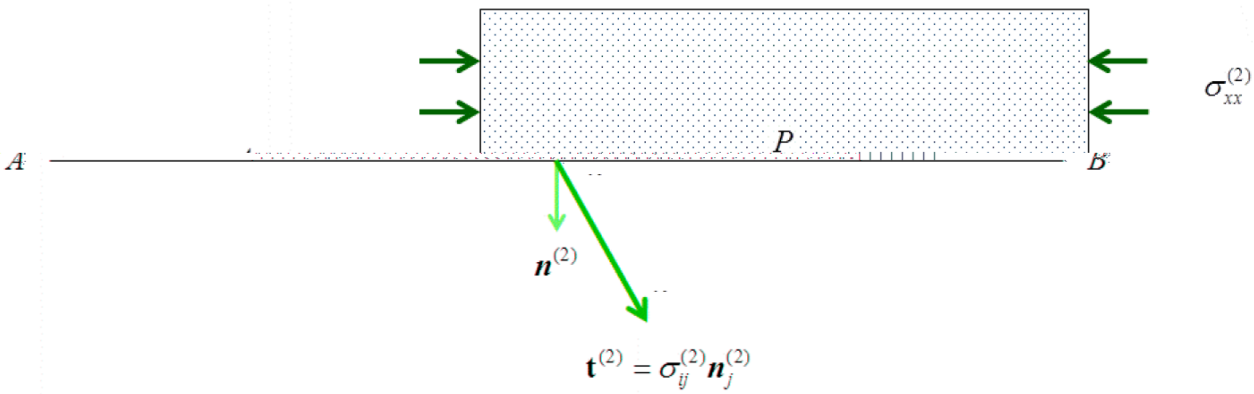


<sol>

At an interface between two bodies, the traction acting on the surface must be the same on both sides of the surface. For the hard material, a surface traction vector $t^{(1)}$ acts on the positive side of the interface, as shown in the figure below. Likewise, a traction vector $t^{(2)}$ acts on the negative side of the interface for the soft material.

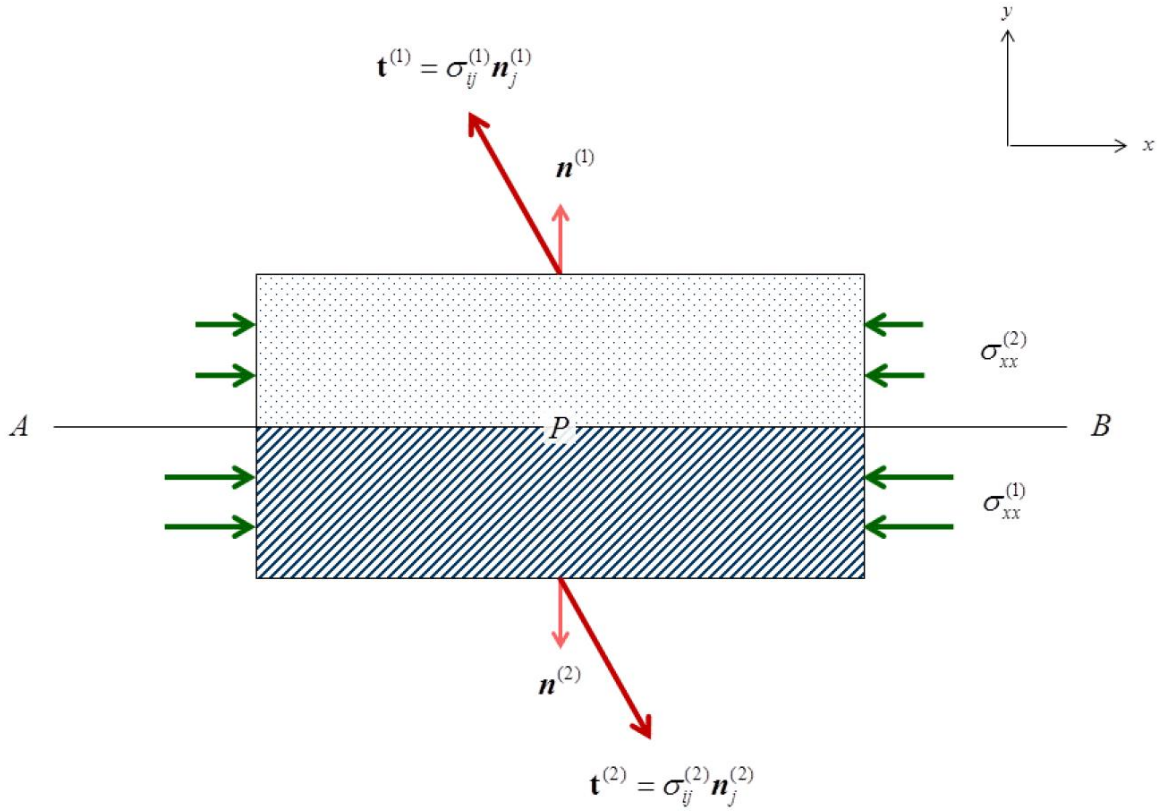


I. Free-body diagram of an infinitesimally thin element of the hard material at P



II. Free-body diagram of an infinitesimally thin element of the soft material at P

Now consider an infinitesimally small element at a point P on the interface as in the figure shown below. Let's denote the unit vectors normal to the interface as $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$, respectively, for the hard material and the soft one. For the problem here, the interface is normal to the y-axis in the Cartesian coordinate; as a result, $\boldsymbol{n}^{(1)} = \boldsymbol{n}_y^{(1)}$ and $\boldsymbol{n}^{(2)} = \boldsymbol{n}_y^{(2)}$. The stress tensor at P in the hard material is $\sigma_{ij}^{(1)}$, while the stress tensor at P in the soft material is $\sigma_{ij}^{(2)}$. Since it requires that $\boldsymbol{t}^{(1)} = \boldsymbol{t}^{(2)}$ to satisfy force equilibrium for the element, the following conditions must be meet: $\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}$, $\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)}$. However, $\sigma_{xx}^{(1)}$ and $\sigma_{xx}^{(2)}$ are not required to be continuous across the boundary to satisfy force equilibrium of the element. Indeed, if the elastic moduli of the two materials are not equal and the compressive strain is uniform, the components of the stress tensors will not be the same. In other words, $\sigma_{xx}^{(1)} \neq \sigma_{xx}^{(2)}$.



III. Free-body diagram of an infinitesimally small element including both materials at P