Homework 5, 04/11/2019 Due: 04/17/2019

A4 professional format, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8

(the solution will be posted usually within a week))

Total 100%

1. **(10%)** Evaluate the integral $\int_3^7 \frac{1}{1.1+x}dx$ by hand using three Gauss points.

<u>sol:</u>

$$f(x) = \frac{1}{1.1+x}$$

$$\xi_1 = 0.07745966692, \quad W_1 = 0.5555555556$$

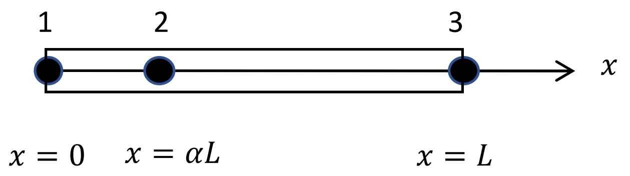
$$\xi_2 = 0, \quad W_2 = 0.8888888889$$

$$\xi_3 = -0.07745966692, \quad W_3 = 0.5555555556 \quad (5\%)$$

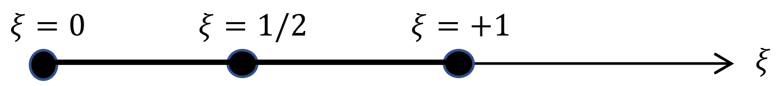
$$\int_3^7 \frac{1}{1.1+x}dx = J\int_{-1}^{1} f(\xi)d\xi = \frac{7-3}{2}\sum_1^3 W_i \cdot f_i(\xi_i \times \frac{7-3}{2} + \frac{7+3}{2}) \approx 0.6809 \quad (5\%)$$

2. **(60%)** Theory of linear elastic fracture mechanics tells us that stresses near a crack tip are inversely proportional to $\sqrt{x}$, where x is the distance from a crack tip. It happens that we can accomplish this by simply placing the mid-side nodes of a quadratic element to their quarter points. A 1D element shown in following figure is the best to illustrate the singular behavior.
   (a) Derive the mapping relationship between the physical coordinate (Cartesian space) x and the parametric coordinate ξ. That is, when $x_1^e = 0,\ x_2^e = \alpha L,\ x_3^e = L.$
   (b) Let us first consider a standard 1D quadratic element with a mid-side node ($\alpha = \frac{1}{2}$). Show the expected linear expression for the strain $\frac{du}{dx}$ using the isoparametric formulation.
   (c) Now if we move the mid-side node at the Cartesian space only to the quarter point ($\alpha = \frac{1}{4}$). Show that we now have a desirable singular expression for the strain $\frac{du}{dx}$ using the isoparametric formulation.

Physical element



Mapped element



sol:

(a) (20pt)

$x = \mathbf{N}(\xi) \cdot \mathbf{x}$

$$= \begin{bmatrix} \dfrac{(\xi - \frac{1}{2}) \cdot (\xi - 1)}{(-\frac{1}{2}) \cdot (-1)} & \dfrac{(\xi) \cdot (\xi - 1)}{(\frac{1}{2}) \cdot (-\frac{1}{2})} & \dfrac{(\xi) \cdot (\xi - \frac{1}{2})}{(1) \cdot (\frac{1}{2})} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \alpha L \\ L \end{bmatrix} \quad (10\%)$$

$$= -4\alpha L(\xi^2 - \xi) + L(2\xi^2 - \xi) \quad (5\%)$$

$$= (2 - 4\alpha)L\xi^2 + (4\alpha - 1)L\xi \quad (5\%)$$

(b) (20pt) If your shape function is wrong, you will lose 10 points.

$$u(\xi) = N(\xi) \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (2u_1 - 4u_2 + 2u_3)\xi^2 + (-3u_1 + 4u_2 - u_3)\xi + u_1$$

For $a = \frac{1}{2}$, we have $\xi = \frac{x}{L}$ form (a) (5%)

$$u(x) = (2u_1 - 4u_2 + 2u_3)(\frac{x}{L})^2 + (-3u_1 + 4u_2 - u_3)(\frac{x}{L}) + u_1 \quad (10\%)$$

$$\frac{du}{dx} = \frac{4x}{L^2}(u_1 - 2u_2 + u_3) + \frac{1}{L}(-3u_1 + 4u_2 - u_3)$$

(5%, If you didn't substitute x for ξ , you will only get 1 point)

(c) (20pt) If your shape function is wrong, you will lose 10 points.

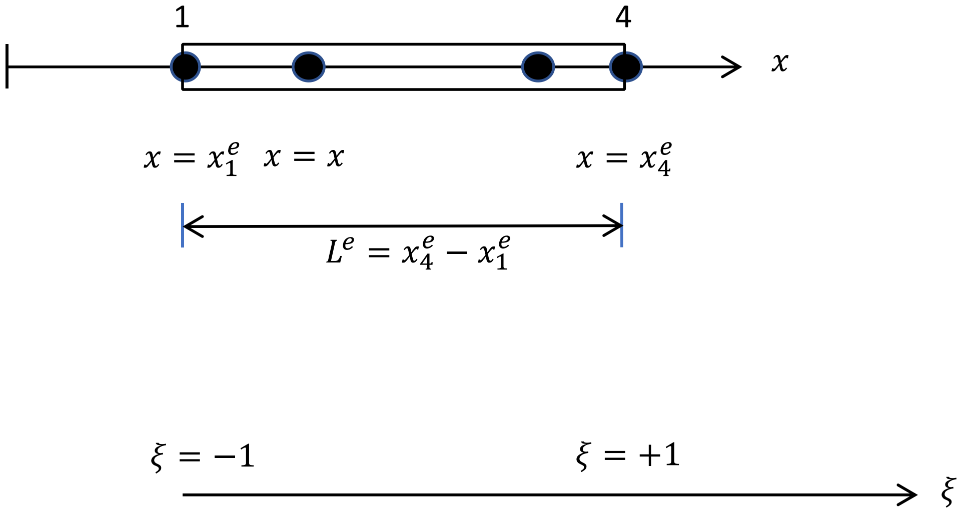
For $a = \frac{1}{4}$, we have $\xi = \sqrt{\frac{x}{L}}$ form (a) (5%)

$$u(x) = (2u_1 - 4u_2 + 2u_3)(\frac{x}{L}) + (-3u_1 + 4u_2 - u_3)\sqrt{\frac{x}{L}} + u_1 \quad (10\%)$$

$$\frac{du}{dx} = \frac{2}{L}(u_1 - 2u_2 + u_3) + \frac{1}{2\sqrt{xL}}(-3u_1 + 4u_2 - u_3)$$

(5%, If you didn't substitute x for ξ , you will only get 1 point)

3. **(30%)** Consider a cubic element below. (a) Derive the Jacobian and (b) show that when the nodes in the physical space are equally distributed, the Jacobian reduces to $\frac{L^e}{2}$.



sol:

(a) (15pt)

$$N(\xi)=\left[\frac{(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1)}{(-\frac{2}{3})\cdot(-\frac{4}{3})\cdot(-2)} \quad \frac{(\xi+1)(\xi-\frac{1}{3})(\xi-1)}{(\frac{2}{3})\cdot(-\frac{2}{3})\cdot(-\frac{4}{3})} \quad \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-1)}{(\frac{4}{3})\cdot(\frac{2}{3})\cdot(-\frac{2}{3})} \quad \frac{(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})}{(2)\cdot(\frac{4}{3})\cdot(\frac{2}{3})}\right]$$

$$=\left[-\frac{9}{16}(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1) \quad \frac{27}{16}(\xi+1)(\xi-\frac{1}{3})(\xi-1) \quad -\frac{27}{16}(\xi+1)(\xi+\frac{1}{3})(\xi-1) \quad \frac{9}{16}(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})\right]$$

(7%)

$$J=\frac{\partial N(\xi)}{\partial\xi}\begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

$$=-\left(\frac{27}{16}\xi^2-\frac{9}{8}\xi-\frac{1}{16}\right)x_1^e+\left(\frac{81}{16}\xi^2-\frac{9}{8}\xi-\frac{27}{16}\right)x_2^e-\left(\frac{81}{16}\xi^2+\frac{9}{8}\xi-\frac{27}{16}\right)x_3^e+\left(\frac{27}{16}\xi^2+\frac{9}{8}\xi-\frac{1}{16}\right)x_4^e$$

(8%)

(b) (15pt)

$Let\ \ x_2^e = x_1^e + \frac{L_e}{3}, x_3^e = x_1^e + \frac{2L_e}{3}, x_4^e = x_1^e + L_e$ (7%)

$J = L_e/2$ (8%)