National Taiwan University | **Finite Element Method**

Department of Civil Engineering | Instructor: C.-S. CHEN

Homework 4, 03/28/2019 Due: 04/10/2019

A4 professional format, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8 (the solution will be posted usually within a week))

Total 120%

1. **(50%)** Consider the concrete pier problem in HW1.

(a) Construct the element stiffness matrix and element external force matrix that takes into consideration of linear variation of cross section. (20%)

$$E = 2\times10^7\ kN/m^2$$

$$A(x) = 1\times(1+x)\,m^2$$

body force：$b(x) = weight \times Area = 24(1+x)\,kN/m$

element 1：(node 1 to node 2)

$x_1^1 = 0, x_2^1 = 1, l^1 = 1$

$$N^1 = \frac{1}{l^1}\begin{bmatrix} x_2^1 - x & x - x_1^1 \end{bmatrix} = \begin{bmatrix} 1-x & x \end{bmatrix}$$

$$B^1 = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$k^1 = \int_{x_1^1}^{x_2^1} B^{(1)^T} A(x) E B^{(1)} dx$$

$$= \int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} 2\times10^7 (1+x) \begin{bmatrix} -1 & 1 \end{bmatrix} dx$$

$$= 2\times10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^1 (1+x)\,dx$$

$$= 2\times10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left(x + \frac{1}{2}x^2 \right)\Big|_0^1$$

$$= 10^7 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \text{ (4\%) } kN/m \text{ (1\%)}$$

/ (1%)

$$F^1 = \int_{x_1^1}^{x_2^1} N^{(1)T} b(x)\,dx + \left(N^{(1)T} A\bar{t} \right)_{x=0}$$

$$= \int_0^1 \begin{bmatrix} 1-x \\ x \end{bmatrix} 24(1+x)\,dx + \left(\begin{bmatrix} 1-x \\ x \end{bmatrix} \times 20\times1\times1 \right)_{x=0}$$

$$= 24\int_0^1 \begin{bmatrix} 1-x^2 \\ x+x^2 \end{bmatrix} dx + \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= 24 \begin{bmatrix} x - \frac{1}{3}x^3 \\ \frac{1}{2}x^2 + \frac{1}{3}x^3 \end{bmatrix}\Bigg|_0^1 + \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= 24 \begin{bmatrix} \frac{2}{3} \\ \frac{5}{6} \end{bmatrix} + \text{(4\%)} \begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 36 \\ 20 \end{bmatrix} \text{ (1\%) (1\%) } kN$$

element 2：(node 2 to node 3)

$x_1^2 = 1, x_2^2 = 2, l^2 = 1$

$N^2 = \frac{1}{l^2}\begin{bmatrix} x_2^2 - x & x - x_1^2 \end{bmatrix} = \begin{bmatrix} 2-x & x-1 \end{bmatrix}$

$B^2 = \begin{bmatrix} -1 & 1 \end{bmatrix}$

$$k^2 = \int_{x_1^2}^{x_2^2} B^{(2)\mathrm{T}} AEB^2 dx$$

$$= \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} 2\times10^7 (1+x) \begin{bmatrix} -1 & 1 \end{bmatrix} dx$$

$$= 2\times10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_1^2 (1+x)\,dx$$

$$= 2\times10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left(x + \frac{1}{2}x^2 \right)\Big|_1^2$$

$$= 10^7 \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$ (6%) (4%) (1%) (1%)

$$F^2 = \int_{x_1^2}^{x_2^2} N^{2^T} b(x)\,dx$$

$$= \int_1^2 \begin{bmatrix} 2-x \\ x-1 \end{bmatrix} 24(1+x)\,dx$$

$$= 24\int_1^2 \begin{bmatrix} 2+x-x^2 \\ -1+x^2 \end{bmatrix} dx$$

$$= 24 \begin{bmatrix} 2x + \frac{1}{2} - \frac{1}{3}x^3 \\ -x + \frac{1}{3}x^3 \end{bmatrix}\Bigg|_1^2$$

$$= 24 \begin{bmatrix} \frac{7}{6} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 28 \\ 32 \end{bmatrix}$$ (4%) kN (1%)

(b) Assemble these two elements to obtain the global stiffness matrix and global external force matrix. Compute the nodal displacements. (15%)

$$K = 10^7 \begin{bmatrix} 3 & -3 & \\ -3 & 8 & -5 \\ & -5 & 5 \end{bmatrix}$$ (3%) kN/m (1%) $$F = \begin{bmatrix} 36 \\ 48 \\ 32 \end{bmatrix}$$ (3%) kN (1%)

$$Kd = F \Rightarrow 10^7 \begin{bmatrix} 3 & -3 & \\ -3 & 8 & -5 \\ & -5 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 48 \\ 32 + r_3 \end{bmatrix}$$

Node 3 is fixed end, so pick the first two row and column.

$$\Rightarrow u_3 = 0 \Rightarrow 10^7 \begin{bmatrix} 3 & -3 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 48 \end{bmatrix}$$ (2%)

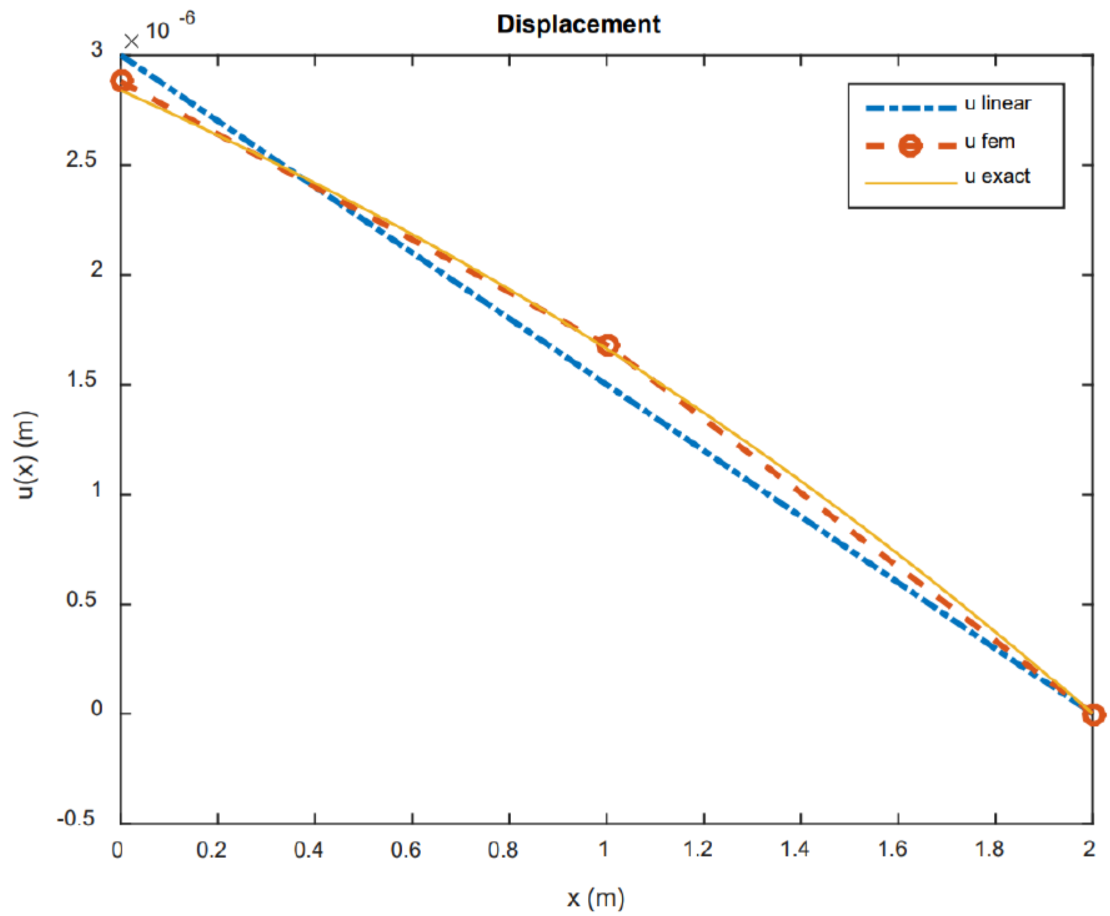
$$K^{-1} = 10^{-7} \times \frac{1}{15} \begin{bmatrix} 8 & 3 \\ 3 & 3 \end{bmatrix}$$

$$K^{-1}F = 10^{-7} \times \frac{1}{15} \begin{bmatrix} 8 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 36 \\ 48 \end{bmatrix} = 10^{-7} \begin{bmatrix} 28.8 \\ 16.8 \end{bmatrix}$$ (5%) m (4%) ∴ (1%)

$$\longrightarrow d = 10^{-7} \begin{bmatrix} 28.8 \\ 16.8 \\ 0 \end{bmatrix} \ (m)$$

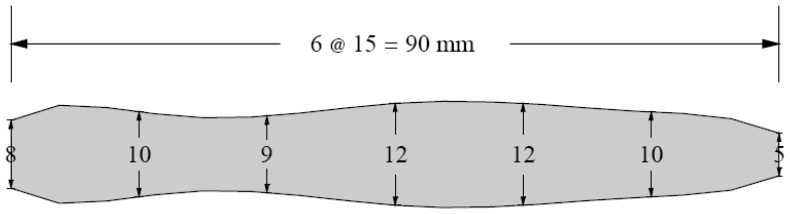
(c) Use MATLAB to plot a comparison of the FEM results with exact and classical linear approximation solutions obtained from HW1. (clear figure 10%, code 5%)

```
clc;clear;close all;
% displacement
x=0:0.01:2;
u1=-15*10^-7*(-2+x); %linear
x_fem=[0 1 2];
u_fem=[2.88*10^-6 1.68*10^-6 0]; %FEM
u_exact=(48+8*log(3)-12.*x-6.*x.^2-8.*log(1+x))/(2*10^7);
figure(1);
plot(x,u1,'-.','LineWidth',2);hold on;
plot(x_fem,u_fem,'--o','LineWidth',2);
plot(x,u_exact,'-','LineWidth',1);
title('Displacement');
xlabel('x (m)')
ylabel('u(x) (m)')
legend('u linear','u fem','u exact');
```



2. **(30%)** A flat aluminum bar has constant thickness of 10 mm and a variable symmetric profile as shown in the figure below.

(a) Construct the Lagrange interpolation functions. (10%)



$$N_i(x) = \prod_{0 \le m \le k, k \ne i} \frac{x - x_m}{x_i - x_m}$$

$$N_1(x) = \frac{(x-15)(x-30)(x-45)(x-60)(x-75)(x-90)}{(0-15)(0-30)(0-45)(0-60)(0-75)(0-90)}$$

$$= \frac{(x-15)(x-30)(x-45)(x-60)(x-75)(x-90)}{8201250000}$$

$$N_2(x) = \frac{x(x-30)(x-45)(x-60)(x-75)(x-90)}{-1366875000}$$

$$N_3(x) = \frac{x(x-15)(x-45)(x-60)(x-75)(x-90)}{546750000}$$

$$N_4(x) = \frac{x(x-15)(x-30)(x-60)(x-75)(x-90)}{-410062500}$$

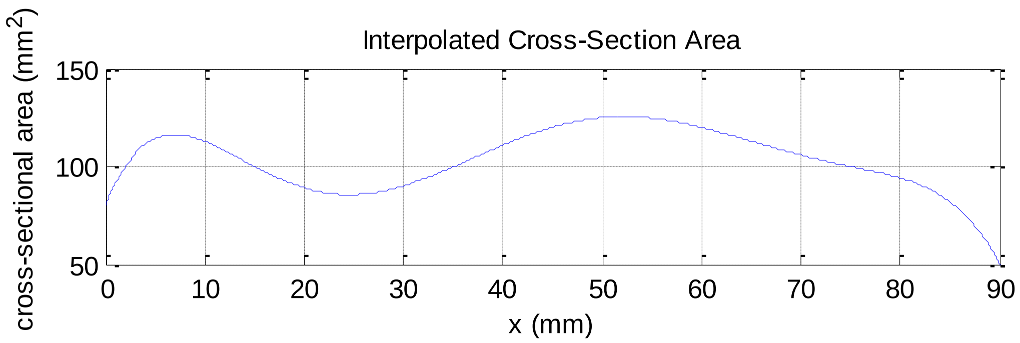
$$N_5(x) = \frac{x(x-15)(x-30)(x-45)(x-75)(x-90)}{546750000}$$

$$N_6(x) = \frac{x(x-15)(x-30)(x-45)(x-60)(x-90)}{-1366875000}$$

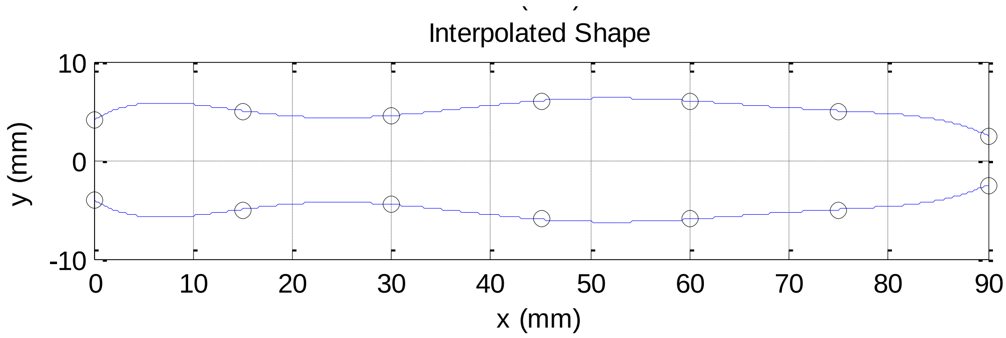
$$N_7(x) = \frac{x(x-15)(x-30)(x-45)(x-60)(x-75)}{8201250000}$$

(b) Use the Lagrange interpolation functions given in (a) and determine the half-area function. Graph the interpolation with a MATLAB plot. The interpolation function should pass through all the seven data points. A sample MATLAB plot is given below. (5% A(x), 3% figure, 2% code)

$$A(x) = \sum_{i=1}^{7} N_i(x) A_i = 80N_1 + 100N_2 + 90N_3 + 120N_4 + 120N_5 + 100N_6 + 50N_7$$



(c) Use the Lagrange interpolation functions given in (a) and determine the interpolation function for its profile. Graph the interpolation with a MATLAB plot. The interpolation function should pass through all the seven data points. A sample MATLAB plot is given below. (6% figure, 4% code)



(b), (c) code

```
close all; clear all; clc;
syms x
N = ones(1,7)*x/x;
A = [8;10;9;12;12;10;5];
for i = 0:6
    for j = 0:6
        if i == j, continue, end
        N(i+1) = N(i+1)*(x-j*15)/((i-j)*15);
    end
end
xp = 0:0.1:90;
subplot(2,1,1)
plot(xp, subs(N*A*10, xp))
grid on
xlabel('x (mm)')
ylabel('cross-sectional area (mm^2)')
title('Interpolated Cross-Section Area')
subplot(2,1,2)
plot(xp, subs(N*A, xp)/2, 'b')
hold on
plot(xp, -subs(N*A, xp)/2, 'b')
plot(0:15:90, A/2, 'ko')
plot(0:15:90,-A/2, 'ko')
grid on
xlabel('x (mm)')
ylabel('y (mm)')
title('Interpolated Shape')
```

3. **(10%)** Evaluate the integral $\int_{-1}^{1}(\xi^2+\sin(\xi/2))d\xi$ by hand using three Gauss points.

$$f(x)=\xi^2+\sin(\xi/2)$$

$$\xi_1=0.7745966692,\quad W_1=0.5555555556$$

$$\xi_2=0,\quad W_1=0.8888888889$$

$$\xi_3=-0.7745966692,\quad W_1=0.5555555556$$

$$\int_{-1}^{1}(\xi^2+\sin(\xi/2))d\xi=\sum_{i=1}^{3}W_i\cdot f(\xi_i)\approx 0.6667$$

4. **(30%)** Solve the weights and abscissa for the Gauss quadrature with $n_{gp}=3$.

$$P_0(\xi)=1$$

$$P_1(\xi)=\xi$$

$$P_2(\xi)=\frac{3}{2}\xi^2-\frac{1}{2}$$

$$P_3(\xi)=\frac{5}{2}\xi^3-\frac{3}{2}\xi=0$$

(15%)

$$\xi_i=0,\quad \pm 0.7745966692$$

(5%)

$$W_1=\frac{2}{(3\times -0.5)^2}=0.8888888889$$

$$W_2=W_3=0.5555555556$$

(10%)