National Taiwan University
Department of Civil Engineering

Finite Element Method
Instructor: C.-S. CHEN

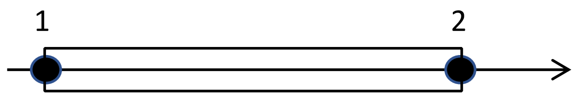
Homework 3, 03/22/2019 Due: 03/27/2019

A4 professional format, collecting at the BEGINNING of class (09:09 am)

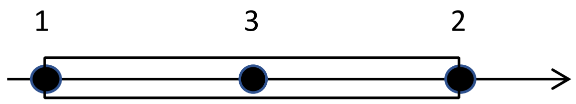
(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8 (the solution will be posted usually within a week))

Total 40%

1. **(40%)** Consider a two-node linear element and a three-node quadratic element shown below with a constant cross-sectional area A^e and Young's modulus E^e subjected to **sine distribution of body forces** $b(x) = 2\sin(\frac{\pi}{L}x)$. Derive the external body force matrix $\mathbf{f}_\Omega^e$ for the linear and quadratic elements, respectively.



$x = 0$ $x = L$



$x = 0$ $x = \frac{L}{2}$ $x = L$

<Sol>

Since $\mathbf{f}_\Omega^\mathbf{e} = \int_{x_1^e}^{x_2^e} \mathbf{N}^\mathbf{eT}\, b(x)\ \mathrm{dx}$, we have to get $\mathbf{N}^\mathbf{eT}$ first and then do integral with the body force.

For 2-node linear element:

$\mathbf{N}^\mathbf{e} = \frac{1}{L}[x_2^e - x \quad x - x_1^e] = \frac{1}{L}[L - x \quad x]$ (5%)

$$\mathbf{f}_\Omega^\mathbf{e} = \int_{x_1^e}^{x_2^e} \mathbf{N}^\mathbf{eT}\, b(x)\ \mathrm{dx}$$

$$= \int_0^L \frac{1}{L}\begin{bmatrix} L - x \\ x \end{bmatrix} 2\sin(\frac{\pi}{L}x)\ \mathrm{dx} \quad (5\%)$$

$$= \begin{bmatrix} \frac{2L}{\pi} \\ \frac{2L}{\pi} \end{bmatrix} (5\%)$$

For 3-node quadratic element:

$$\mathbf{N^e} = \left[\frac{(x - x_3^e)(x - x_2^e)}{(x_1^e - x_3^e)(x_1^e - x_2^e)} \quad \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)} \quad \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}\right] \text{(5\%)}$$

$$= \left[\frac{(x - \frac{L}{2})(x - L)}{(0 - \frac{L}{2})(0 - L)} \quad \frac{(x - 0)(x - L)}{(\frac{L}{2} - 0)(\frac{L}{2} - L)} \quad \frac{(x - 0)(x - \frac{L}{2})}{(L - 0)(L - \frac{L}{2})}\right]$$

$$= \frac{2}{L^2}\left[(x - \frac{L}{2})(x - L) \quad -2(x)(x - L) \quad (x)(x - \frac{L}{2})\right] \text{(5\%)}$$

$$\mathbf{f}_\Omega^e = \int_{x_1^e}^{x_2^e} \mathbf{N^{eT}}\, b(x)\ \mathrm{dx}$$

$$= \int_0^L \frac{2}{L^2}\begin{bmatrix} (x - \frac{L}{2})(x - L) \\ -2(x)(x - L) \\ (x)(x - \frac{L}{2}) \end{bmatrix} 2\sin(\frac{\pi}{L}x)\ \mathrm{dx} \text{(10\%)}$$

$$= \frac{2L}{\pi^3}\begin{bmatrix} \pi^2 - 8 \\ 16 \\ \pi^2 - 8 \end{bmatrix} \text{(5\%)}$$