Chapter 5: Nonlinear Finite Element for Solids

5.1 Introduction

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. E

2D 3D . M ,

. H ,

. I ,

. T

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M . A

/ . A

. F ,

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M US

V .

Finite Element Method

06/03/2019

I ____________________, ,

. I ____________,

. I ________________________________,

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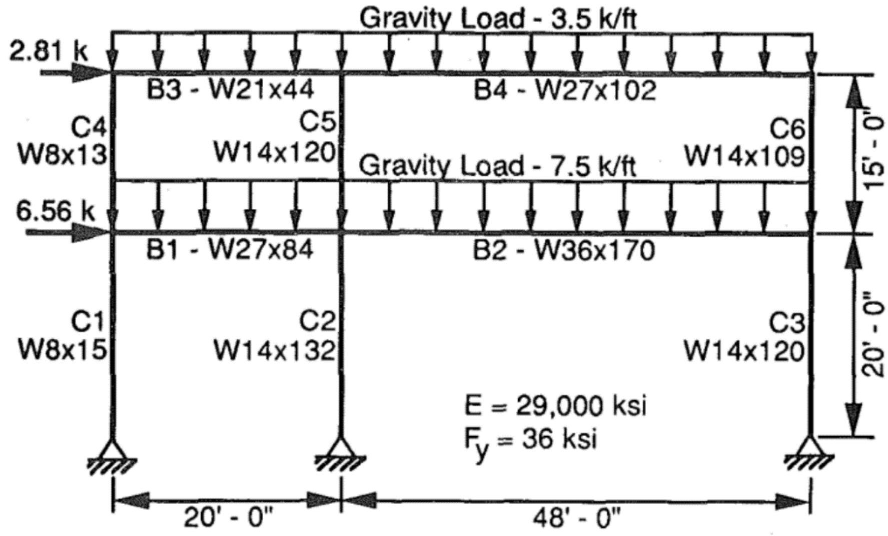
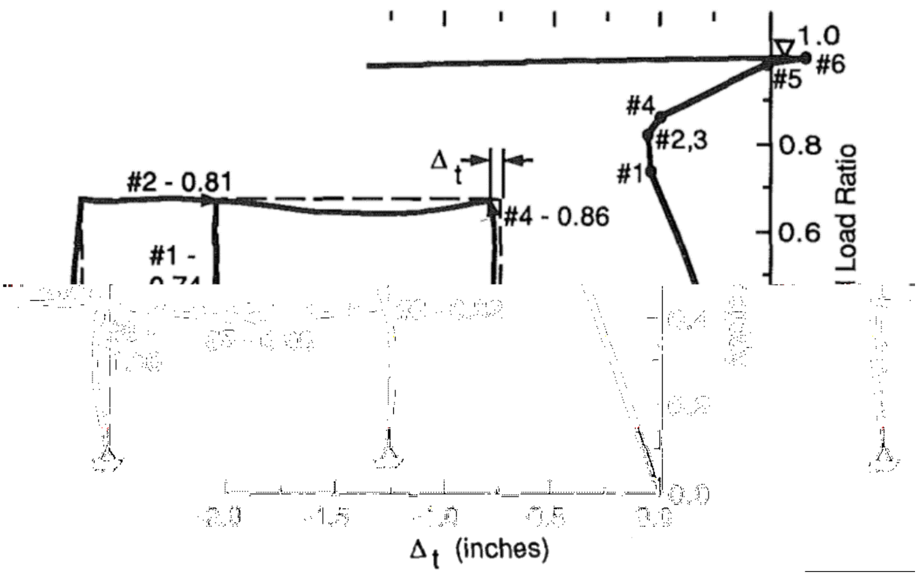


FIG. 5. Loads and Dimensions for Frame U-P36H



P ________________________________.

5.2 Nonlinear Phenomena

M :

- **Geometrical nonlinearity** , *the strains are still small.* G .
- **Finite deformations** . H . C , , .
- **Material nonlinearity**: M . A - , - . T material .
- **Stability problems** : . G includes bifurcations like buckling of frames or shells but can also be connected to limit points which indicate snap-through behaviour of a structure. Material instabilities come along with necking or shear bands in metals but also geo-materials. The origin of these lies either in an instability of the equilibrium equations or in the loss of positive definiteness related to the incremental constitutive tensor of the material. Both instabilities react in a very sensitive way to imperfections.
- **Contact problems** are characterized by stemming from the boundary are associated with **contact** between two bodies or deformation dependent loading.
- **Coupled problems** occur when different interacting fields which describe e.g. solids, heat in solids or fluids are needed to formulate a complex physical problem. Examples are thermomechanical coupling, fluid-structure-interaction or problems in which chemical reactions, heat generation and conduction and mechanical stresses have to be coupled to model an engineering process, like the design of a new material. In all cases, nonlinearities in each of the different field equations have to be considered.

F

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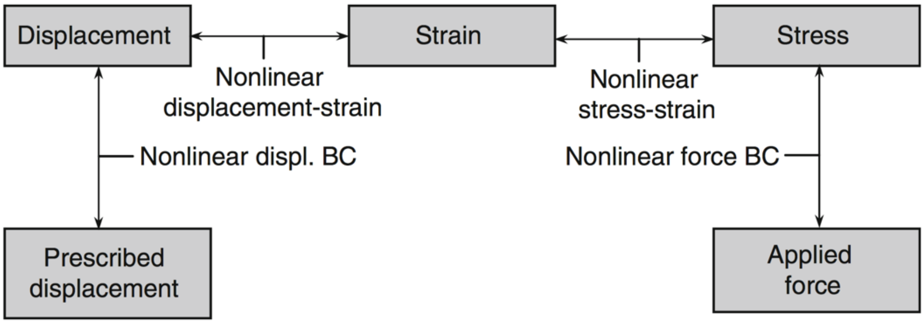
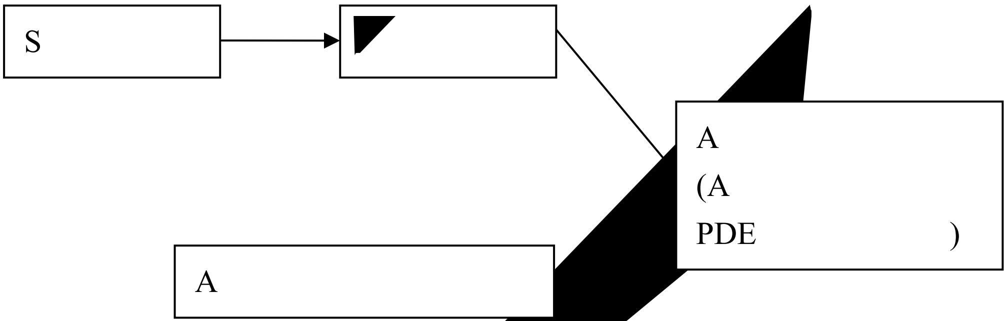


P ________, _____ ,
_______________________________________.



T , (E . (2.34) (2.38)):

(2.34) $\mathbf{w}^T\left(\mathbf{K}\mathbf{d}-\mathbf{f}\right)=0$

(2.38) $\mathbf{K}_F\mathbf{d}_F$ ... $_E$ $\Rightarrow$ $\mathbf{d}_F=\left(\mathbf{K}_F\right)^{-1}(\mathbf{f}_F-\mathbf{K}_{EF}^T\overline{\mathbf{d}}_E)$

T ( ) **K** **f**
d. **cannot** **d** **K**
f . A **K**
f $\mathbf{K}_F\mathbf{d}_F$ $\mathbf{f}_F-\mathbf{K}_{EF}^T\overline{\mathbf{d}}_E$.

nonlinear algebraic equations.

I ,
. S ______________________ . I
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, - .

5.3 Solution Methods for Nonlinear Algebraic Equations

I - - (-

) $\mathbf{K}_F\mathbf{d}_F = \mathbf{f}_F$ ($\bar{\mathbf{d}}_E = \mathbf{0}$) $\mathbf{d}_F$

$\mathbf{K}_F$ $\mathbf{d}_F$. T

.

I - ,

, :

(5.1) $g\ u\ x\ =$ $u = u(x)$

u . A

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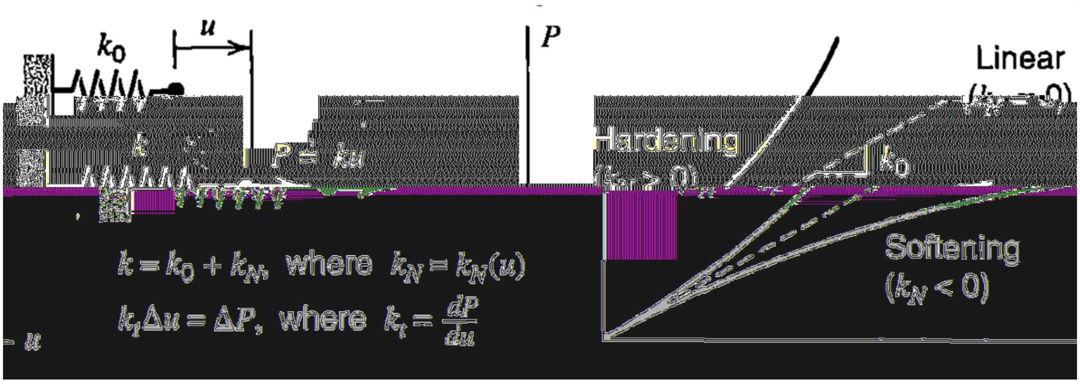


T P u

(5.2) $ku = P$ $(k_0 + k_N)u = P$ $k_N = k_N(u)$

I - . . . - . . .

$\mathbf{K}_F\mathbf{d}_F = \mathbf{f}_F$, k, ku,

u, u P . I u

, .

C **tangent stiffness**, :

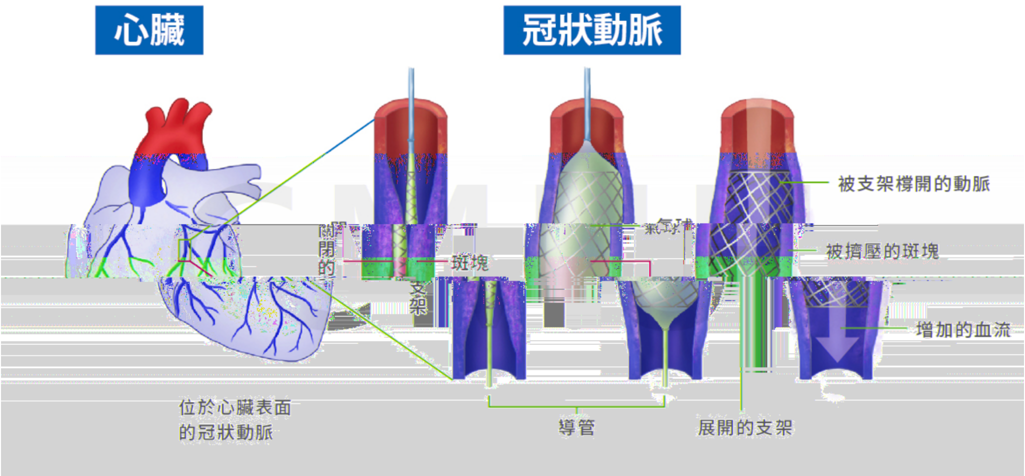
(5.3) $k_t = \dfrac{dP}{du} = k + \dfrac{\partial k_N}{\partial u} u$

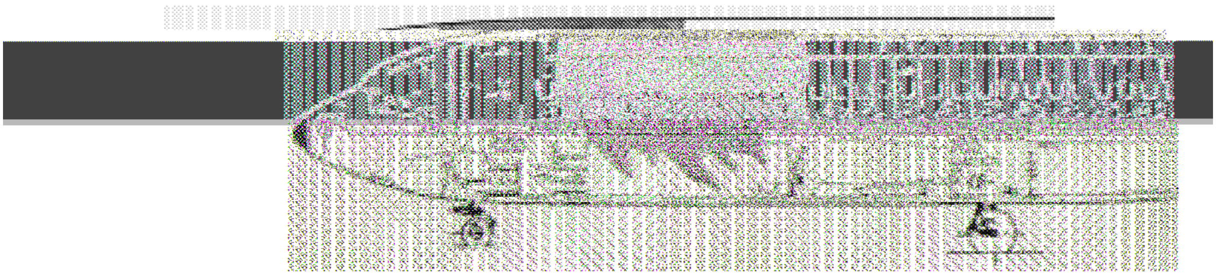
_____ P u .

Remarks:

1. N **tangent stiffness** k_t **NOT** k (, **tangent stiffness matrix** $\mathbf{K}_T$ $\mathbf{K}_F$)
2. I - , u, P, u. I , $\mathbf{K}_F$ $\mathbf{f}_F$ $\mathbf{d}_F$. E

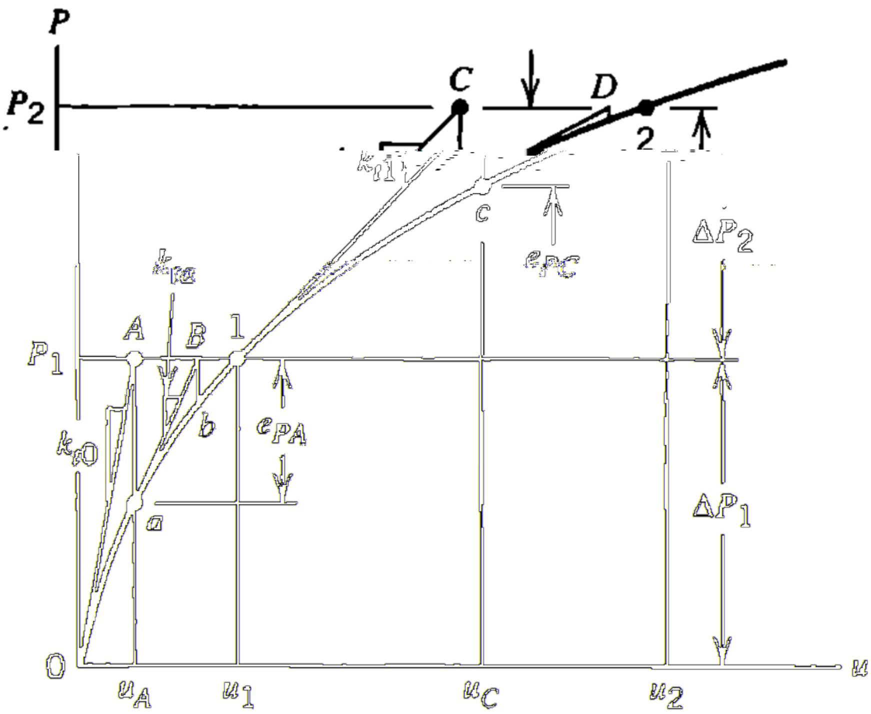
(. ., ________________: , ,).





pa gure 1.2: 1988 Aloha Airlines accident–the shaded area illustrates the fuselage lost at cruising altitude.

5.3.1 Newton-Raphson Method for Single DOF



I , N ' ________,

. H P

u , ______________________.

I E . (5.2), $u=0$. T P_1

u_1 .

T k [redacted] ,

; $\Delta P = P$ [redacted] .

:

(5.4) $k_{t0}\Delta u = \Delta P_1$ Δ [redacted] $)^{-1}\Delta P_1$ $u_A = 0+\Delta u$

H [redacted] u_1 . T ______________

P_1 .

Q: [redacted] ( ) e_{PA}?

A:

(5.5)

ku

______________________________, .

equilibrium iter

( ). P_1

, ***a*** **tangent**

a. T , u_B:

(5.6) $k_{ta}\Delta u = e_{PA}$ $)^{-1} e_{PA}$ $u_B = u_A + \Delta u$

T P_1.

Q: ( ) e_{PB}?

A:

(5.7)

T ***b***

Δu $u_B + \Delta u$. A

,

, Δu ,

u_1.

A ΔP_2 ,

u P. E

E . (5.4) (5.5)

(5.8) $_1$ = $_2$ $\Delta u = \left(k_t\right)^{-} \Delta P$ $u_C = u_1 + \Delta u$

(5.9) $e_{PC} = P_2 - ku_C$ $k = k(u)$ u_C.

C , , ,

1, 2, 3, ...

P u . T ____

________________________________.

Finite Element Method

06/03/2019

Remark: N -R :

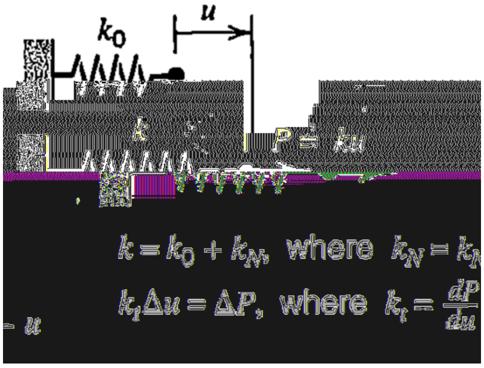
prediction (. ., E . (5.8)) *the tangent stiffness*

. T **correction** (. ., E . (5.9))

.

E : N -R

: $k_0 = 50$, $k_N = \ 5u$, $P = 100$.



(Solution)

T :

$$k = k_0 + k_N = 50 - 5u$$

T :

$$P = ku = (50 - 5u)u$$

Q: ?

A:

L $P_1 = P = 100$:

I 1

Prediction (E . (5.4)):

$$k_{t0} = 50$$

$$k_{t0}\Delta u = 100 \qquad \Delta u = \frac{1}{50}100 \qquad u_A = 0 + \Delta u = 2.0$$

Correction (E . (5.5)):

$$k = 50 - 5u_A = 40$$

$$e_{PA} = P_1 - ku_A = 20$$

<u>I 2</u>

Prediction (E . (5.6)):

$k_{ta} = 50 - 10u_A = 30$

$k_{ta}\Delta u = e_{PA}$ $\Delta u =$ [redacted] $= 0.666$[redacted] $u_A + \Delta u = 2.6667$

C[redacted]*orrection* [redacted](5.7)):

[redacted] $u_B = 36.666$[redacted]

[redacted] P_1 $ku_B = 2.22$[redacted]2

Q: [redacted] ?

A: [redacted]

MATLAB :

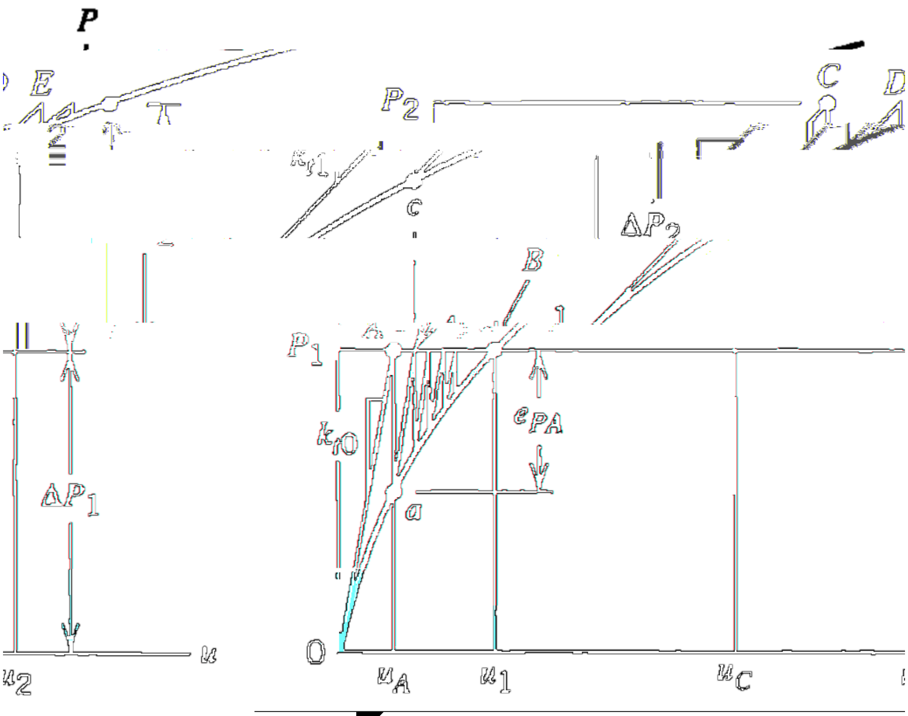
```
%
% a single nonlinear spring (Newton-Raphson method)
%
u = 0;
P = 100;
e = P;
conv = e ^ 2 / P ^ 2;
tol = 1.0e-12;
iter = 0;
while conv > tol && iter < 20
    iter = iter + 1;
    uold = u;
    kt = 50 - 10*uold;
    delta_u = kt\e;
    u = uold + delta_u;
    k = 50 - 5*u;
    e = P - k*u;
    conv = e ^ 2 / P ^ 2;
    fprintf('%3d %10.5f %7.5e \n',iter, e, conv);
end
```

T :

```
  1   20.00000 4.00000e-02
  2    2.22222 4.93827e-04
  3    0.04535 2.05676e-07
  4    0.00002 4.21494e-14
```

I 4 $\frac{\|e_P\|}{\|P\|}$ 1×10^{-12}.

6.3.2 Modified Newton-Raphson Method for Single DOF



T N [redacted] N -R

k_t

Δu [redacted] . T

[redacted] , k_{t0}

[redacted] k_{t1} k_{t1} P_2 .

[redacted] MATLAB :

```
%
% a single nonlinear spring (modified Newton-Raphson method)
%
u = 0;
P = 100;
e = P;
conv = e ^ 2 / P ^ 2;
tol = 1.0e-12;
iter = 0;
while conv > tol && iter < 30
    iter = iter + 1;
    uold = u;
    kt = 50;
    delta_u = kt\e;
    u = uold + delta_u;
    k = 50 - 5*u;
    e = P - k*u;
```

```
    conv = e ^ 2 / P ^ 2;
    fprintf('%3d %10.5f %7.5e \n',iter, e, conv);
end
```

T

```
  1   20.00000 4.00000e-02
  2    8.80000 7.74400e-03
  3    4.37888 1.91746e-03
  4    2.29435 5.26403e-04
  5    1.23276 1.51970e-04
  6    0.67106 4.50328e-05
  7    0.36786 1.35317e-05
  8    0.20241 4.09698e-06
  9    0.11161 1.24558e-06
 10    0.06161 3.79552e-07
 11    0.03403 1.15801e-07
 12    0.01880 3.53555e-08
 13    0.01039 1.07986e-08
 14    0.00574 3.29889e-09
 15    0.00317 1.00791e-09
 16    0.00175 3.07964e-10
 17    0.00097 9.41013e-11
 18    0.00054 2.87541e-11
 19    0.00030 8.78635e-12
 20    0.00016 2.68485e-12
 21    0.00009 8.20414e-13
```

I 21 [redacted] $\frac{\|e_P\|}{\|P\|}$ 1×10^{-12}.

Remark: C [redacted] N -R .
A N [redacted] -R N -
R , -
, ,
[redacted] .

[redacted]ewton-Raphson Method for Multiple DOFs

.
T ,
u:

(5.10) $\mathbf{K}(\mathbf{u})\mathbf{u} = \mathbf{f} \quad \Rightarrow \quad \Psi(\mathbf{u}) = \mathbf{f}$

I E . (2.38), $\mathbf{K}(\mathbf{u}) = \mathbf{K}_F,\ \mathbf{u} = \mathbf{d}_F,\ \mathbf{f} = \mathbf{f}_F,\ \bar{\mathbf{d}}_E = \mathbf{0}$.

S i $\mathbf{u}^i$. T

- T :

(5.10) $\Psi(\mathbf{u}^{i+1}) \approx \Psi(\mathbf{u}^i) + \mathbf{K}_T^i(\mathbf{u}^i) \cdot \Delta\mathbf{u}^i$

i :

(5.11) $\mathbf{K}_T^i(\mathbf{u}^i) \equiv \left(\frac{\partial \Psi}{\partial \mathbf{u}}\right)^i$

T $\Delta\mathbf{u}^i$ $\mathbf{u}^{i+1}$. C E . (5.10)
(5.10), ________________ :

(5.12) $\mathbf{K}_T^i \Delta\mathbf{u}^i = \mathbf{f} - \Psi(\mathbf{u}^i)$

Remark: E (5.12) , :

(1) , $\mathbf{K}_T^i(\mathbf{u}^i)$, , $\mathbf{u}^i$;

(2) , $\Delta\mathbf{u}^i$, , $\mathbf{u}$;

(3) - ,
. T . A
, $\Delta\mathbf{u}^i$, :

(5.13) $\mathbf{u}^{i+1} = \mathbf{u}^i + \quad \mathbf{u}^i$

I ,
:

(5.14) $\mathbf{R}^{i+1} = \mathbf{f} - \Psi(\mathbf{u}^{i+1})$

I , , $\mathbf{u}^{i+1}$,
, . O ,
.

<u>E</u>: $\mathbf{K}(\mathbf{u})\mathbf{u} = \mathbf{f}$
:

$$\mathbf{K}_T = \left(\frac{\partial \Psi}{\partial \mathbf{u}}\right) = \left(\frac{\partial (\mathbf{K}\mathbf{u})}{\partial \mathbf{u}}\right) = \mathbf{K} + \frac{\partial \mathbf{K}}{\partial \mathbf{u}}\mathbf{u}$$

T **K** **u**

DOF. F , 2 :

$$\begin{bmatrix} K & K \\ K & K \end{bmatrix}\begin{bmatrix} u \\ u \end{bmatrix} = \begin{bmatrix} f \\ f \end{bmatrix}$$

L

$$\Psi_1(u_1, u_2) = K_{11}u_1 + K_{12}u_2$$
$$\Psi_2(u_1, u_2) = K_{21}u_1 + K_{22}u_2$$

Q: E . (**hint**: $\mathbf{K}_T = \left(\frac{\partial \Psi}{\partial \mathbf{u}}\right) = \left(\frac{\partial (\mathbf{Ku})}{\partial \mathbf{u}}\right) = \mathbf{K} + \frac{\partial \mathbf{K}}{\partial \mathbf{u}}\mathbf{u}$)

A:

F n ,
:

$$\mathbf{K}_T = \mathbf{K} + \frac{\partial \mathbf{K}}{\partial \mathbf{u}}\mathbf{u} \quad \Rightarrow \left(K_T\right)_{ij} = K_{ij} + \sum_{m=1}^{n} \frac{\partial K_{im}}{\partial u_j} u_m$$

Remarks:

:

1. A ( ) . A .
2. T . T . H , , . S , N -R , .