$\mathbf{f}^e$ $\mathbf{f}^e_\Gamma$

T e e l fo of equilv le t od l fo ces f o t ctio s is:

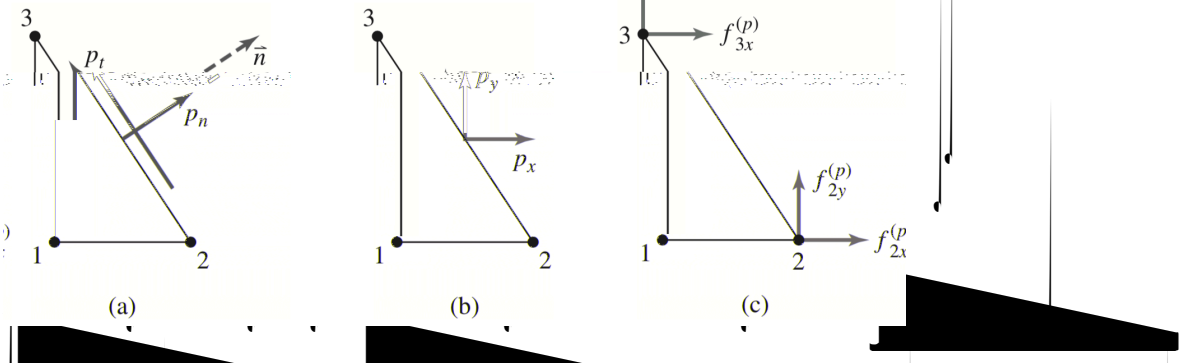
(.68) $\mathbf{f}^e_\Gamma = t\int_{\Gamma^e_t} \mathbf{N}^{eT}\overline{\mathbf{t}}\, d\Gamma$

A i you c c y out t is y u e io l i te tio d fo co st t d li e t ctio s, w e i te tio lytic lly.

essu e d s e i

F eque tly, t e ou d y ist i uted lo d s y ise f o lied essu e (o l st ess) o s e i lo ds. L l e st ess, t ese dist i uted lo ds ct o ele e t ed es t t lie o t e lo l ou d y.

As e e l e le, co side T ele e t vi o l d t e ti l lo ds p_n d p_t cti lo t e ed e defi ed y ele e t odes 2 d .



Ele e t t lc ess is de ote t, d t e lo ds e ssu ed to e e essed i te s of fo ce e u it e . We see to e l ce t e dist i uted lo ds wit equiv le t od l fo ces cti t odes 2 d .

Fi st, t e dist i uted lo ds e co ve ted to equiv le t lo di s i t e lo l coo di te di ectio s, s i Fi u e ( ), vi

$$p_x = p_n n_x - p_t n_y$$
$$p_y = p_n$$

wit n_x d n_y co es o di to t e co o e ts of t e u it outw d o l vecto to ed e

2- . He e, we tio p fo suc lo di s, s t e u its e t ose of essu e.

Fo e e l essu e d s e i co ditio , (.68) c e w itte s:

$$(.69) \quad \begin{bmatrix} \left(f_\Gamma^e\right)_{x1} \\ \left(f_\Gamma^e\right)_{y1} \\ \left(f_\Gamma^e\right)_{x2} \\ \left(f_\Gamma^e\right)_{y2} \\ \left(f_\Gamma^e\right)_{x3} \\ \left(f_\Gamma^e\right)_{y3} \end{bmatrix} = t \int_{\Gamma_t^e} [N^e] \begin{bmatrix} p_x \\ p_y \end{bmatrix} d\Gamma$$

w e e

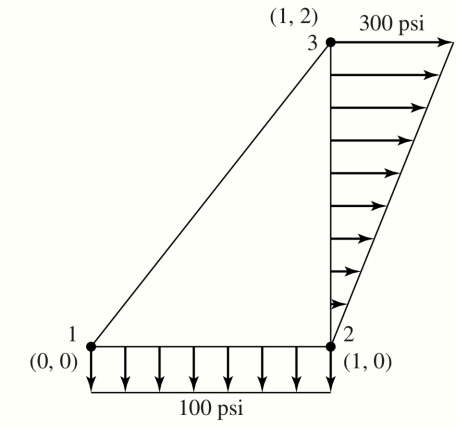
$$[N^e]^T = \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix}$$

ee n t t I te o tio u ctio N_1^e is ze o lo ed e 2- , t e equiv le t od l fo ces s de icted i Fi u e (c) e:

$$\begin{bmatrix} \left(f_!^e\right)_{x1} \\ \left(f_!^e\right)_{y1} \\ \left(f_!^e\right)_{x2} \\ \left(f_!^e\right)_{y2} \\ \left(f_!^e\right)_{x3} \\ \left(f_!^e\right)_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t\left(\int_2^3 N_2^e p_x \, d!\right) \\ t\left(\int_2^3 N_2^e p_y \, d!\right) \\ t\left(\int_2^3 N_3^e p_x \, d!\right) \\ t\left(\int_2^3 N_3^e p_y \, d!\right) \end{bmatrix}$$

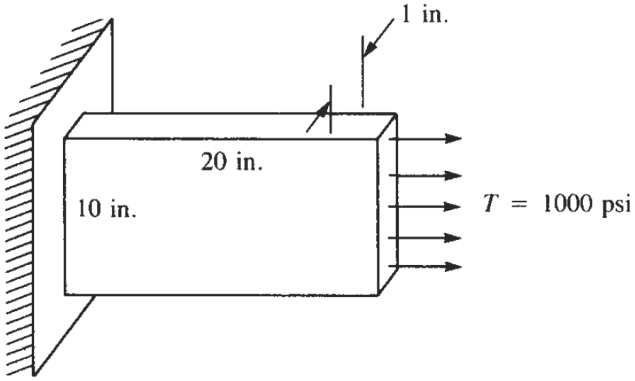
Give t e t i ul l e st ess ele e t s ow i Fi u e elow, dete i e t e

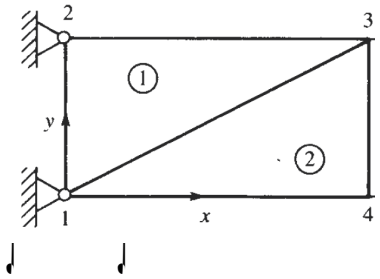
od l fo ces equiv le t to t e dist i uted lo ds. Ele e t t ic ess is 0.2 i . d u ifo .



Let us ow o t ou si le e ie wit det ned solutio of T ele e ts to illust te t e fi ite ele e t et c st ess o le .

Let us co side t i te su jected to t e su f ce t ctio s ow elow. We would li e to dete i e t e od l dis l ce e ts d t e ele e t st esses usi two T ele e ts. T e l te t ic ess $t = 1$ in., $E =$ $\times$, d $\nu = 0.3$.





T e fi ite ele e t es co sisti of two t i ul ele e ts is s ow i t e Fi u e ove. To e se ou life w ile e fo i d c lcul tio s wit t e fi ite ele e t fo ul tio , we u e t ose de ees of f eedo t t e esse ti l ou d y co ditio s fi st.

$$\mathbf{d} = \begin{bmatrix} \bar{\mathbf{d}}_E \\ \mathbf{d}_F \end{bmatrix} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix}$$

Ou o l is to solve $\mathbf{d}_F$.

We do ot eed to wo y out t is u e i sc e e w e we do MATLA

i le e t tio .

```
displacements=        (GDof,prescribedDof,stiffness,
force)
% function to find solution in terms of global displacements
      D  =       ( 1:GD    ,          D   );
 =       (     D   ,    D  )    (      D  );
displacements=zeros(GDof,1);
displacements(activeDof)=U;
```

I MATL ecto co t i i , ll `activeDof` de ees o y ,

setti u t e diffe e ce etwee ll de ees of f eedo (**d**) d t e esc i ed de ees of f eedo ($\bar{\mathbf{d}}_E$).

T e $\mathbf{B}^e$ t i fo t e t ee- o de t i le is ive y (see (.61))

$$\mathbf{B}^e = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix}$$

w e e t e e of t e t i le

$$A^e = \frac{1}{2}\begin{vmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix} = \frac{1}{2}\det\left(\mathbf{M}^e\right) = (x_2^e y_3^e - x_3^e y_2^e) + (x_3^e y_1^e - x_1^e y_3^e) + (x_1^e y_2^e - x_2^e y_1^e)$$

A d t e ele e t stiff ess t i :

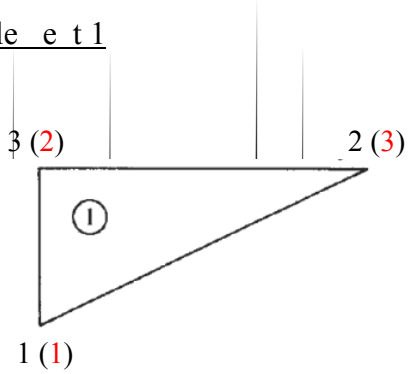
$$\mathbf{K}^e = tA^e \mathbf{B}^{eT} \mathbf{D}^e \mathbf{B}^e .$$

w e e t e co stitutive t i $\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$ fo l e st ess o le s.

Su stituti You s odulus $E =$ × d oisso tio $\nu = 0.3$ i to t e $\mathbf{D}^e$ t i , we ve:

$$\mathbf{D}^{e} = \frac{}{}\begin{bmatrix} & \\ & \end{bmatrix}$$

Ele e t 1



T e l c u e i dic tes t e ode u e i (u e ed cou te cloc wise) d t e ed fo ode u e i .

$$A^{(1)} = 100$$

$$\mathbf{B}^{(1)} = \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} (\text{in.}^{-1})$$

$$\mathbf{K} \quad \frac{}{}\begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

you s ould i e t tely o se ve t t e ele e t stiff ess t i $\mathbf{K}^{(1)}$ is sy et ic t i . If you ut t e u e s i to MATLA , you e lize t t $\mathbf{K}^{(1)}$ is wit ze o dete i t.

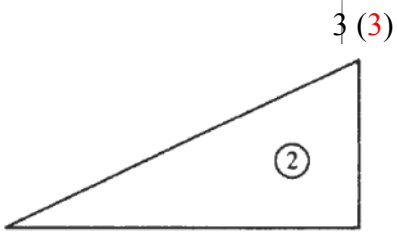
```
>> K = [140 0 0 -70 -140 70;
0 400 -60 0 60 -400;
0 -60 100 0 -100 60;
-70 0 0 35 70 -35;
-140 60 -100 70 240 -130;
70 -400 60 -35 -130 435];
```

```
>>     ( )

ans =

     0
```

Ele e t 2



A i , t e l c u e i dic tes t e loc l ode u e i d t e ed fo lo l ode u e i .

$$A^{(2)} = 100$$

$$\mathbf{B}^{(2)} = \frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} (\text{in.}^{-1})$$

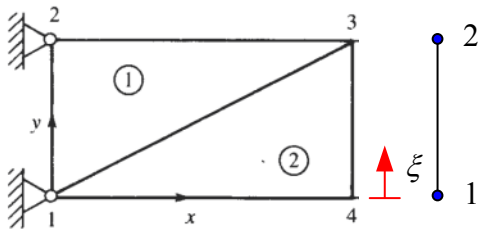
$$\mathbf{K}^{(2)} = \frac{75000}{0.91}\begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & 400 & 0 & 400 \end{bmatrix} (\text{lb/in.})$$

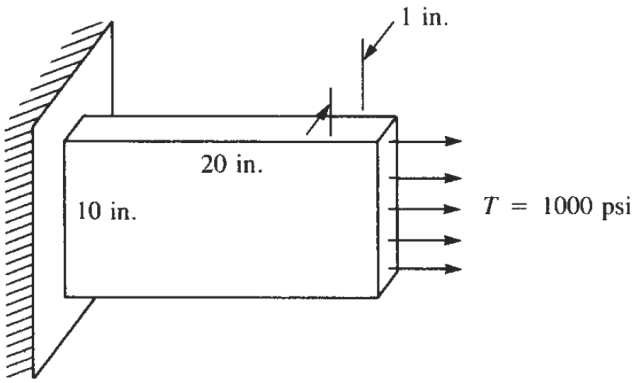
T e lo l stiff ess t i is o t i ed y di ect sse ly of t e two ele e t stiff ess t ices:

$$\mathbf{K} = \int_{-1}^{1} \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \det \mathbf{J}\, d\xi$$

Let us now consider the element force vector $\mathbf{f}^e$. In this case, we only have $\mathbf{f}^e$ due to the action of [illegible] edge-4 and we shall list the local coordinate interpolation functions of the edge parameter ξ:

$$N_1^{edge43} = N_1 = 1 - \xi \quad \text{and} \quad N_2 = \xi$$





$$\mathbf{f}_{\Gamma}^{e}=\int_{\Gamma_t^e}\mathbf{N}^{eT}\bar{\mathbf{t}}\,d\Gamma=\int_0^1\begin{bmatrix}N_1\bar{t}_x\\N_1\bar{t}_y\\N_2\bar{t}_x\\N_2\bar{t}_y\end{bmatrix}Jd\xi=\int_0^1\begin{bmatrix}(1-\xi)\times1000\\0\\\xi\times1000\\0\end{bmatrix}Jd\xi=\begin{bmatrix}\frac{1}{2}\\0\\\frac{1}{2}\\0\end{bmatrix}1000\times10=\begin{bmatrix}5000\\0\\5000\\0\end{bmatrix}$$

Now etu to ou e le d t e esulti lo l syste of equ tio s is ive y:

$$\frac{375000}{0.91}\begin{bmatrix}48&0&-28&14&0&-26&-20&12\\0&87&12&-80&-26&0&14&-7\\-28&12&48&-26&-20&14&0&0\\14&-80&-26&87&12&-7&0&0\\0&-26&-20&12&48&0&-28&14\\-26&0&14&-7&0&87&12&-80\\-20&14&0&0&-28&12&48&-26\\12&-7&0&0&14&-80&-26&87\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\d_5\\d_6\\d_7\\d_8\end{bmatrix}=\begin{bmatrix}0+r_1\\0+r_2\\0+r_3\\0+r_4\\5000\\0\\5000\\0\end{bmatrix}$$

$$\begin{bmatrix}d_5\\d_6\\d_7\\d_8\end{bmatrix}=\frac{375000}{0.91}\left(\begin{bmatrix}48&0&-28&14\\0&87&12&-80\\-28&12&48&-26\\14&-80&-26&87\end{bmatrix}\right)^{-1}\begin{bmatrix}5000\\0\\5000\\0\end{bmatrix}$$

$$\begin{bmatrix}d_5\\d_6\\d_7\\d_8\end{bmatrix}=\begin{bmatrix}609.6\\4.2\\663.7\\104.1\end{bmatrix}\times10^{-6}(\text{in.})$$

St ess lcul tio s

We ow dete i e t e st esses i e c ele e t y ec lli (.20) d (.4):

(.20) **D**

(.4) $\boldsymbol{\varepsilon}^e=\nabla_S\mathbf{u}^e=\nabla_S\mathbf{N}^e\mathbf{d}^e=\mathbf{B}^e\mathbf{d}^e$

$\Rightarrow\boldsymbol{\sigma}=\mathbf{D}\mathbf{B}^e\mathbf{d}^e$

Ele e t 1

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\frac{30(10^6)}{0.91}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}\times\frac{1}{200}\begin{bmatrix}0 & 0 & 10 & 0 & -10 & 0\\ 0 & -20 & 0 & 0 & 0 & 20\\ -20 & 0 & 0 & 10 & 20 & -10\end{bmatrix}\begin{bmatrix}0\\ 0\\ 609.6\\ 4.2\\ 0\\ 0\end{bmatrix}\times 10^{-6}$$

$$=\begin{bmatrix}1005\\ 301\\ 2.4\end{bmatrix}(\text{psi})$$

Ele e t 2

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\frac{30(10^6)}{0.91}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}\times\frac{1}{200}\begin{bmatrix}-10 & 0 & 10 & 0 & 0 & 0\\ 0 & 0 & 0 & -20 & 0 & 20\\ 0 & -10 & -20 & 10 & 20 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 663.7\\ 104.1\\ 609.6\\ 4.2\end{bmatrix}\times 10^{-6}$$

$$=\begin{bmatrix}995\\ -1.2\\ -2.4\end{bmatrix}(\text{psi})$$

o t e co ute st esses σ_{xx}, σ_{yy}, σ_{xy} i e c ele e t e se se

0 /01/2019

We [illegible] T ele e ts to solve 2 [illegible] o le s (o seco d-o de [illegible] equ tio s). T e sic ste s i volved i t e fi ite ele e t lysis of ty ic l o le :

: isc etiz tio of t e ive do i i to [illegible] eselected fi ite ele e ts. We t us [illegible] e fi ite ele e t es of eselected ele e ts.

: e v tio of ele e t equ tio s fo l ty ic l ele e ts i t e es [illegible] ele e t, co [illegible] t i $\mathbf{K}^e$, d fo ce vecto $\mathbf{f}^e$.

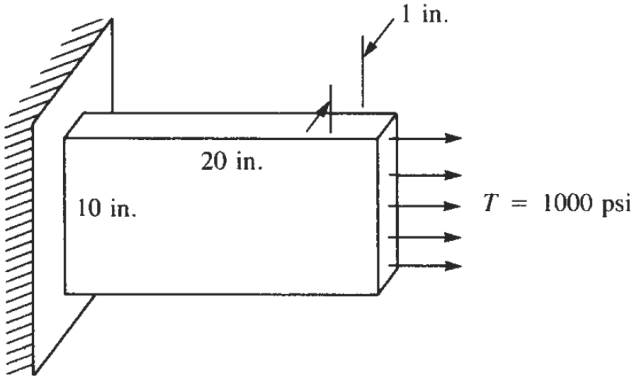
: Asse [illegible] ly of ele e t equ tio s to o t i equ tio s of t e w o [illegible] te $\mathbf{Kd} = \mathbf{f}$.

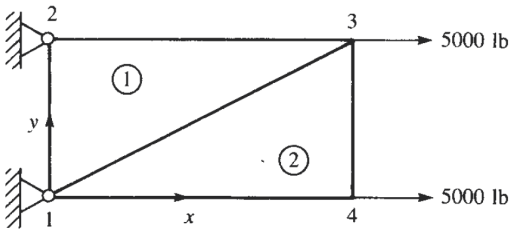
: I ositio of esse ti l ou d y co ditio s of t e o le .

: Soluti [illegible] equ tio s d o t i t e lo l dis l ce e ts $_F$.

: ost ocessi of t e esults. Fo e c ele e t, ev lu te t e st s i s d st esses d ot e s (ost- ocessi).

We will co side t e s e T e le to de o st te ou MATLA i le e t tio .





: isc etiz tio of t e ive do i i to [illegible] collectio of eselected fi ite ele e ts. We t us co [illegible] e t es of eselected ele e ts.

T e fi st ste is e ocessi : t e i o [illegible] is to den e o [illegible] i tes d set of ele e ts co ected t odes (d ot e te i l o e ties s well).

```
% A Thin Plate Subjected to Uniform Traction
% T3 Implementation
% 2 elements
% clear memory and close all figures
clear all;
```

```
clc;
close all;

% materials
E  = 30e6;     poisson = 0.30;  thickness = 1;

% matrix D
D=E/(1-poisson^2)*[1 poisson 0;poisson 1 0;0 0 (1-poisson)/2];

% preprocessing
% numberElements: number of elements
numberElements = 2;
% numberNodes: number of nodes
numberNodes = 4;
% coordinates and connectivities
elementNodes=[1 3 2; 1 4 3];
nodeCoordinates=[0, 0; 0, 10; 20, 10; 20, 0];
drawingMesh(nodeCoordinates,elementNodes,'T3','k                    all;
```

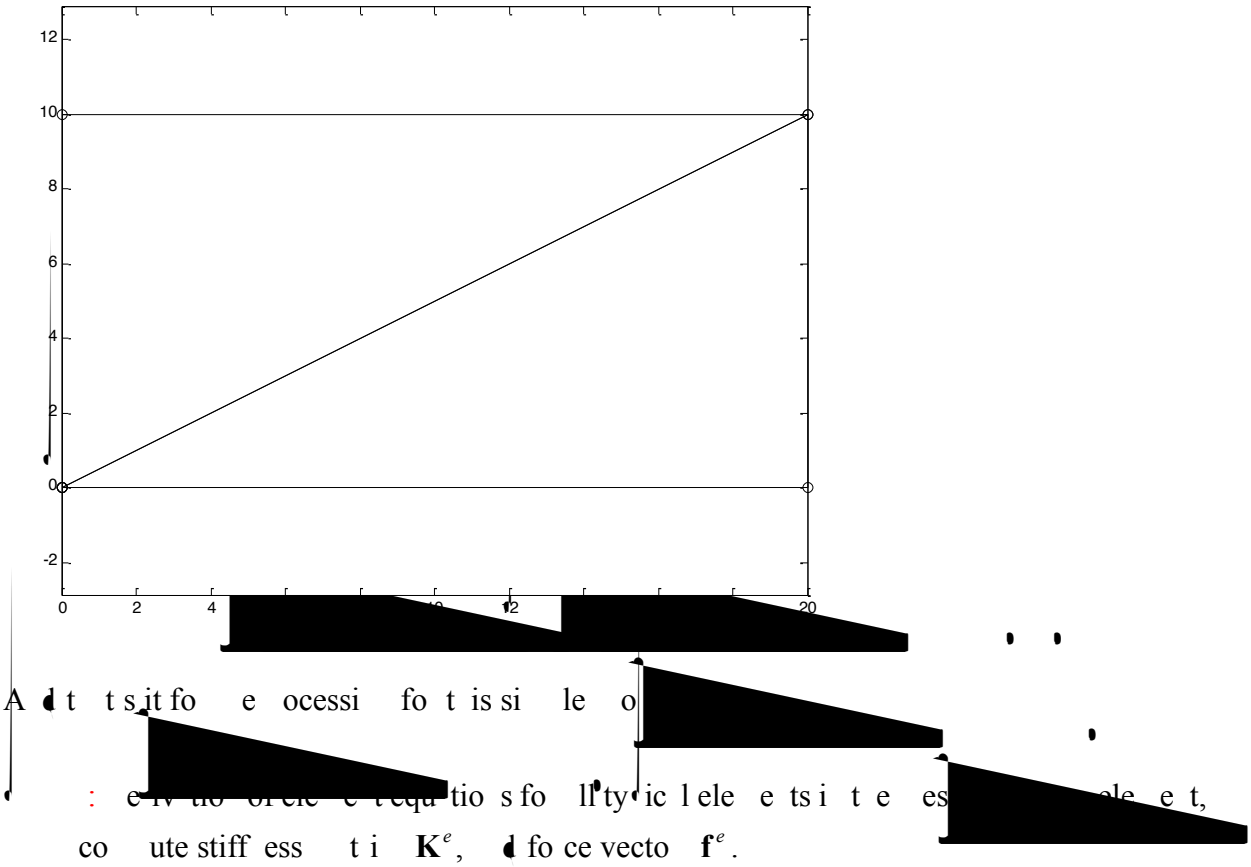
Si ce it is t e e sie to e u i te tio l ist es w ile ty i t e coo di tes d co ectivities i 2 , it would e ette to d w t e es sc iz it. We ve w itte e te si le fu ctio `drawingMesh.m` t t is useful to visu lize you es i MATLA . T e cu e t ve sio of `drawingMesh.m` o ly su o ts T d Q4 ele e ts ut it will e e te ded l te fo ot e ele e ts.

```
function drawingMesh(nodeCoordinates, elementNodes, type, format)

    case 'T3'
        seg1 = [1,2,3,1];
    case 'Q4'
        seg1 = [1,2,3,4,1];
    otherwise
     disp('Type is not supported yet')


for e = 1:length(elementNodes(:,1))
        (    C          (            ( ,   1),1),
             C          (            ( ,   1),2),      )
    hold on
end

axis equal
hold off
```



A t t s it fo e ocessi fo t is si le o

: e iv tio of ele e t equ tio s fo ll ty ic l ele e ts i t e es ele e t, co ute stiff ess t i $\mathbf{K}^e$, d fo ce vecto $\mathbf{f}^e$.

: Asse ly of ele e t equ tio s to o t i equ tio s of t e w ole o le .

Fo t e T ele e t, t e ele e t stiff ess t i is $\mathbf{K}^e = tA^e\mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e$. I MATLA , we c dele te ele e t stiff ess t i c lcul tio s d sse ly of ele e t stiff ess i to si le fu ctio `formStiffness2D.m`

```
function stiffness=              2D(GDof,numberElements,...
  elementNodes,numberNodes,nodeCoordinates,D,thickness)

% compute stiffness matrix for T3 elements

stiffness=zeros(GDof);

for e=1:numberElements
  numNodePerElement = length(elementNodes(e,:));
  numEDOF = 2*numNodePerElement;
  elementDof=zeros(1,numEDOF);
  for i = 1:numNodePerElement
    elementDof(2*i-1)=2*elementNodes(e,i)-1;
    elementDof(2*i)=2*elementNodes(e,i);
  end

  %  B matrix
  x1 = nodeCoordinates(elementNodes(e,1),1);
  y1 = nodeCoordinates(elementNodes(e,1),2);
```

```
  x2 = nodeCoordinates(elementNodes(e,2),1);
  y2 = nodeCoordinates(elementNodes(e,2),2);
  x3 = nodeCoordinates(elementNodes(e,3),1);
  y3 = nodeCoordinates(elementNodes(e,3),2);
  A = 1/2*det([1 x1 y1; 1 x2 y2; 1 x3 y3]);
  B = 1/(2*A).*[y2-y3 0 y3-y1 0 y1-y2 0;
                0 x3-x2 0 x1-x3 0 x2-x1;
                x3-x2 y2-y3 x1-x3 y3-y1 x2-x1 y1-y2]

% stiffness matrix
  stiffness(elementDof,elementDof)=...
    stiffness(elementDof,elementDof)+...
    A*          *B'*D*B;
end
```

Ge e 11 le e t tio oi e te l fo ce vecto s $\mathbf{f}^e$ due to t ctio is t e i volved d we s ll ost o e t e discussio . Fo ow let us just use t e ow od l fo ce esults di ectly:

```
% GDof: global number of degrees of freedom
GDof = 2*numberNodes;

% boundary conditions
prescribedDof = [1 2 3 4]';
% force vector
force = zeros(GDof,1);
      (5) = 5000;
      (7) = 5000;

% calculation of the system stiffness matrix
stiffness = formStiffness2D(GDof,numberElements,...
    ele          numberNodes,nodeCoordinat          kness);
```

: I ositio of esse ti l ou d y co ditio s of t e o le .

: Solutio of t e sse led equ tio s d o t i t e lo l dis l ce e ts $_F$.

```
% solution
displacements=solution(GDof,prescribedDof,stiffness,force);

% output displacements
outputDisplacements(displacements, numberNodes, GDo
```

Notice t t t e fu ctio `solution.m` e i s t e s e fo l d 2 o le s. I is ec oes ou e ly de iv tio co clusio s o fi ite ele e t fo ul tio (see t e fte (. 0)).

T e `outputDisplacements.m` is just fo tted i ti fo dis l ce e ts.

```
function outputDisplacements...
    (displacements,numberNodes,GDof)

% output of displacements in tabular form

disp('Displacements')
jj=1:numberNodes; format
f=[jj; displacements(1:2:GDof)'; displacements(2:2:GDof)'];
fprintf('Node          UX          UY\n')
fprintf('%4d                %10.4e\n',f)
```

O ce we u t e sc i t, we et:

```
Displacements
Node          UX          UY
   1   0.0000e+000    0.0000e+000
   2   0.0000e+000    0.0000e+000
   3   6.0958e-004    4.1633e-006
   4                  1.0408e-004
```

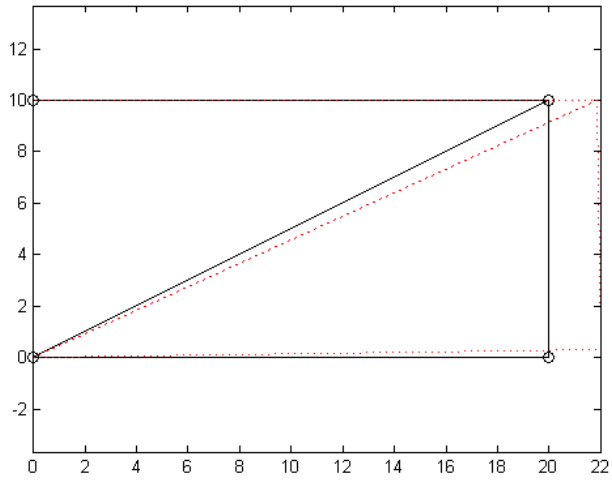
I co iso wit t e solutio s we d:

$$\begin{bmatrix} d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} = \begin{bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{bmatrix} \times 10^{-6} (\text{in.})$$

: ost ocessi o I e esults. To e c ele e t, ev lu te t e st i s d st esses d ot e s (ost- ocessi

We will let you fi is t is t i L

si `drawingMesh.m`, we c lso lot t e defo ed es e sily:



We will i let you fi is t is t i L

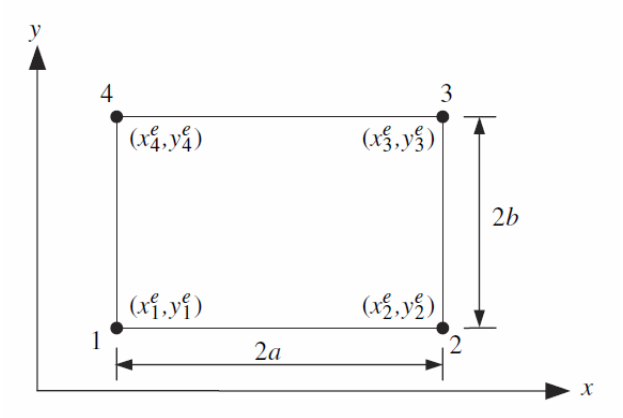
We will ow develo t e fou - oded ect ul ele e t. T e 4- oded ect le is t e si lest qu d si ult eously to t e - oded t l le. T e ect ul ele e t is ot ve y useful to oel co lic ted s es. It eve t eless se ves elude to e t c te o iso et ic fo ul tio of qu d il te l ele e ts.

As we will o se ve l te , t e iou - oded ect ul d qu d il te l ele e t obte ives ette o i tio of dis l ce e ts d st esses i co iso wit t e t ee- oded t i ul ele e t (T). To st t t e de iv tio s, we will follow i t e

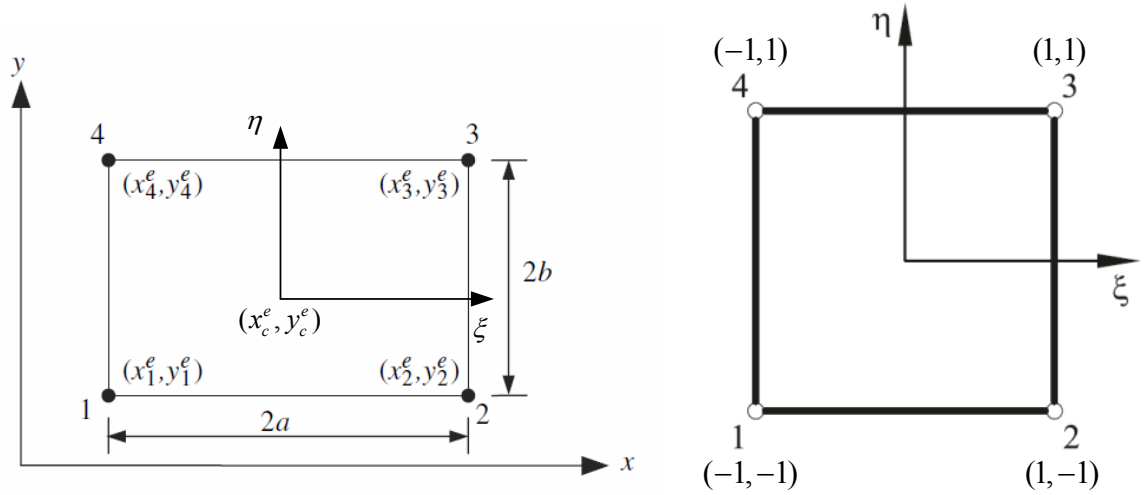
1. o st uctio of s e fu ctio s t l N .
2. Fo ul tio of t e st i -dis l ce e t t i $\mathbf{B}^e$.
. lcul tio of $\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega$ d $\mathbf{f}^e = \mathbf{f}^e_\Omega + \mathbf{f}^e_\Gamma = \int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega + \int_{\Gamma^e_t} \mathbf{N}^{eT}\overline{\mathbf{t}}\, d\Gamma$ usi $\mathbf{N}^e$ d $\mathbf{B}^e$.

$\mathbf{N}^e$

A ect ul ele e t wit di e sio s $2a \times 2b$ is s ow i t e fi u e clow (il l te lo les e 90°) wit co e odes 1–4 (l eled cou te cloc wise), es ectively.



Alt ou we c o e d est lis t e e iv tio s l ysic l coo di tes, t e e iv tio s e uc e sie ose e t coo di te s (,) A iu it e te syste (ξ, η) wit its o l l loc ted t t e ce t e of t e ect ul ele e t is defi ed. To e e ele e t, t ese e t coo di tes e el ted to t e ysic l coo di tes y si le t sl tio d sc li .



e (,) we ve t e followi i el tio s i etwee ysic l d e t coo di tes:

(.70)

Note t t wit t is c e of v i les

(.71) $$d\xi = \frac{dx}{a}$$ $$d\eta = \frac{dy}{b}$$

1. Equ tio s ( ) (.71) defi e ve y si le etwee ysic l d e t c ste s fo ect ul ele e ts. I is i o e i volved fo qu i te ele e t s ow i e t c te .
2. T e use of e t coo di tes will e t e co st uctio of t e s e fu ctio s d ev lu tio of i te tio s ve y u G uss q It is e t e ely owe ful w e used fo develo i ele e ts of co le s es.

As t e e e two Of's t ode, ect ul ele e t s tot l of ei t OFs. T e dis l ce e t field is o i ted y:

(.72) $$\mathbf{u}^e(\xi,\eta) = \mathbf{N}^{Q4}(\xi,\eta)\mathbf{d}^e$$

(.72)

$$\begin{bmatrix} u_x^e \\ u_y^e \end{bmatrix} = \begin{bmatrix} N_1^{Q4}(\xi,\eta) & 0 & N_2^{Q4}(\xi,\eta) & 0 & N_3^{Q4}(\xi,\eta) & 0 & N_4^{Q4}(\xi,\eta) & 0 \\ 0 & N_1^{Q4}(\xi,\eta) & 0 & N_2^{Q4}(\xi,\eta) & 0 & N_3^{Q4}(\xi,\eta) & 0 & N_4^{Q4}(\xi,\eta) \end{bmatrix} \begin{bmatrix} u_{x1}^e \\ u_{y1}^e \\ u_{x2}^e \\ u_{y2}^e \\ u_{x3}^e \\ u_{y3}^e \\ u_{x4}^e \\ u_{y4}^e \end{bmatrix}$$

As t e ele e t s fou odes, it is ecess y to st t wit oly o i l e sio s, t e dditio l te s ould co e f o t e t i d ow of t e sc l t i le.

1

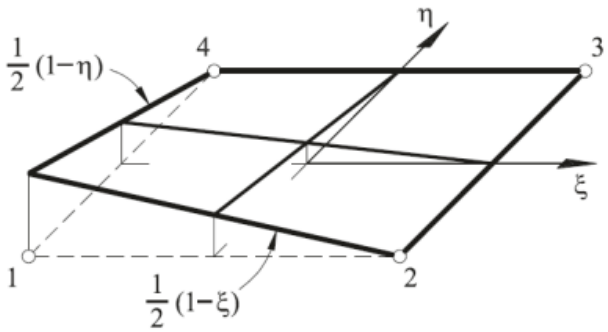
ξ η

ξ^2 $\xi\eta$ η^2

W ic of t e t ee te s i t e t i d ow s ould e selected d w y

T e s $(i=1,2$ st ucted y t e

T is o c is se o t f oducts of lowe di e sio l s e fu ctio s de loiti t e K o ec e delt o e ty of s e fu ctio s.



Fo e le, $N_1^{Q4}(\xi,\eta)$ c e o t i ed f o t e oduct of o e-di e sio l $N_1(\xi)$ d $N_1(\eta)$.

$$N_1^{Q4}(\xi,\eta)=N_1(\xi)\times N_1(\eta)=\frac{1}{2}(1-\xi)\times\frac{1}{2}(1-\eta)$$

oes $_1{}^4(\ ,\ )$ ve t e K o ec e delt o e ty fo t e fou - ode ect ul ele e ts

W t e t e s e fu ctio s N_i^Q $i=$ y i s ectio

A i Equ tio (.7) s ould e c s e fu ctio to e e ct, s it v ie li e ly i ot t e ξ d η di ectio s y i y di ectio ot e t t es two ξ d η di ectio s. e oti t e e t coo di tes of ode i y (ξ_i,η_i), t e ili e s e fu ctio $N_i^{4Q}(\xi,\eta)\ (i=1,\ 2,\ 3,\ 4)$ c e e-w itte i co cise fo :

(.74) $$N_i^{Q4}(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

$\mathbf{N}^{Q4}$

si t e s e occu e s fo t e c se of t e t i ul T ele e t, we c o t i t est i - dis l ce e t t i $\mathbf{B}^e$:

(.7) $$\mathbf{B}^e=\nabla_S\mathbf{N}^{Q4}$$

w ite t e t i fo fo (.7).

e ive $\mathbf{B}^e(1,1)$ f o $\mathbf{B}^e$ t i fo t e ili e ect ul ele e t f o t e ove

You c e ive ll t e te s fo $\mathbf{B}^e$ t i d o t i t e followi :

(.76) $$\mathbf{B}^e = \frac{1}{4}\begin{bmatrix} -\frac{1-\eta}{a} & 0 & \frac{1-\eta}{a} & 0 & \frac{1+\eta}{a} & 0 & -\frac{1+\eta}{a} & 0 \\ 0 & -\frac{1-\xi}{b} & 0 & -\frac{1+\xi}{b} & 0 & \frac{1+\xi}{b} & 0 & \frac{1-\xi}{b} \\ -\frac{1-\xi}{b} & -\frac{1-\eta}{a} & -\frac{1+\xi}{b} & \frac{1-\eta}{a} & \frac{1+\xi}{b} & \frac{1+\eta}{a} & \frac{1-\xi}{b} & -\frac{1+\eta}{a} \end{bmatrix}$$

[redacted]

It is ow cle t tt e st i- is i ce e t t i lo ili e ect ul ele e t is o lo e co st t t i . T is i lies t tt e st i , d e ce t e st ess, wit i li e ect ul ele e t is ot co st t.

T e ele e t stiff ess t i c e o t i ed f o (.47):

(.77) $$\mathbf{K}^e = \int \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega = t\int_{-1}^{+1}\int_{-1}^{+1} ab\,\mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\xi d\eta$$

[redacted]

Fo t is si le ili it is ossi le to o t i closed fo fo t e ele e t stiff ess t i y c yi tio lytic lly. As t e etwee ysic l d e t coo i te syste s is o e i volved (s we s ll see i t e qu d il te l ele e t i e t c te), we will eed two-di e sio l G uss Qu d tu e di ectly e te ded f o l .

T e ele e t e te l fo ce t i of ect ul ele e t wit t ic ess *t* co sists of two te s: o e f o t e o y fo ces d t e ot e f o t e t ct io s:

- $\mathbf{f}^e$ $\mathbf{f}^e_\Omega$

(.78) $$\mathbf{f}^e_\Omega = t\int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega$$

We c c y out t is y u e ic l i te tio . Fo co st t o y fo ces d li e o y fo ces, we c de ive t e i te tio lytic lly. Fo co st t dist i uted o y fo ce, $\bar{\mathbf{b}} = \begin{bmatrix} \bar{b}_x \\ \bar{b}_y \end{bmatrix}$ i w ic $\bar{b}_x$ d $\bar{b}_y$ e co st t v lues, we ve:

$$\mathbf{f}_{\Omega}^{e} = t\left[\int_{\Omega^e} \mathbf{N}^{e}\, d\Omega\right]^{T} \bar{\mathbf{b}} = tab\left[\int_{-1}^{+1}\int_{-1}^{+1} \mathbf{N}^{Q4} d\xi d\eta\right]\bar{\mathbf{b}}$$

Followi si il ocess s we did i T ele e t, we c fi d eve y o de s $\begin{bmatrix} - \\ - \end{bmatrix}$ f o t e co st t o dy fo ce.

- $\mathbf{f}^{e}$ $\mathbf{f}_{\Gamma}^{e}$

(.79) $\mathbf{f}_{\Gamma}^{e} = t\int_{\Gamma_t^e} \mathbf{N}^{eT}\bar{\mathbf{t}}\, d\Gamma$

A i t is will follow w t we did i T ele e t.

ec ll t e G uss Qu d tu e i l

$$\hat{I} = \int_{-1}^{1} f(\xi)d\xi = W_1 f(\xi_1) + W_2 f(\xi_2) + \ldots = \sum_{i=1}^{n_{gp}} W_i f(\xi_i)$$

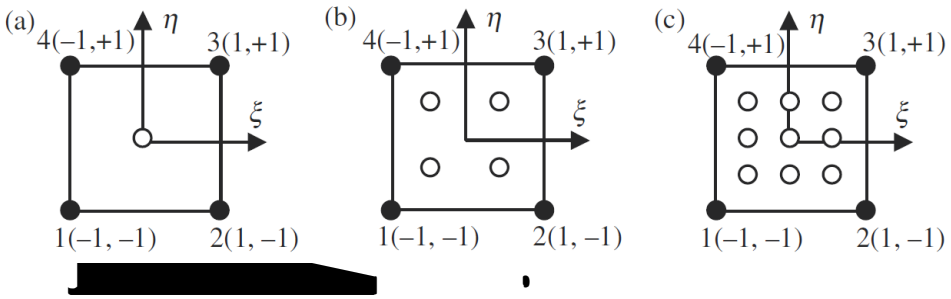
i w ic W_i e y e t e i te tio , oi ts ( ). T e n_{gp} u e of G uss oi ts. Fo two-di e sio l i te tio s, t e G uss i te tio is s led i two di ectio s, s follows:

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)d\xi d\eta = \int_{-1}^{+1}\left(\sum_{i=1}^{n_{gp}} W_i f(\xi_i,\eta)\right)d\eta$$

(.80)

$$= \sum_{i=1}^{n_{gp}}\sum_{j=1}^{n_{gp}} W_i W_j f(\xi_i,\eta_j)$$

T us, t e i te ls ev lu te u e ic lly y dou le su tio , usi t e s e wei ts qu d tu e oi ts s i o e-di e sio l qu d tu e. T e fi u e elow s ows i te tio oi ts i squ e e io .



E ess t e ele e t stiff ess t i (.77) i te s of G uss qu d tu e.

(.81)

If we ssu e co st t co stitutive t i , ow y i te tio oi ts do we eed to o t i t e e ct solutio of t e ele e t stiff ess t i fo t e ect ul ele e t (.7)

We c dete i e t e st esses i e c ele e t y ec lli (.20) d (.4):

(.20) **D**

(.4) $\boldsymbol{\varepsilon}^e = \nabla_S \mathbf{u}^e = \nabla_S \mathbf{N}^{Q4} \mathbf{d}^e = \mathbf{B}^e \mathbf{d}^e$

$\Rightarrow \boldsymbol{\sigma} = \mathbf{D}$

Mo e so istic ted oced u e to dete I e o e ccu te st esses f o G uss oi ts will e el o ted i t e e t c te .

We e ow e ul ele e ts to solve 2 ti e el sticity o le s (o seco d-o de ti I me e ti I equ tio s). T e sic ste s i volved i t e fi ite ele e t lysis of ty ic l o le :

: isc etiz tio of t e ive do i i to [illegible] fi ite ele e ts. We t us [illegible] fi ite ele e t es of [illegible] ele e ts.

: e v tio of ele e t equ tio s fo ll ty ic l ele e ts i t e es [illegible] ele e t, co [illegible] t i $\mathbf{K}^e$, d fo ce vecto $\mathbf{f}^e$.

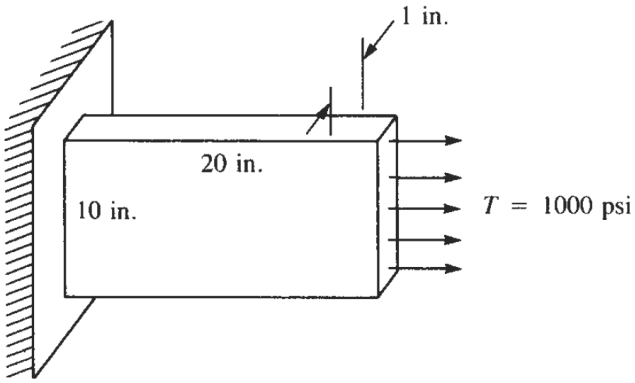
: Asse [illegible] ly of ele e t equ tio s to o t i equ tio s of t e w o [illegible] $\mathbf{Kd} = \mathbf{f}$.

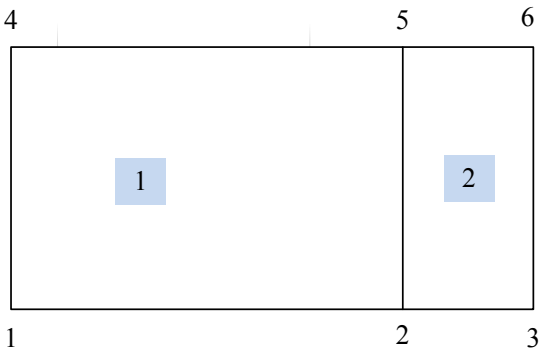
: I ositio of esse ti l ou d y co ditio s of t e o le .

: Soluti [illegible] d equ tio s d o t i t e lo l dis l ce e ts $_F$.

: ost ocessi of t e esults. Fo e c ele e t, ev lu te t e st i s d st esses d ot e s (ost- ocessi).

Let us co sid e t e s e t i l te su jected t [illegible] ow elow s we did fo T . We would li e to dete i e t e od l dis l [illegible] e t st esses usi two Q4 ele e ts i ste d of two T ele e ts. T e l te t ic ess $t = 1$ in., $E =$ × , d $\nu = 0.3$.





T e fi ite ele e t es co sisti of two ect ul ele e ts is s ow i t e Fi u e ove. I ste d of fi i t e ou d y, we ow llow ode 4 to ove i t e y-di ectio . T us, we ve

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si le lytic l solutio fo t e i l st i d dis l ce e t:

ε_{xx} σ_{xx} E × × −

u_x $L\varepsilon_{xx}$ × −

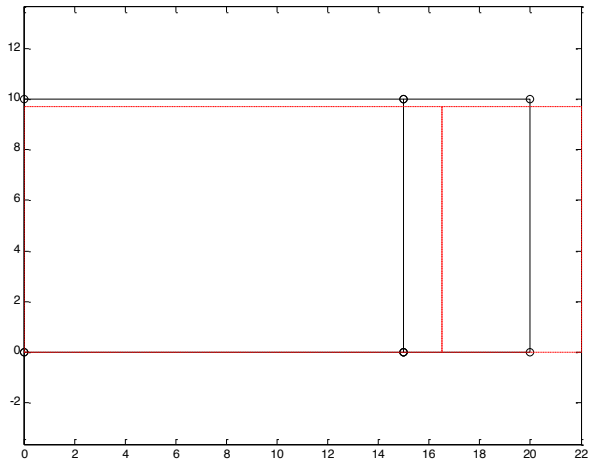
MATLA I le e t tio

```
% A Thin Plate Subjected to Uniform Traction
% Rectangular Element Implementation
% 2 elements
% clear memory
clear all;
clc;
close all;

% materials
E  = 30e6;    poisson = 0.30;  thickness = 1;

% matrix D for plane stress
D=E/(1-poisson^2)*[1 poisson 0;poisson 1 0;0 0 (1-poisson)/2];

% preprocessing
% numberElements: number of elements
numberElements = 2;
% numberNodes: number of nodes
numberNodes = 6;
% coordinates and connectivities
elementNodes=[1 2 5 4; 2  3 6 5];
nodeCoordinates=[0, 0; 15, 0; 20 0; 0 10; 15 10; 20 10];
          (    C          ,            ,' 4',' - ');

% GDof: global number of degrees of freedom
GDof = 2*numberNodes;

% boundary conditions
         D   =  1 2 7 ';
% force vector
force = zeros(GDof,1);
     (5) = 5000;
     (11) = 5000;

% calculation of the system stiffness matrix
stiffness =                 (GDof,numberElements,...
   elementNodes,nodeCoordinates,D,thickness);

% solution
displacements = solution(GDof,prescribedDof,stiffness,force);

% output displacements
outputDisplacements(displacements, numberNodes, GDof);
```

```
 =    (   (    C          )-
   (    C          ))/   (   (             ))*0.1;
% reform displacement vector to fit nodeCoordinates
     =    2   (             ,2)* ;
hold on
                C          +    ,            ,' 4',' --');
```

T e dis l ce e ts e (i o!):

```
Displacements
Node          UX          UY
   1   0.0000e+00   0.0000e+00
   2   5.0000e-04   -3.2903e-19
   3   6.6667 -04   -5.4760e-19
   4   0.0000e+00   -1.0000e-04
   5   5.0000e-04   -1.0000e-04
   6   6.6667 -04   -1.0000e-04
```



Most of t e oodies e idde i t e formStiffnessRec d its el ted fu ctio s. We will o t ou t e elow. Notice t t ose i ii ted i l c e fu ctio s d t ose i li ted i lue e oo i ee i eu istic of fi ite ele e ts o MATLA .

```
function stiffness=                (GDof,numberElements,...
    elementNodes,nodeCoordinates,D,thickness)

% compute stiffness matrix
% for plane stress rectangular elements

stiffness=zeros(GDof);

% 2 by 2 quadrature
[gaussWeights,gaussLocations]=     2 ('2x2');

for e=1:numberElements
  numEDOF = 8;
```

```
  elementDof=zeros(1,numEDOF);
  for i = 1:4
     elementDof(2*i-1)=2*elementNodes(e,i)-1;
     elementDof(2*i)=2*elementNodes(e,i);
  end
  %
  % THIS IS A HACK: we assume node 1 and node 2 align
  % with x-axis and node 2 and node3 align with y-axis
  %
  a = 0.5*abs(nodeCoordinates(elementNodes(e,2),1) -
nodeCoordinates(elementNodes(e,1),1));
  b = 0.5*abs(nodeCoordinates(elementNodes(e,3),2) -
nodeCoordinates(elementNodes(e,2),2));
  % cycle for Gauss point
  for q=1:size(gaussWeights,1)
    GaussPoint=gaussLocations(q,:);
    xi=GaussPoint(1);
    eta=GaussPoint(2);

% shape functions and derivatives
    [shapeFunction,naturalDerivatives]=shapeFunctionQ4(xi,eta);
    XYderivatives(1,:) = 1/a * naturalDerivatives(1,:);
    XYderivatives(2,:) = 1/b * naturalDerivatives(2,:);

%  B matrix
    B=zeros(3,numEDOF);
    B(1,1:2:numEDOF)  = XYderivatives(1,:);
    B(2,2:2:numEDOF)  = XYderivatives(2,:);
    B(3,1:2:numEDOF)  = XYderivatives(2,:);
    B(3,2:2:numEDOF)  = XYderivatives(1,:);

% stiffness matrix
    stiffness(elementDof,elementDof)=...
        stiffness(elementDof,elementDof)+...
        thickness*a*b*B'*D*B*gaussWeights(q);
  end
end
```

uss2d.

```
   function [weights,locations]=     2 (option)
    % Gauss quadrature in 2D
    % option '2x2'
    % option '1x1'
    % locations: Gauss point locations
    % weights: Gauss point weights

    switch option
       case '2x2'

       locations=...
         [ -0.577350269189626 -0.577350269189626;
            0.577350269189626 -0.577350269189626;
            0.577350269189626  0.577350269189626;
           -0.577350269189626  0.577350269189626];
```

```
        weights=[ 1;1;1;1];

        case '1x1'

        locations=[0 0];
        weights=[4];
    end

    end function gauss2d
```

s eFu ctio Q4.

```
% .....................................................................
    function [shape,naturalDerivatives]=     F        4(xi,eta)

    % shape function and derivatives for Q4 elements
    % shape : Shape functions
    % naturalDerivatives: derivatives w.r.t. xi and eta
    % xi, eta: natural coordinates (-1 ... +1)

    shape=1/4*[ (1-xi)*(1-eta), (1+xi)*(1-eta), ...
           (1+xi)*(1+eta), (1-xi)*(1+eta)];
    naturalDerivatives=...
          1/4*[-(1-eta),  1-eta, 1+eta, -(1+eta);
              -(1-xi), -(1+xi),  1+xi,  1-xi];

    end % end function shapeFunctionQ4
```

T is c te e sic co ce ts fo t e lysi ids wit t e FEM. T e ste s followed i t e lo ul tio of t e i e tic v i les, t e st i st ess fields, t e equili iu e essio s vi t e st o d we fo s. T e ticul fo of ele e t stiff ess t ices d ele e t e te l fo ce t ices fo t e - oded li e t i le (T), d t e 4- oded ect le ve ee ive .

T e de iv tio of i e o de t i ti te ele e ts of e equi es syste tic oced u e to o t i t e s e lu ctio s, d u e ic l i te tio . T ese to ics will e studied i t e e t c te .