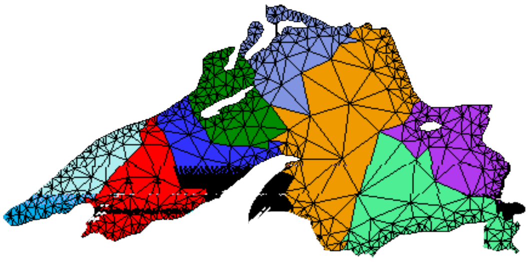
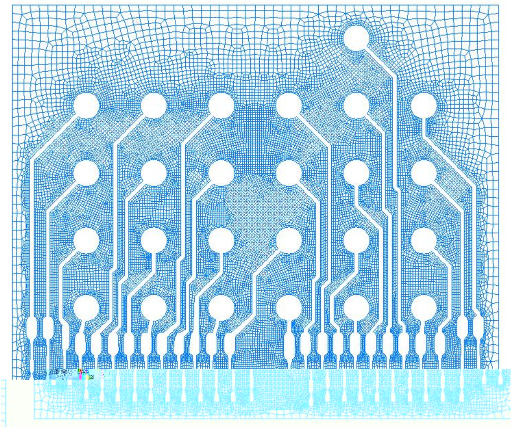
3.6 Trial Solutions and Weight Functions in 2D: An Overview

$$\boxed{\text{Continuity}} + \boxed{\text{Completeness}} \rightarrow \boxed{\text{Convergence}}$$

As we have mentioned in 1D, the trial solutions have to be constructed so that the polynomial expansion for each element is **complete** and the global approximation is C^0 continuous or, in other words, satisfying interelement compatibility in the displacement field for stress analysis. In multi-dimensions, the requirements remain the same, but the construction of trial solutions and weight functions presents several challenges as **the construction of continuous fields for arbitrary meshes** of **quadrilaterals** and **triangles** becomes more complicated.





3.6.1 Completeness and Continuity in 2D

<u>Completeness</u>

$$\begin{array}{ccccccccl}
 & & & 1 & & & & & \text{constant} \\
 & & x & & y & & & & \text{linear} \\
 & x^2 & & xy & & y^2 & & & \text{quadratic} \\
x^3 & & x^2y & & xy^2 & & y^3 & & \text{cubic}
\end{array}$$

Pascal's triangle shown above is used to determine the **completeness of polynomial expansion** for trial solutions in 2D. Each row of the triangle gives the monomials that must be included in a finite element approximation to provide an element with the **order** of completeness indicated to the right. If any of the terms in a row are missing, then the element will not be complete to that degree and will not have the convergence rate associated with that row of the expansion.

Finite Element Method

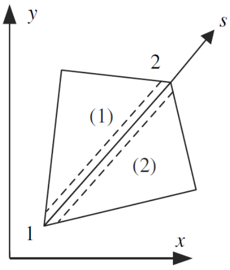
04/24/2019

Q: Given an element-wise trial solution $u_x^e(x,y)$ (you can do similar things for u_y^e), write down the complete polynomial expansion with unknown coefficients α up to a quadratic order.

A:

Continuity

We next consider the issue of C^0 continuity in 2D. Consider the two adjacent elements shown in the figure below, each with a complete linear polynomial expansion:



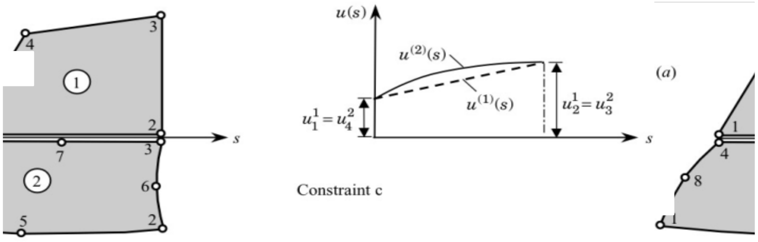
$$u_x^{(1)}(x,y) = \alpha_0^{(1)} + \alpha_1^{(1)} x + \alpha_2^{(1)} y$$

$$u_x \quad x \;\; y \;\; = \alpha \quad + \alpha \quad x + \alpha \quad y$$

where the superscripts indicate the element number as usual. Each polynomial is obviously C^0 continuous within the element. However, for the function to be globally C^0, it must also be C^0 continuous **at every point on the interfaces between the elements (not just at the nodes)**. In other words, for the specific example that we are considering, it is necessary that

$$(3.51) \qquad u_x^{(1)}(s) = u_x^{(2)}(s)$$

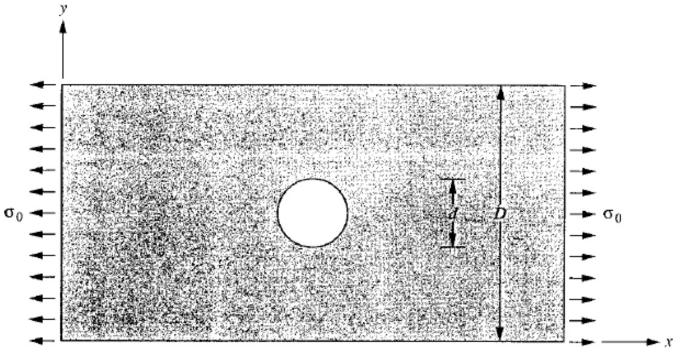
Same things hold for $u_y \quad s \quad u_y \quad s$. Figure below contains an example that does not satisfy the C^0 continuity along the interfaces between elements.

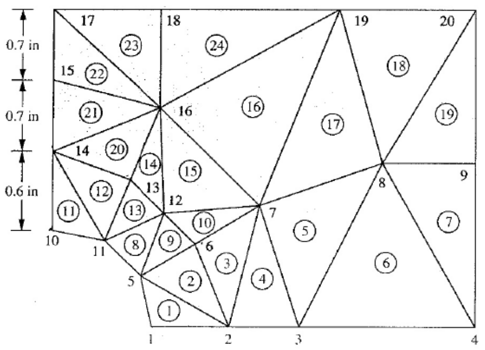


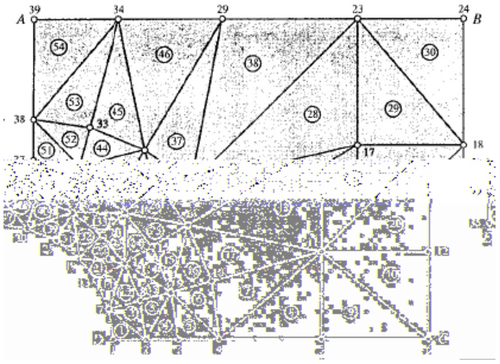
In the next section, we will describe how to construct continuous and complete shape functions for a simple **three-node triangular element** (T3, the simplest element in 2D).

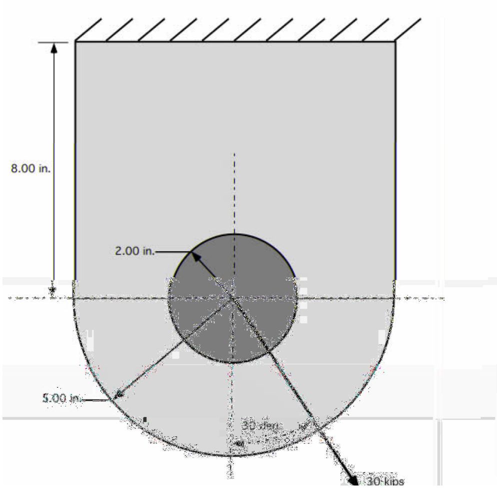
3.7 Three-Node Triangular Element (T3)

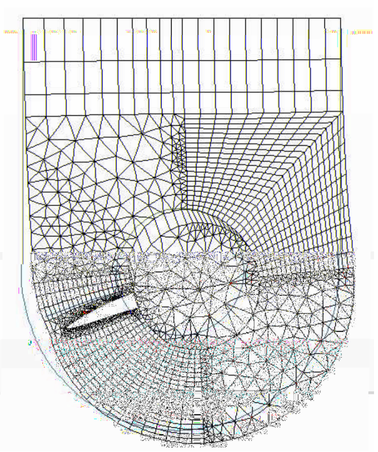
The three-node triangular element is one of the most versatile and simplest elements in two dimensions. One can easily represent **almost any geometry with triangular elements** and, without too much trouble, construct meshes that have more elements in areas of high stress gradients, so that greater accuracy can be obtained with the same number of elements.









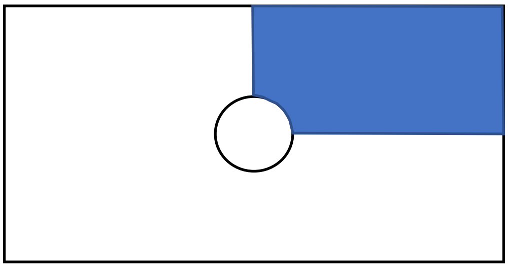


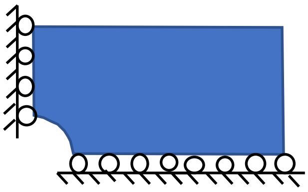
Finite Element Method

04/24/2019

Remarks:

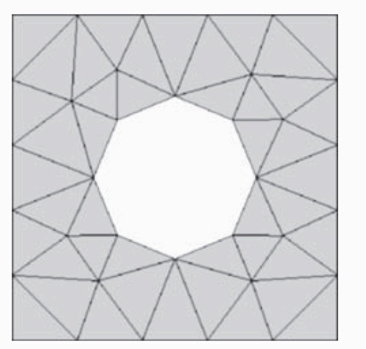
1. The "plate with a circular hole subjected to uniform tension" also demonstrates how to use symmetry to reduce degrees of freedom. We can take advantage of symmetry and model $\frac{1}{4}$ of the region shown below. All nodes along the horizontal plane of symmetry can move only in the horizontal direction and those on the vertical plane of symmetry can move only in the vertical direction.
2. In addition to reducing the model size, the use of symmetry provides enough boundary conditions so that there is no rigid-body motion in the model.





A **disadvantage** of the three-node triangle **(T3) is that it is a relatively inaccurate element** (in comparison with the Q4 quadrilateral element), and in fact the element is not recommended for production analysis with commercial finite element software.

Let us consider a finite element mesh consisting of three-node triangular elements shown in the following figure and make a few remarks:



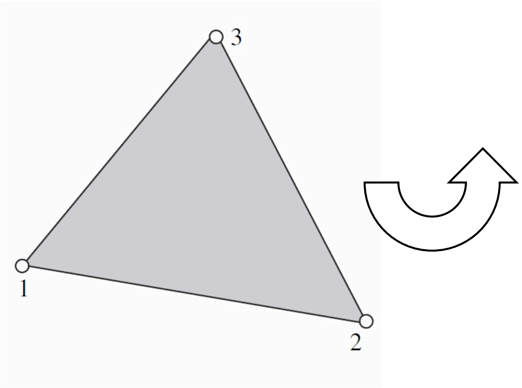
(1) Nodes are placed at the corners of all elements.

(2) An arbitrary number of elements can be joined to a node. There are no restrictions on the topology of a finite element mesh, though for reasonable accuracy, the angles of any element **should NOT be very acute.**

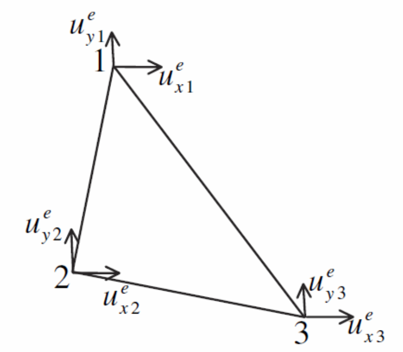
(3) As the sides of a triangular element are straight, curved edges of the body must be approximated. Thus, the curved sides of the hole are approximated by straight segments, which introduce an error in the geometry of the finite element model. The finite element

solution will be the solution to the geometry with the straight edges, so some error arises due to this approximation of the shape. However, in most cases, if a sufficient number of elements are used, this error is quite small.

3.7.1 Shape Functions: T3



A typical T3 element is shown above. You should pay attention on the node-numbering scheme: **the nodes need to be numbered counterclockwise.** The formulations given below can also be developed for clockwise numbering, but most finite element programs, use counterclockwise numbering, and it is important to adhere to this convention, as otherwise some crucial signs will be wrong.



Remark: For 2D stress analysis, the interpolation is applied to both displacement components, u_x and u_y. This is a vector field element. For heat conduction, the interpolation is applied to temperature. This is a scalar field element. For instruction purpose, we shall first use the trial solution for u_x to illustrate the interpolation and the corresponding shape functions. A formal matrix formulation will then apply to the vector field u_x and u_y.

The trial solution in each triangular element is approximated by a linear function of the spatial coordinates *x* and *y*:

(3.52) $$u_x^e(x,y)=\alpha_0^e+\alpha_1^e x+\alpha_2^e y$$

Or

(3.52b) $$(\ ,\ )=[1\qquad]\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix}$$

$$=\mathbf{p}(\ ,\ )\boldsymbol{\alpha}$$

Notice that **the number of unknown coefficients** that describe the field in a triangular element is equal to **the number of nodes**, so we should be able to uniquely express the **coefficients** α_i^e in terms of the nodal values.

We will now construct shape functions for the element following the same procedure that we used in one dimension. There are two ways to do it: the hard way and the simple way. The hard way is to do the inverse of the matrix. The simple way is to utilize the Kronecker delta property of the shape functions. We shall do BOTH below.

The Hard Way to Construct T3 Shape Functions

Let us first express the **unknown coefficients** α_0^e, α_1^e and α_2^e in terms of the **unknown nodal values**:

(3.53a) $$u_{x1}^e(x,y)=\alpha_0^e+\alpha_1^e x_1+\alpha_2^e y_1$$

(3.53b) $$u_{x2}^e(x,y)=\alpha_0^e+\alpha_1^e x_2+\alpha_2^e y_2$$

(3.53c) $$u_{x3}^e(x,y)=\alpha_0^e+\alpha_1^e x_3+\alpha_2^e y_3$$

Or

(3.54) $$\begin{bmatrix}u_{x1}^e\\ u_{x2}^e\\ u_{x3}^e\end{bmatrix}=\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{bmatrix}\alpha_0^e\\ \alpha_1^e\\ \alpha_2^e\end{bmatrix}\quad\Rightarrow\quad \mathbf{d}_x^e=\mathbf{M}^e\boldsymbol{\alpha}^e$$

(3.55) $$\boldsymbol{\alpha}^e = \left(\mathbf{M}^e\right)^{-1} \mathbf{d}_x^e$$

(3.56) $$u_x^e(x,y) = \mathbf{N}^e(x,y)\mathbf{d}_x^e \qquad \text{where} \qquad \mathbf{N}^e(x,y) = \mathbf{p}(x,y)\left(\mathbf{M}^e\right)^{-1}$$

where the row matrix $\mathbf{N}^e(x,y) = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]$ is the **element shape function** matrix.

To develop a closed form expression for the shape functions, it is necessary to invert the matrix $\mathbf{M}^e$. This can be done analytically or using MATLAB's Symbolic Toolbox, which gives:

(3.57) $$\left(\mathbf{M}^e\right)^{-1} = \frac{1}{2A^e}\begin{bmatrix} x_2^e y_3^e - x_3^e y_2^e & x_3^e y_1^e - x_1^e y_3^e & x_1^e y_2^e - x_2^e y_1^e \\ y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

where A^e is the area of the element e that is equal to half of the determinant of the matrix $\mathbf{M}^e$:

(3.58) $$2A^e = \det\left(\mathbf{M}^e\right) = (x_2^e y_3^e - x_3^e y_2^e) + (x_3^e y_1^e - x_1^e y_3^e) + (x_1^e y_2^e - x_2^e y_1^e)$$

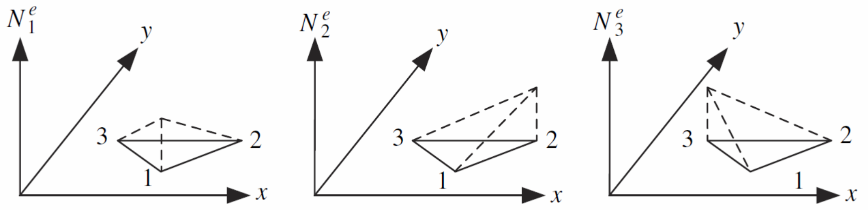
Thus the shape functions are:

(3.59a) $$N_1^e(x,y) = \frac{1}{2A^e}((x_2^e y_3^e - x_3^e y_2^e) + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y)$$

(3.59b) $$N_2^e(\ ,\ ) = \frac{1}{2A^e}((\ {}^{e}_{3}\ {}^{e}_{1} - \ {}^{e}_{1}\ {}^{e}_{3}) + (\ {}^{e}_{3} - \ {}^{e}_{1})\ + (\ {}^{e}_{1} - \ {}^{e}_{3})\ )$$

(3.59c) $$N_3^e(x,y) = \frac{1}{2A^e}((x_1^e y_2^e \quad x_2^e y_1^e) + (y_1^e \quad y_2^e)x + (x_2^e \quad x_1^e)y)$$

The shape functions are drawn for a typical triangular element in the figure below. It can be seen that each shape function vanishes at all nodes except one and that node is the number on the shape function.



In other words, these shape functions have the Kronecker delta property:

(3.60) $N_I^e(x_J^e, y_J^e) = \delta_{IJ}$

The Simple Way to Construct Shape Functions

We can use the Kronecker delta property to determine the shape functions. Let us first consider the shape function for node 1:

$$N_1^e(x,y) = \beta_0^e + \beta_1^e x + \beta_2^e y$$

We have **3 conditions**: $N^e \;\; x^e \;\; y^e \;\; = \;\; , \;\; N_1^e(x_2^e, y_2^e) = N_1^e(x_3^e, y_3^e) = 0$

$$\Rightarrow \begin{bmatrix} 1 & {}^e_1 & {}^e_1 \\ 1 & {}^e_2 & {}^e_2 \\ 1 & {}^e_3 & {}^e_3 \end{bmatrix} \begin{bmatrix} \beta_0^e \\ \beta_1^e \\ \beta_2^e \end{bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad {}^e_0 = \frac{1}{2A^e} a_1 \;\; , \;\; \beta_1^e = \frac{1}{2A^e} b_1 \;\; , \;\; \beta_2 = \frac{1}{2} \;\; {}_1$$

where $a_1 = x_2^e y_3^e - x_3^e y_2^e$

$b_1 = y_2^e - y_3^e$

$c_1 = x_3^e - x_2^e$

$$A^e = \text{ area of the triangle } = \frac{1}{2} \begin{vmatrix} 1 & {}^e_1 & {}^e_1 \\ 1 & {}^e_2 & {}^e_2 \\ 1 & {}^e_3 & {}^e_3 \end{vmatrix} = \frac{1}{2} \det\left(\mathbf{M}^e\right)$$

Similarly,

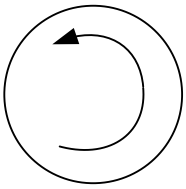
$$N_2^e(x,y) = \frac{1}{2A^e}(a_2 + b_2 x + c_2 y) \;\; , \;\; N_3^e(x,y) = \frac{1}{2A^e}(a_3 + b_3 x + c_3 y)$$

$$\text{where} \quad \begin{cases} a_2 = x_3^e y_1^e - x_1^e y_3^e \\ b_2 = y_3^e - y_1^e \\ c_2 = x_1^e - x_3^e \end{cases} , \qquad \begin{cases} a_3 = x_1^e y_2^e - x_2^e y_1^e \\ b_3 = y_1^e - y_2^e \\ c_3 = x_2^e - x_1^e \end{cases}$$

Notice the cyclic repetition and regularity

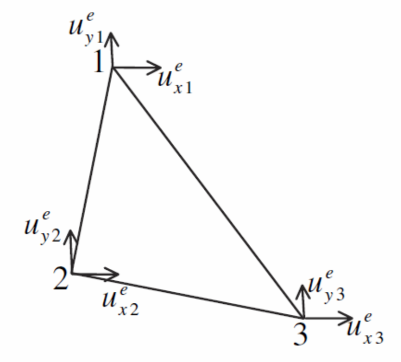
$$N_i^e(x,y) = \frac{1}{2A^e}(a_i + b_i x + c_i y)$$

$$\begin{cases} a_i = x_j^e y_k^e - x_k^e y_j^e \\ b_i = y_j^e - y_k^e \\ c_i = x_k^e - x_j^e \end{cases}$$



3.7.2 Element Matrices: T3

We now follow three-step procedures described previously to develop element matrices for a T3 element.



1. Construction of shape functions matrix $\mathbf{N}^e$

$$\mathbf{u}^e(\qquad = \mathbf{N}^e(\qquad \mathbf{d}^e \tag{3.39}$$

2. Formulation of the strain-displacement matrix $\mathbf{B}^e$

$$\boldsymbol{\varepsilon}^e \quad \nabla_S \mathbf{u}^e \quad \nabla_S \mathbf{N}^e \mathbf{d}^e \quad \mathbf{B}^e \mathbf{d}^e \tag{3.43}$$

(3.61) $$\mathbf{B}^e = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix}$$

3. Calculation of $\mathbf{K}^e$ and $\mathbf{f}^e$ using $\mathbf{N}^e$ and $\mathbf{B}^e$

(i) Calculation of $\mathbf{K}^e$ using $\mathbf{N}^e$ and $\mathbf{B}^e$

The element stiffness matrix is given by (3.47) $\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega$. If the material properties are assumed to be constant in the element and for a T3 element with a thickness *t*, we have:

$$\mathbf{K}^e = tA^e\mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e$$

Q: What is the dimension of $\mathbf{K}^e$ matrix for T3?

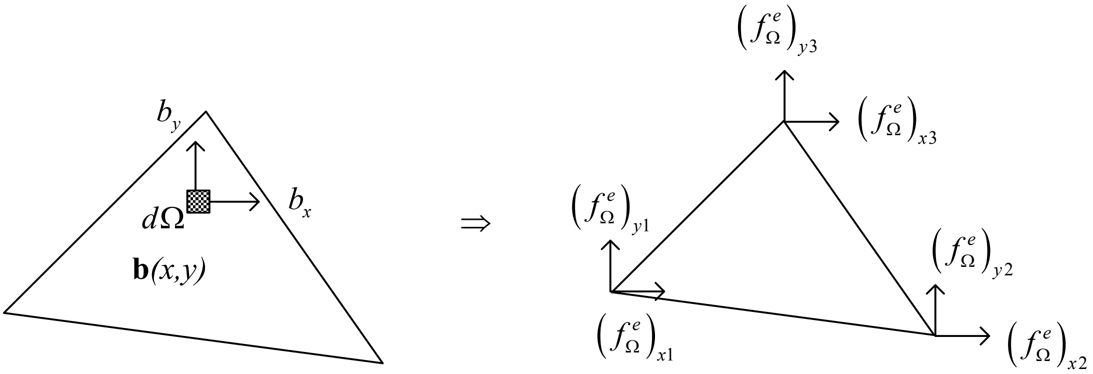
A:

(ii) Calculation of $\mathbf{f}^e$ using $\mathbf{N}^e$ and $\mathbf{B}^e$

The element external force matrix of a T3 element with a thickness *t* consists of two terms: one from the body forces and the other from the tractions:

$$\mathbf{f}^e = \mathbf{f}_\Omega^e + \mathbf{f}_\Gamma^e = t\int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega + t\int_{\Gamma_t^e} \mathbf{N}^{eT}\bar{\mathbf{t}}\, d\Gamma .$$

3.7.2.1 $\mathbf{f}^e$ due to body forces ($\mathbf{f}^e$)



$$\mathbf{f}_{\Omega}^{e} = t \int_{\Omega^e} \mathbf{N}^{eT} \mathbf{b} \; d\Omega$$

We can carry out this by numerical integration. For constant body forces and linear body forces, we can derive the integration analytically.

The simple case: constant body forces $\overline{\mathbf{b}}$

For a constant body force $\overline{\mathbf{b}} = \begin{bmatrix} \overline{b}_x \\ \overline{b}_y \end{bmatrix}$ in which $\overline{b}_x$ and $\overline{b}_y$ are constant values, we have:

(3.62a) $\mathbf{f}_{\Omega}^{e} = t \left[\int_{\Omega^e} \mathbf{N}^{e} \; d\Omega \right]^T \overline{\mathbf{b}}$

Q: Let us write the matrix explicitly!

A:

(3.62b)

Finite Element Method

04/24/2019

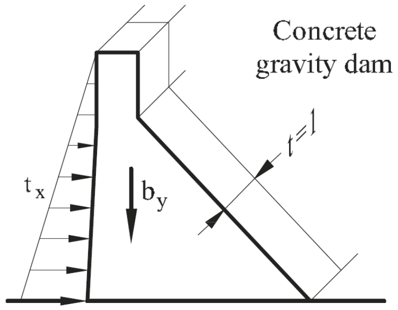
Q: What is the geometric interpretation of $\int_{\Omega} \quad {}_{1} \quad \Omega$?

A:

Thus the element body force matrix for constant body forces $\bar{\mathbf{b}}$ are:

$$\begin{array}{c} \left(f^e\right)_{x1} \\ \left(f^e\right)_{y1} \\ \left(f^e\right)_{x2} \\ \left(f^e\right)_{y2} \\ \left(f^e\right)_{x3} \\ \left(f^e\right)_{y3} \end{array} = \frac{tA^e}{3} \begin{array}{c} \bar{b}_x \\ \bar{b}_y \\ \bar{b}_x \\ \bar{b}_y \\ \bar{b}_x \\ \bar{b}_y \end{array} \tag{3.63}$$

Example: Consider a single T3 element to represent a concrete gravity dam. What is the body force for this case?



A:

The more involved case: linear body forces

If the body forces vary linearly within the element, we can use the shape function to interpolate the field:

(3.64) $\mathbf{b}(\ \ ,\ \ ) = \mathbf{N}\ (\ \ ,\ \ )\mathbf{b}$

in which $\mathbf{b}^e$ is the nodal body forces:

$$\mathbf{b}^e = \begin{bmatrix} b^e_{x1} \\ b^e_{y1} \\ b^e_{x2} \\ b^e_{y2} \\ b^e_{x3} \\ b^e_{y3} \end{bmatrix}$$

(3.65) $$\mathbf{f}^e_{\Omega} = t\left(\int_{\Omega^e} \mathbf{N}^{eT}\mathbf{N}^e \, d\Omega \right)\mathbf{b}^e$$

Knowing the fact that we have the following analytical integration for T3:

(3.66) $$\int_{\Omega^e} \left(N_1^e\right)^a \left(N_2^e\right)^b \left(N_3^e\right)^c \, d\Omega = \frac{a!\,b!\,c!}{(a+b+c+2)!} 2A^e$$

The element body force matrix thus yields:

(3.67) $$\mathbf{f}^e = \frac{tA^e}{12} \begin{array}{c} 2b_{x1} + b_{x2} + b_{x3} \\ 2b_{y1} + b_{y2} + b_{y3} \\ b_{x1} + 2b_{x2} + b_{x3} \\ b_{y1} + 2b_{y2} + b_{y3} \\ b_{x1} + b_{x2} + 2b_{x3} \\ b_{y1} + b_{y2} + 2b_{y3} \end{array}$$

Remark: for the special case $b_{x1} = b_{x2} = b_{x3} = \bar{b}_x$ and $b_{y1} = b_{y2} = b_{y3} = \bar{b}_y$, (3.67) reduces to (3.63).