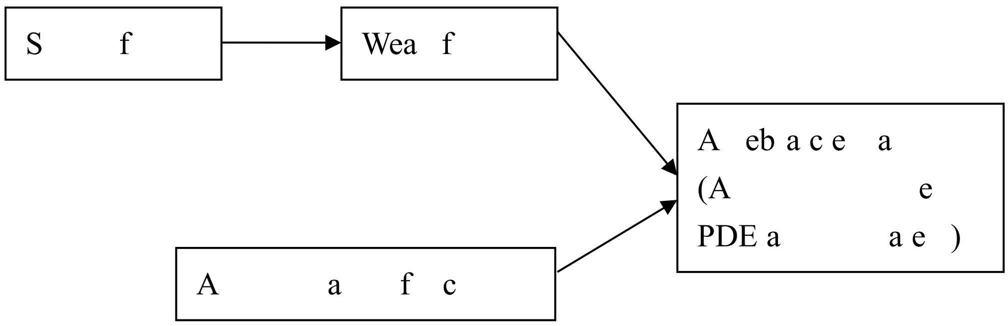
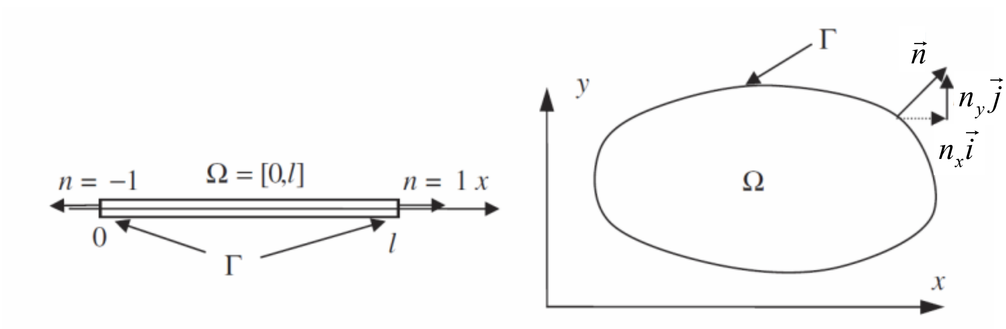
04/17/2019

# Chapter 3: Linear Elasticity and 2D Triangular and Rectangular Finite Element Formulation

## 3.1 Introduction




| D e | D a Ω | B da Γ |
|---|---|---|
| 1D | L e | T e d |
| 2D | A ea | C ed e |
| 3D | V e | C ed face |

I C a e, e e ace e a e a e d d f C a e 2 a d e e d 1D f - d e a b e . We a a f e ad a ab e, a e de e e f e f a d ea f f **linear elasticity** b e .

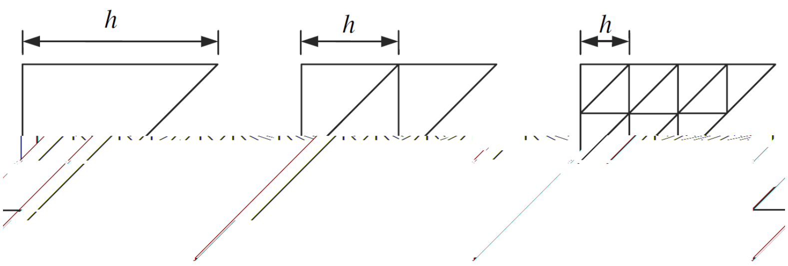
A ca be ee f e ad a , e f e de e a f e e e e e d de e e e e a a d b da c d , c a e e f . We ee a d e , a bef e, e a e e e a a d a a b da c d . U a f a a <u>e a b a</u>, e e de e a ea f . F a , e e f e e e e a a f a b a a e e f b e ea f .

O e a ec a e e e e e d e **similarity** e e-d e a f a .

**Remark 1:** T e e e f e a d ea f d e a e de ca e f ee d e . I e ee ac ce da , a a e a e d e ee d e , e ac a e f ee d e a b e . Ne e ee , e e e f f e e e e f a f ee d e a a .

**Remark 2:** I 2D a d 3D b e a **the power of the FEM becomes clearly apparent.** T e a -d e d ffe e f a e d e a b e , a e e ac f e a a d ffe e a e a -d e fea b e f b e e d a e b da c d . T , **numerical solution of the partial differential equations is generally the ONLY possibility for practical problems.** O e e a d, e a ac f f e e e e e d **remains the same**: a a e e e f c a d a b **finite element shape functions** a **as the number of elements is increased, the quality of the solution is improved.**

I e a $h \to 0$ ($h$ be e e e e e) a e a de c ea ed, e f e e e e d **converge to the exact solution** f e a a a e ff c e a d c e e.




Ma e a c P e a

O e c ca e e d e e d d e e **notation.** I d e , a ab e c a d ace e a e **vectors.** We de e ec b e ed a . F e a e, a d ace e ec $\vec{u}$ ca be e e ed a :

(3.1) $$\vec{u} = u_x\vec{i} + u_y\vec{j}$$

c $\vec{i}$ a d $\vec{j}$ a e e ec e $x$ a d $y$ d ec a d $u_x$ a d $u_y$ a e e $x$ a d $y$ c e f e d ace e ec , e ec e .

W e e e e de a ff e e e e e a , bec e c e e e **matrix notation**. A c a ca be ed de c be a ec $\vec{u}$ b e c e f e ec e de a be :

(3.2) $\mathbf{u} = \begin{bmatrix} \quad \end{bmatrix}$

A a e a ec e d e ca a d c ( e d c ). T e ca a d c f ec Ca e a c d a e e f e d c f e c e f e ec ; e ca a d c f a ec $\vec{u}$ a ec $\vec{r}$ e b :

(3.3) $\vec{\ } \cdot \vec{\ } = \quad + \quad$

**Q:** H e e e ca a d c (3.3) e f a f e e $\mathbf{u} = \begin{bmatrix} \quad \end{bmatrix}$ a d $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$?

**A:**

(3.4)

A e a e a ec e d e **gradient**. T e ad e de **a measure of the slope of a field, so it is the 2D counterpart of a derivative in 1D**. T e ad e ec e a def ed b

(3.5) $\vec{\nabla} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})$

T e ca a d c f e **gradient operator** **a vector field** e e **divergence** f e ec f e d. T e e d e e ce a ed f d ec a c , e e efe e f ea a .

**Q:** W a e d e e ce f a ec $\vec{u}$?

**A:** (3.6)

T e **matrix form** f ad e e a :

$$(3.7) \qquad \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

N e a ∇ b d a d a ed f a f .

**Q:** T e a f f e d e e ce e b e ac e d e ca a d c b a

a e e a . S e e d e e ce f $\mathbf{u} = \begin{bmatrix} \ \end{bmatrix}$ a f ?

**A:**

(3.8)

## 3.2 Linear Elasticity

L ea e a c ed f d a e a a e , a d c de e a c d a e e ec ed de a e a e e c ea e . We e e e ba c a a d e e a ( f ) f ea e a c be .

T e e f ea e a c e e e f f a :

1. def a a e e a ;
2. e be a f e a e a ea ;
3. d a c effec a e e ec ed;
4. a e a cc d e def a f e d.

T e e a eed be f ed. F e a e, e ec def a f a b d e de e ce ad a e a , b e ec e def a f d a a a e a e a .

T e e e e f a ea e a a a e c e e a ed e a . T e e e e a e:

1. e b d be e b ; e b e e a e f e f ce a a f e d a .
2. a f e e a a a d e c de c a e a .
3. e def a be e a a d .

I add , de e a e a a , e eed a ea e f e a a e e e e a e f e def a , c ca ed e a d ace e e a ( e a c ).

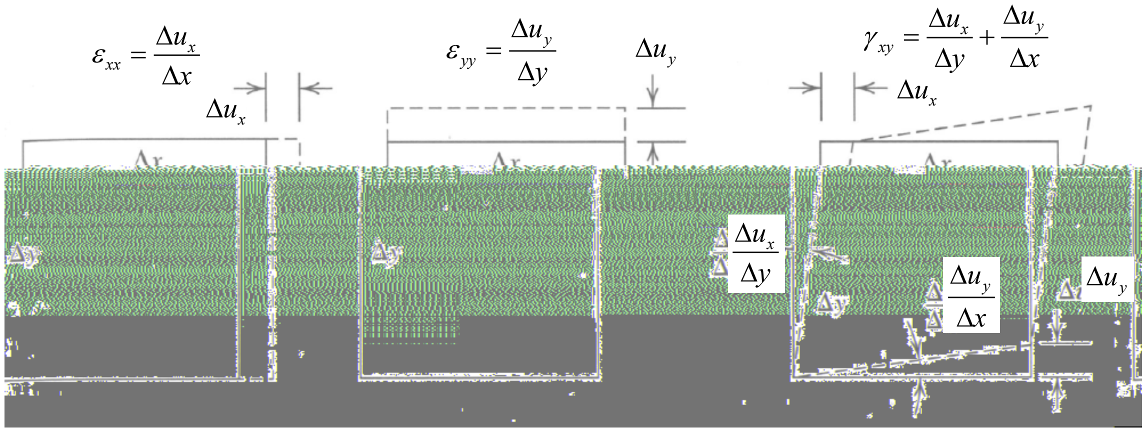
Ba ed e e e e e , e a d c a -d ace e e a ( e a c ), ac a d e (Ca c f a: f a e a be ee e , a a d ac ), e b e a a d c ee a f ea e a c .

<u>K e a c (S a -D ace e E a )</u>

T e **displacement vector** d e a ec c e . We e a Ca e a c d a e e , e c e f e d ace e a e e $x$-c e a d e $y$-c e . I ca be e a a d ec f a :

(3.1 a d 3.2) $\mathbf{u} = \begin{bmatrix} \\ \end{bmatrix}$ ( a f ) $\vec{u} = u_x \vec{i} + u_y \vec{j}$ ( ec f )

U de e a f **small displacement gradients**, e ca e ee de e de a ab e de c be e def a f a c e 2D. T e e a ab e c e d e **strains**. F e be de c e def a f a c e $\Delta x \times \Delta y$ e $x$ a d $y$ d ec .



T e a a (e e a a ) $\varepsilon$ a d $\varepsilon_{yy}$ a e e c a e e d ded b

e a e ; e e ee ea a $\gamma_{xy}$ e a f c a e a a e.

Le $\Delta x$ a d $\Delta y$ a ac e , e a e:

$$(3.9) \qquad \varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

**Q:** Reca 1D, e a e a a e e $\varepsilon(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{du}{dx}$ (E . (1.2)).

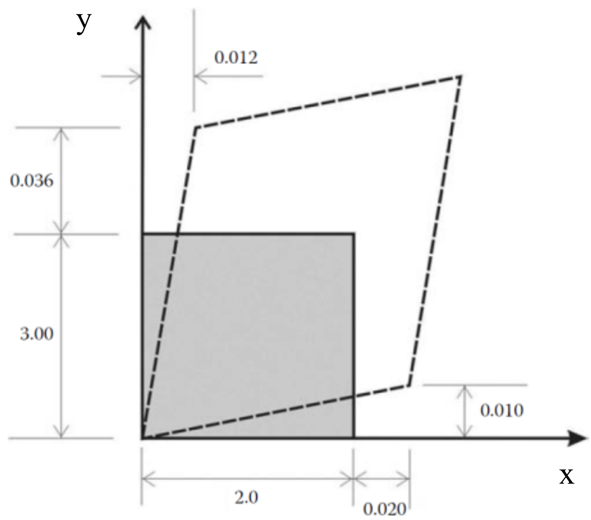
W d e eed a a de a e 2D?

**A:**

<u>E a e:</u>

A -d e a d def ed a F e be . U de e e c f a def a e , de e e e a a d ea a . T e d e e e e e def ed a e.

<u>**Answer**</u>



**Remark:**

T f f e ea a a ea c f e e e e f a e: e e ee ea a $\gamma_{xy}$ e ab e a d e e ea a $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$.

I f e e e e e d , e a a e a a a ed a c a :

$$(3.10) \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

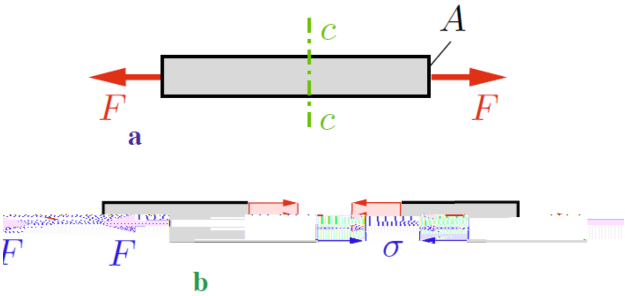
A d a -d ace e e a ca be e e ed b :

$$(3.11) \qquad \boldsymbol{\varepsilon} = \nabla_S \mathbf{u} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

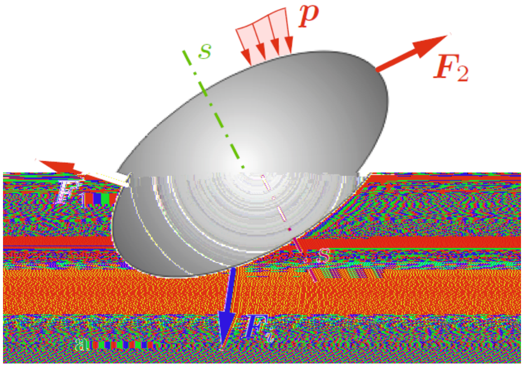
e e $\nabla_S$ **a symmetric gradient matrix operator**.

<u>I e a F ce, S e Vec a d S e</u>

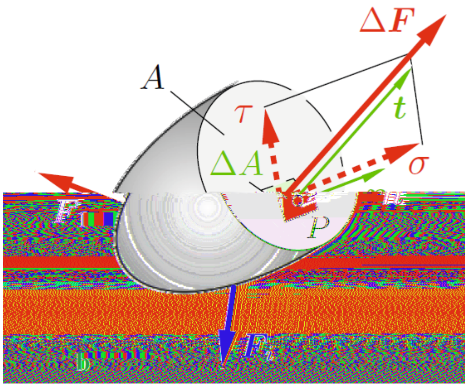
T e e e a ad ca e e a f ce . I 1D, e e a f ce ca be a ed b a a a c f e ba . <u>S e e a e c a e e c ec</u>




Le e e d e dea 2D a d 3D. F e e c de a b d c aded a b a , e. . b f ce $\vec{F}_i$ a d d b ed f ce $\vec{p}$ a e F (a) be .



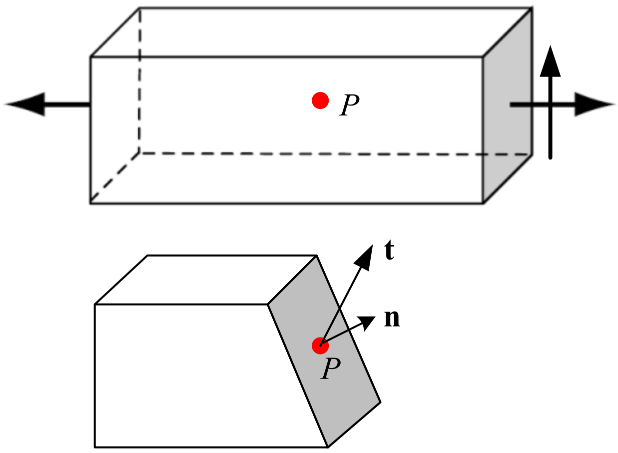

T e e e a ad e e a e e a f ce . I a a a ec *s-s* e b d , e e a a ea f ce ( e e ) a e d b ed e e e e a ea A. I c a e ba e e e e e e a e c a e e c ec , <u>e e e e a a e c ec</u> .




S ce e e e e a e e e e e e c ec , **<u>it must be defined at an arbitrary point</u> *P* of the cross section** ( ee F (b) ab e). T e a ea e e e $\Delta A$ c a *P* b ec ed e e a e a f ce $\Delta \vec{F}$. T e a e a e e e a ea e e e def ed a e a $\frac{\Delta \vec{F}}{\Delta A}$ (f ce e a ea). We a e a e a $\frac{\Delta \vec{F}}{\Delta A}$ e $\Delta A \to 0$ e d <u>a f e a e</u>:

$$\vec{t} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A} = \frac{d\vec{F}}{dA}$$

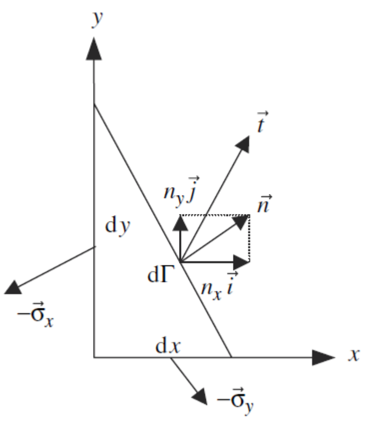
T a e $\vec{t}$ ca ed **stress vector**. S e ec e e e <u>e f ce e a ea</u> ac a ec f c face. A a f ed *P* f eac e ec $\vec{n}$, e e a e ec $\vec{t}$ ac e a e a $\vec{n}$. N e a $\vec{t}$ , e e a , e d ec f $\vec{n}$.




T e e a e a a $P$ f e ec e ff c e de e ed b $\vec{t}$. If e c e d ffe e ec e $P$, d ffe e e ec ( $\vec{t}$ ) ac e . A e a a e e a e ec a f c f e a ec a face. T e ef e, e e e a de e d e e a f e ec c c a ac e ed b e a ec $\vec{n}$.

I de e def e e e a e f a a e a $P$, e eed de c be e e e C d a e e ead f a b a e ed a e . We ced e be :

We f c de a f e a a e f a a e a $P$ a d def e e **stress vectors** $\vec{\sigma}_x$ a d $\vec{\sigma}_y$ ac e a e a e $x$ a d $y$ d ec .



T e f ce e b f a a b d ab e e e :

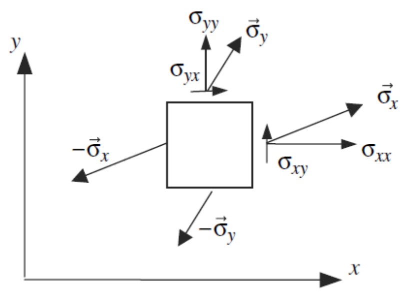
$\vec{t}d\Gamma - \vec{\sigma}_x dy - \vec{\sigma}_y dx = \vec{0}$

S ce

$dy = n_x d\Gamma$ and $dx = n_y d\Gamma$

We e a e:

(3.12) $\vec{t} - \vec{\sigma}_x n_x - \vec{\sigma}_y n_y = \vec{0}$




N c de a a e f a a e a $P$, e e 2D c e d e **stress vectors** $\vec{\sigma}_x$ a d $\vec{\sigma}_y$:

(3.13) $\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$

(3.14) $\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}$

**Stresses are thus mathematical description of internal forces.** T e e a e a - d e a b d de c bed b a e e $\sigma$ a d $\sigma_{yy}$ a d ea e e $\sigma_{xy}$ a d $!_{yx}$. F e e b a a e, ca be a $\sigma_{xy} = \sigma_{yx}$. T e f ed a d a e f a ed ( ca f d e f a a C Mec a c e b ).

T e e a e a a $P$ a 2D b d **uniquely** de e ed b $\sigma$ , $\sigma_{yy}$, $\sigma_{xy}$.

I f e e e e e d , e e e a e a a a ed a c a a a :

(3.15) $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$

Occa a , c e e a a e e c e a $2 \times 2$ e c a $\boldsymbol{\tau}$ a

(3.16) $\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$

Re (3.12) a d $\vec{i}$ a d $\vec{j}$ (3.13) a d (3.14) e d :

(3.17)
$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y = \vec{\sigma}_x \cdot \vec{n}$$
$$t_y = \sigma_{xy} n_x + \sigma_{yy} n_y = \vec{\sigma}_y \cdot \vec{n}$$

O a f

(3.18) $\mathbf{t} = \boldsymbol{\tau}\mathbf{n}$ $\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$

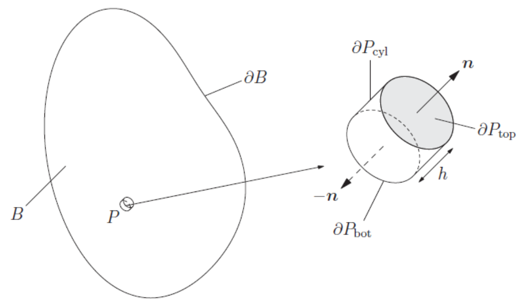
E a (3.18) e fa Ca c f a: e f a e a be ee e , a a d e ec de ed b A Ca c . T e Ca c f a a d:

> K e c e $\sigma$ , $\sigma_{yy}$, $\sigma_{xy}$, e ca e d a ce e e ec $\vec{t}$ ac a face e a ec $\vec{n}$.

**Remark:** Ca c 1822 ea ed a e e e e d ffe e ce be ee e e a f ce ac e ca face f a b d a d e a f ce ac ac a face e b d . T ea a a a ea be a e e, a a , b e a . H e e , c ea ed e c f e f ea 100 ea f fa ed a d a fa ed a e de a d e a f ce a eceded Ca c a d a ed e a f e c e f d a d f d .

<u>E a e:</u> C de a b - a ed b d de f a a e b d B. S e b a e a e a $P$, e e ec e de f a face a e e a a d e. T a ,

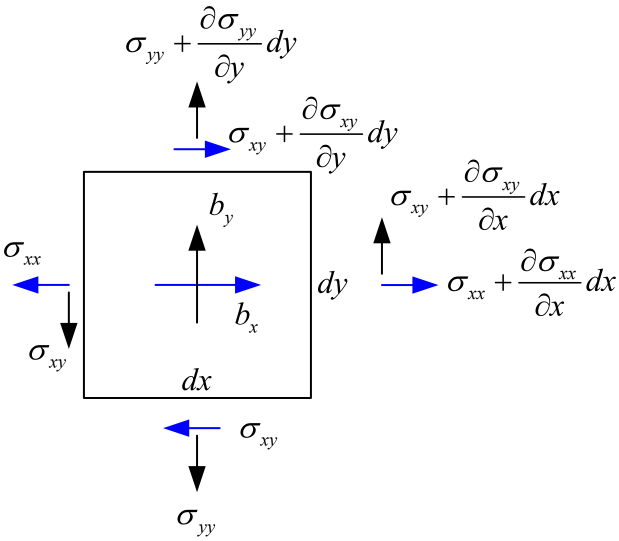
$\vec{}( \; , \vec{}) = -\vec{}( \; , -\vec{})$




**(Answer)**

<u>E b E a</u>

F e be e e a ac a d ffe e a e e e f a c e 2D.




**Q:** W a e a c e b f f ce e *x* d ec ?

**A:**

We ca d e a e f e *y*-d ec . T e d ffe e a e a f e b f a 2D b e a e

(3.19)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x \quad 0$$

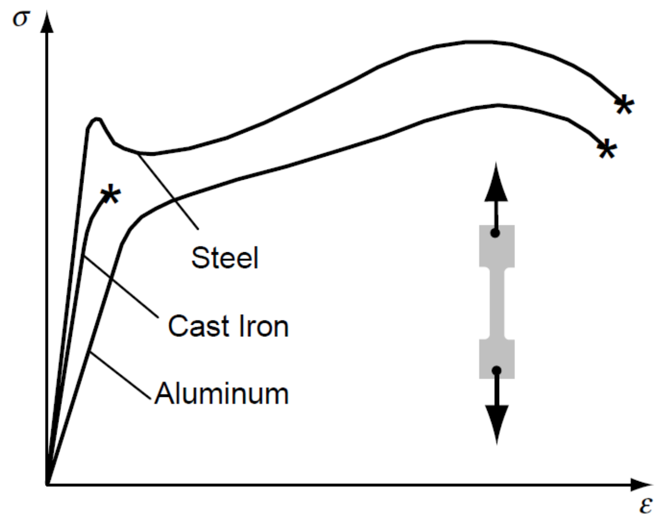
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y \quad 0$$

**Q:** We ca a e e ee b e a **in matrix form**. If e c de e a e f e e c ad e e a e (3.11) a d c a f f e e , a e a f f e b ?

**A:**

(3.20)

C e E a (S e -S a E a )



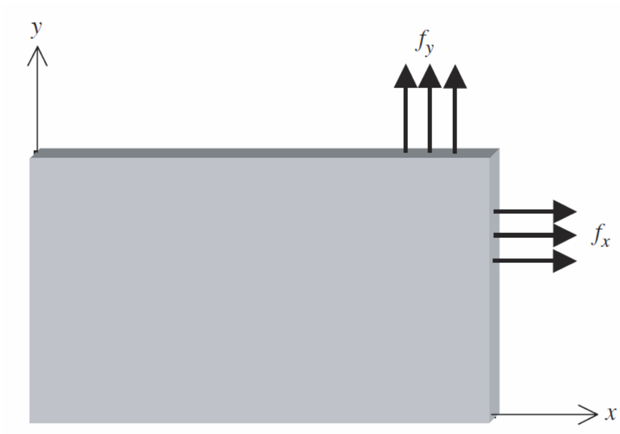

I d e , e e e a ea e a be ee e e a d a a ce ca be e a

(3.21) $= \mathbf{D}$

e e $\mathbf{D}$ a $3\times3$ a a d $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$, $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$. T e e ca ed e e e a ed

H e a . I a a a e c, e-def e a . I -d e a b e , e a **D** de e d e e e a e a a e e a a e a c d . T e e a de e e e de f ed f a ee-d e a ca b d a -d e a de .

## (a) Plane stress problem




T e a e e $(t = l_z << l_x \approx l_y)$ a d b ec ed ad e $x$-$y$ a e ( a e ad ).

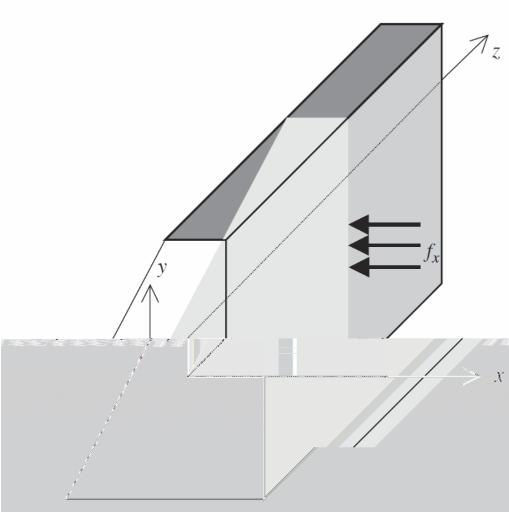
$$\Rightarrow \sigma_z = 0,\ \sigma_{zx} = 0,\ \sigma_{zy} = 0 \quad (\qquad e \quad a\, e)$$

$$\sigma_z = 0 \ \Rightarrow \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \neq 0 \qquad e\ e\ a$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

**(b) Plane strain problem**




L ad a f c f *x* a d *y*.

$$l_z >> l_x \approx l_y$$

$$w = u_z = 0 \Rightarrow \varepsilon_z = \varepsilon_{zx} = \varepsilon_{zy} = 0$$

$_z$ ( $_x$ $_y$) 0 e e a

**Remark:** T e a **D** a e e a d a e a be ca be b a ed f e e e a ed H e a f 3D c a e a .

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

e e $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$ $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$ a d ce a e ea d a e a e ea e a d

ea a $G = \frac{E}{2(1+\nu)}$. C T e , S. a d G d e , J. *Theory of Elasticity*, 3 d Ed , 1970 DL L a , *A First Course in Finite Element Method, 4th Edition*, 2011 (A e d C, a a ab e f e c e eb e) f f e e ab a .

## 3.3 Linear Elasticity: Strong Form

Reca (3.19), e d ffe e a e a f e b f a 2D b e a e:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x \quad 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y \quad 0$$

T e G e d ffe e a e a ca be e e ed ec f b eca e e ec $\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$ a d $\vec{\sigma}_y = \sigma_{xy}\vec{i} + \sigma_{yy}\vec{j}$ (f (3.13) a d (3.14)) a d e def f d e e ce (3.6):

$$\text{(3.22)} \qquad \begin{aligned} \vec{\nabla} \cdot \vec{\sigma}_x + b_x = 0 \\ \vec{\nabla} \cdot \vec{\sigma}_y + b_y = 0 \end{aligned}$$

A e d e , e c de e f b da c d : T e f e b da e e e ac e c bed de ed b $\Gamma_t$, a d e f e b da e e e d ace e e c bed de ed b $\Gamma_u$. T e ac b da c d e a ( ee (3.18)):

$$\boldsymbol{\tau}\mathbf{n} = \bar{\mathbf{t}}$$

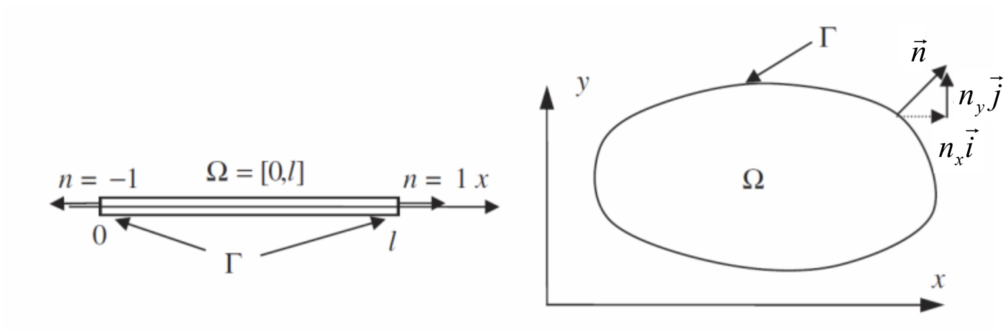
ec f ( ee (3.17)):

$$\vec{\sigma}_x \cdot \vec{n} = \bar{t}_x \qquad (3.23)$$
$$\vec{\sigma}_y \cdot \vec{n} = \bar{t}_y$$

T , e f f ea e a c b e a ed be :

$$(3.24) \qquad \vec{\nabla}\cdot\vec{\sigma}_x + b_x = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{\sigma}_y + b_y = 0 \quad \text{on} \quad \Omega$$
$$\vec{\sigma}_x \cdot \vec{n} = \bar{t}_x \quad \text{and} \quad \vec{\sigma}_y \cdot \vec{n} = \bar{t}_y \quad \text{on} \quad \Gamma_t$$
$$\vec{u} = \bar{\vec{u}} \quad \text{on} \quad \Gamma_u$$

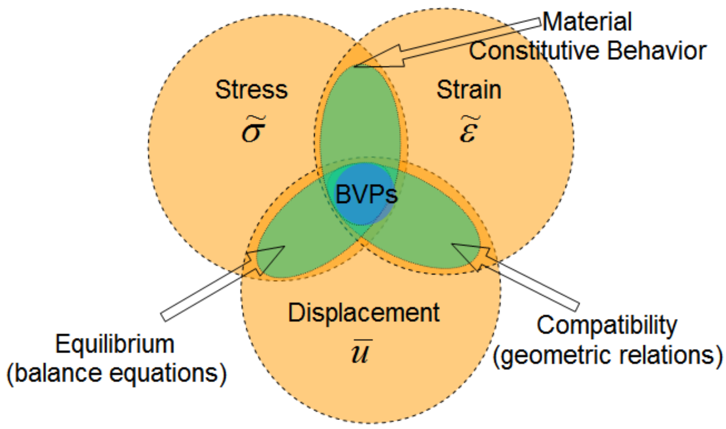
T e a be ee 1D a d 2D e d a Ω a d b da Γ a e a a a ed a d ab a ed be .




| D e | D a Ω | B da Γ |
|---|---|---|
| 1D | L e | T e d |
| 2D | A ea | C ed e |
| 3D | V e | C ed face |

S a

T e e ed e f ea ea c a e e e , a , a d d ace e . K e a c def e e c a b ( e e c e a be ee d ace e a d a ). C e de def e e a e a c e be a a d e a e e e a . F a , e b e ab e e G e d ffe e a e a a d e f e e d ace e a b da a e b e (BVP).




## 3.4 Linear Elasticity: Weak Form

We f e **basic three steps** a f e 1D b e b a e ea f . Bef e e ca ceed, e a d ce e G ee e e : e 2D a d 3D c e a f e e a b a 1D.

Ma e a c P e

G ee e e : e 2D a d 3D c e a f e e a b a 1D. C de **ANY** ca a f c $\phi(x,y)$ (f e a e, $u_x$ a d $u_y$ a e ca a f c a d $\vec{u}$ a ec f c ). T e e e ce f G ee e e e a e **an integral of a gradient of a scalar function over an area (volume) to a contour (surface) integral** b :

$$(3.25) \qquad \int_{\Omega} \vec{\nabla}\phi \, d\Omega = \int_{\Gamma} \phi \vec{n} \, d\Gamma$$

T e f ed a d a e f a ed ( ca f d e f a a E ee Ma e a c e b ).

04/17/2019

**Q:** We ca f e ee a d e G ee e e 2D e a , e f e $x$-d ec a d e e f e $y$-d ec . W a a e e ? ( : $\vec{\nabla}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})$)

**A:**

(3.26a)

(3.26b)

We ca e (3.26a) a d (3.26b) de e e d e e ce e e c e a e **an area (volume) integral of the divergence of a vector field to the contour (surface) integral of a vector field**:

$$(3.27)\qquad \int_{\Omega}\vec{\nabla}\cdot\vec{u}\,d\Omega=\int_{\Gamma}\vec{u}\cdot\vec{n}\,d\Gamma$$

B e ac e ca a f c $\phi(x,y)$ (3.26a) $u_x$ a d (3.26b) $u_y$:

$$\int_{\Omega}(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y})\,d\Omega=\int_{\Gamma}(u_x n_x+u_y n_y)\,d\Gamma$$

c e d e e ce e e e (3.27).

**Remark**: D e e ce e e a a e f e a e a c f e ee , a c a e ec a c a d f d d a c . C f f e e ab a : ://e . ed a. / /D e e ce_ e e .

F a e a e ead e 2D a d 3D c e a f e a b a :

$$(3.28)\qquad \int_{\Omega}\vec{\nabla}\cdot{}^{\rightarrow}\,d\Omega=\int_{\Gamma}{}^{\rightarrow}\cdot|^{\rightarrow}\,d\Gamma-\int_{\Omega}\vec{\nabla}\cdot{}^{\rightarrow}\,d\Omega$$

We a f eca e de f a ec a e f a e **derivative of a product**

1D $\frac{d}{dx}(wf)=w\frac{df}{dx}+f\frac{dw}{dx}$. **The power of the gradient operator** $^{\rightarrow}$ **is shown by the**

**following product rule in 2D and 3D**:

(3.29) $\vec{\nabla}\cdot(w\vec{u}) = \vec{\nabla}w\cdot\vec{u} + w\vec{\nabla}\cdot\vec{u}$

**(Derivation Exercise)** Ve f e de f (3.29) 2D ( : eed e a d )
**Answer:**

I e a (3.29) e e d a Ω e d :

(3.30) $\int\limits_{\Omega}\vec{\nabla}\cdot(w\vec{u})\,d\Omega = \int\limits_{\Omega}\vec{\nabla}w\cdot\vec{u}\,d\Omega + \int\limits_{\Omega}w\vec{\nabla}\cdot\vec{u}\,d\Omega$

**Q:** W a a e f e a e d e e ce e e e e e LHS (3.30)?
**A:**
(3.31)

S b (3.31) (3.30), e e a e:

$w\,\vec{u}\quad \vec{n}\,d\ =\ w\ \vec{}\quad \vec{u}\,d\ +\ \vec{}\ w\quad \vec{u}\,d$

c e d e G ee f a (3.28), e c e a f e a b a 2D a d 3D.

<u>Wea F f L ea E a c</u>

We a e ead de e e **weak form** f ea e a c b e f e **three-step** ced e e e ed 1D.

**<u>Step 1</u>**

T e f e e d ffe e a e a (3.24) b e c e d e f c a d e ae e e d a .

(3.32a) $\int_{\Omega} w_x \ \vec{\nabla}\cdot\vec{\sigma}_x \ d\Omega + \int_{\Omega} w_x b_x \ d\Omega = 0 \quad \forall w_x \in U_0$

(3.32b) $\int_{\Omega} w_y \ \vec{\nabla}\cdot\vec{\sigma}_y \ d\Omega + \int_{\Omega} w_y b_y \ d\Omega = 0 \quad \forall w_y \in U_0$

e e $U_0$ e e f f c a a e e e a b da e (d ace e

b da e ) $U_0 = \{ w_i(x,y) | w_i = 0 \text{ on } \Gamma_u \}$ a d

$\mathbf{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ ( a f ) $\vec{w} = w_x\vec{i} + w_y\vec{j}$ ( ec f )

**Step 2**

I e ec d e , e d b e e d ffe e a a $\vec{u}$ and $\vec{w}$ e a . T ac e e , e d e G ee f a (3.28), e c e a f e a b a 2D a d 3D.

**Q:** W a a e f e a G ee f a (3.28) e f e (3.32a) a d (3.32b)?
**A:**
(3.33a)

(3.33b)

**Step 3**

I e e e a b da c d : add (3.33a) a d (3.33b) a d eca e e f c a $\Gamma_u$ e d :

(3.34a) $\int_{\Omega} (\vec{\nabla} w_x \ \cdot \vec{\sigma}_x + \vec{\nabla} w_y \ \cdot \vec{\sigma}_y) \ d\Omega = \int_{\Gamma_t} (w_x\vec{\sigma}_x \cdot \vec{n} + w_y\vec{\sigma}_y \cdot \vec{n}) \ d\Gamma + \int_{\Omega} (w_x b_x + w_y b_y) \ d\Omega$

O e RHS e ec f e :

(3.34b) $$\int_{\Omega} (\vec{\nabla} w_x \ \cdot \vec{\sigma}_x + \vec{\nabla} w_y \ \cdot \vec{\sigma}_y)\, d\Omega = \int_{\Gamma_t} \vec{w} \cdot \vec{\bar{t}} \ d\Gamma + \int_{\Omega} \vec{w} \cdot \vec{b} \ d\Omega$$

**Q:** W a $\vec{\nabla} \quad \cdot \vec{\sigma}$ a d $\vec{\nabla} w_y \ \cdot \vec{\sigma}_y$ f e ca e ca a d c ?

**A:**

T , e e a d e LHS f (3.34b) ca be e a f

$$\vec{\nabla} w_x \ \cdot \vec{\sigma}_x + \vec{\nabla} w_y \ \cdot \vec{\sigma}_y = \begin{bmatrix} \dfrac{\partial w_x}{\partial x} & \dfrac{\partial w_y}{\partial y} & \dfrac{\partial w_x}{\partial y} + \dfrac{\partial w_y}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

(3.35) $$= (\nabla_S \mathbf{w})^T \boldsymbol{\sigma}$$

$$= (\nabla_S \mathbf{w})^T \mathbf{D} \nabla_S \mathbf{u}$$

c $\boldsymbol{\nabla}_S$ **the symmetric gradient matrix operator** def ed (3.11) e e e a ed

a d ace e 2D a d $\boldsymbol{\nabla}_S \quad \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$.

T , e ea f 2D ca be e a ( a f ):

(3.36)

F d $\mathbf{u} \in U$ c a

$$\int_{\Omega} (\nabla_S \mathbf{w})^T \mathbf{D} \nabla_S \mathbf{u} \, d\Omega = \int_{\Gamma_t} \mathbf{w}^T \bar{\mathbf{t}} \, d\Gamma + \int_{\Omega} \mathbf{w}^T \mathbf{b} \, d\Omega \qquad \forall \mathbf{w} \in U_0$$

e e $U = \{\mathbf{u} | \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_u\}$ a d $U_0 = \{\mathbf{w} | \mathbf{w} = 0 \text{ on } \Gamma_u\}$.

E e e ea f (3.36) f a e a e e e e d a $\Omega$ a f e a e e e e d a , e a e e e e e e ea f :

$$\text{(3.37)} \qquad \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} (\nabla_S \mathbf{w}^e)^T \mathbf{D}^e \left(\nabla_S \mathbf{u}^e\right) d\Omega - \int_{\Omega^e} \mathbf{w}^{eT} \mathbf{b} \, d\Omega - \int_{\Gamma_t^e} \mathbf{w}^{eT} \bar{\mathbf{t}} \, d\Gamma \right\} = 0$$

**Remark:**

Reca e ea f 1D (E . (1.11))

F d $u(x)$ a e f c a a f $u = \bar{u} \text{ on } \Gamma_u$ c a

$$\int_{\Omega} \left(\frac{dw}{dx}\right)^T AE \frac{du}{dx} dx - \int_{\Omega} w^T b \, dx - (w^T A \bar{t})\Big|_{\Gamma_t} = 0 \qquad w = 0 \text{ on } \Gamma_u$$

A d e e e e ea f 1D:

(2.31) $\quad 1_{e=1}^{n_{el}} \,\#\, )\, 2_{/ \, e} + \% \frac{dw^e}{dx} \&^T , \; A^e E^e + \% \frac{du^e}{dx} \& , \; dx \,(\, 2_e \, w^{eT} b \, dx \,(\, (w^{eT} A^e \bar{t})_{!_t^e} \; 0 \; = 0$

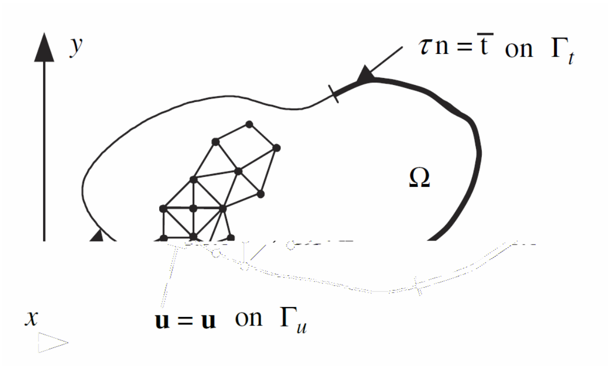
Y ca ed a e b e e e a be ee e-d e a a d -d e a ca e . T , ed a e FE f a (a d e e a ) a e a f a e .

## 3.5 Linear Elasticity: Finite Element Formulation from Weak Form

E e e Ma ce

We de e f e e e e e a f e ea f e e a f 2D e a a .

C de a d a Ω a b da Γ d c e ed 2D e e e ( a e ad a e a ) be :




T e e a e **two degrees-of-freedom per node** c e d e c e f e ba d ace e , e **global nodal displacement matrix** :

(3.38) $\mathbf{d} = \left[\ {}_1\ \ {}_1\ \ {}_2\ \ {}_2\ \cdots\ \right]$

e e $n_{np}$ e be f de e f e e e e e .

I add , e f e e e e a a f e a a d e f c **each element** ca be e e ed a :

(3.39)
$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\mathbf{d}^e$$
$$\mathbf{w}^{eT}(x,y) = \hat{\mathbf{w}}^{eT}\mathbf{N}^e(x,y)^T$$

e e e e e e a e f c a $\mathbf{N}^e$ (3.39) e a :

(3.40)
$$\mathbf{N}^e \quad \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \cdots & N_{n_{en}}^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \cdots & 0 & N_{n_{en}}^e \end{bmatrix}$$

e e $n_{en}$ be f de f a e e e .

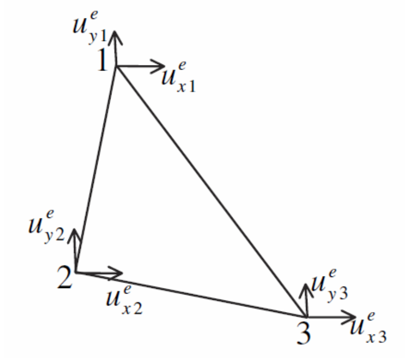
T e e e e da d ace e a $\mathbf{d}^e$ (3.39) :

(3.41) $\mathbf{d}^e \;\; \left[u_x^e \quad u_y^e \quad u_x^e \quad u_y^e \quad \cdots \quad u_{xn_{en}}^e \quad u_{yn_{en}}^e\right]^T$

F e a e, a ea a a e e e , $n_{en} = 3$ a d e c a $\mathbf{d}^e$ c f

e :

$$\mathbf{d}^e = \begin{bmatrix} u_{x1}^e \\ u_{y1}^e \\ u_{x2}^e \\ u_{y2}^e \\ u_{x3}^e \\ u_{y3}^e \end{bmatrix}$$




S a , e a f e e e e da a e f e f c $\hat{\mathbf{w}}^e$ (3.39) :

(3.42) $\hat{\mathbf{w}}^e = \left[\hat{w}_{x1}^e \quad \hat{w}_{y1}^e \quad \hat{w}_{x2}^e \quad \hat{w}_{y2}^e \quad \cdots \quad \hat{w}_{xn_{en}}^e \quad \hat{w}_{yn_{en}}^e\right]^T$

**Q:** H f d e e e ff e a a d e e e e e a f ce a ?

**A:**

A e e c ad e e a $\mathbf{N}^e$ e $\mathbf{B}^e$:

(3.43) $\boldsymbol{\varepsilon}^e = \nabla_S \mathbf{u}^e = \nabla_S \mathbf{N}^e \mathbf{d}^e = \mathbf{B}^e \mathbf{d}^e$

T e a -d ace e a $\mathbf{B}^e$ :

$$(3.44) \qquad \mathbf{B}^e = \nabla_S \mathbf{N}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \cdots & \frac{\partial N_{n_{en}}^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \cdots & 0 & \frac{\partial N_{n_{en}}^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_{n_{en}}^e}{\partial y} & \frac{\partial N_{n_{en}}^e}{\partial x} \end{bmatrix}$$

S a , e de a e f e f c a e:

$$(3.45) \qquad \left(\nabla_S \mathbf{w}^e\right)^T = \left(\mathbf{B}^e \hat{\mathbf{w}}^e\right)^T = \hat{\mathbf{w}}^{eT} \mathbf{B}^{eT}$$

S b (3.44) a d (3.45) e e e e e ea f (3.37) a d eca $\mathbf{d}^e$ $\mathbf{L}^e\mathbf{d}$ a d $\hat{\mathbf{w}}^e$ $\mathbf{L}^e\hat{\mathbf{w}}$ e d :

$$(3.46) \qquad \hat{\mathbf{w}}^T \left\{ \sum_{e\ 1}^{n_{el}} \mathbf{L}^{eT} \left\{ \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{D}^e \mathbf{B}^e d\Omega \, \mathbf{L}^e \mathbf{d} - \int_{\Gamma_t^e} \mathbf{N}^{eT} \bar{\mathbf{t}} \, d\Gamma - \int_{\Omega^e} \mathbf{N}^{eT} \mathbf{b} \, d\Omega \right\} \right\} \quad 0 \qquad \forall \hat{\mathbf{w}}_F$$

$\hat{\mathbf{w}}_F$ e f $\hat{\mathbf{w}}$ c e d de a a e **NOT** a e e a b da .

We ca def e **element matrices** a 1D:

( ) e e e ff e a

$$(3.47) \qquad \mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{D}^e \mathbf{B}^e d\Omega$$

( ) e e e e e a f ce a

$$(3.48) \qquad \mathbf{f}^e = \mathbf{f}_\Omega^e + \mathbf{f}_\Gamma^e = \int_{\Omega^e} \mathbf{N}^{eT} \mathbf{b} \, d\Omega + \int_{\Gamma_t^e} \mathbf{N}^{eT} \bar{\mathbf{t}} \, d\Gamma$$

**Summary:** f e e e e a e e e ac ce!

T e e a e f a 2D e e e e c e a c a ed e 1D e e e beca e f e e d e . T e ced e f de e e e e a ce , e e , e a a f e 1D e e e . T e e e ca be a ed e f **three-step procedure for developing element matrices**:

1. C c f a e f c a $\mathbf{N}^e$.
2. F a f e a -d ace e a $\mathbf{B}^e$.
3. Ca c a f $\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^{eT}\mathbf{D}^e\mathbf{B}^e d\Omega$ a d $\mathbf{f}^e = \mathbf{f}_\Omega^e + \mathbf{f}_\Gamma^e = \int_{\Omega^e} \mathbf{N}^{eT}\mathbf{b}\, d\Omega + \int_{\Gamma_t^e} \mathbf{N}^{eT}\bar{\mathbf{t}}\, d\Gamma$ $\mathbf{N}^e$ a d $\mathbf{B}^e$.

A e b a d S L ea A eb a E a

T e ea f E . (3.46) ca e be e a :

(3.49) $$\hat{\mathbf{w}}^T\left\{\left(\sum_{e=1}^{n_{el}}\mathbf{L}^{eT}\mathbf{K}^e\mathbf{L}^e\mathbf{d}\right)-\left(\sum_{e=1}^{n_{el}}\mathbf{L}^{eT}\mathbf{f}^e\right)\right\}=0 \qquad \forall\hat{\mathbf{w}}_F$$

O

(3.50) $$\hat{\mathbf{w}}^T\left(\mathbf{Kd}-\mathbf{f}\right)=0 \qquad \forall\hat{\mathbf{w}}_F$$

We **exactly** e d ce E . (2.34) 1D!!! T e e f f e e e e f a a d a e f e ac e a e a e a e 1D (**see Section 2.3**).