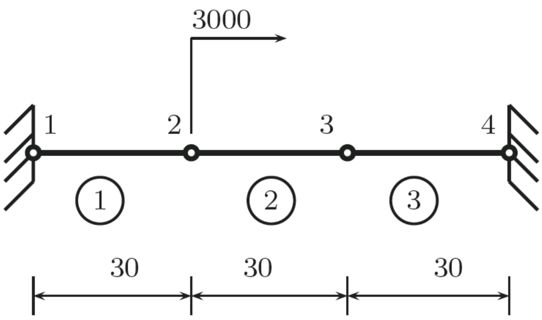
$\mathbf{K}^e$ $\mathbf{f}^e$

$\mathbf{Kd} = \mathbf{f}$

$\mathbf{d}_F$

$\mathbf{K}^e$ $\mathbf{f}^e$ *elementwise*




```
% clear memory
clear all

% E: modulus of elasticity
```

```
% A: area of cross section
% L: length of bar
E=30e6; A=1; L=[30 30 30];

% numberElements: number of elements
numberElements=3;
% numberNodes: number of nodes
numberNodes=4;
% generation of coordinates and connectivities
elementNodes=[1 2;2 3;3 4];
nodeCoordinates=[0 30 60 90];
```

```
clear all
```

elementNodes

```
elementNodes=[1 2;2 3;3 4];
```

$\mathbf{K}^e$ $\mathbf{f}^e$

$$\mathbf{K}^e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

```
% for structure:
   % displacements: displacement vector
   % force : force vector
   % stiffness: stiffness matrix
force=zeros(numberNodes,1);
stiffness=zeros(numberNodes,numberNodes);
% applied load at node 2
force(2)=3000.0;

% computation of the system stiffness matrix
for e=1:numberElements;
  % elementDof: element degrees of freedom (Dof)
  elementDof=elementNodes(e,:) ;
  k(e)=E*A/L(e) ;
  stiffness(elementDof,elementDof)=...
      stiffness(elementDof,elementDof)+k(e)*[1 -1;-1 1];
end
```

zeros

```
force=zeros(numberNodes,1);
stiffness=zeros(numberNodes,numberNodes);
```

```
force(2)=3000.0;
```

```
stiffness([1 2],[1 2])= stiffness([1 2],[1 2])+ k(e)*[1 -1;-1 1];
```

```
stiffness([2 3],[2 3])= stiffness([2 3],[2 3])+ k(e)*[1 -1;-1 1];
```

```
>> A=[2 3 5; 7 11 13; 17 19 23]
A =
     2     3     5
     7    11    13
    17    19    23
>> A([1 3], [1 3])
ans =
     2     5
    17    23
```

```
>> A=[2 3 5; 7 11 13; 17 19 23]
A =
     2     3     5
     7    11    13
    17    19    23
>> A([1 3], [1 3]) = [0 1; 2 3]
A =
     0     3     1
     7    11    13
     2    19     3
```

$\mathbf{d}_F$

```
% boundary conditions and solution
% prescribed dofs
prescribedDof=[1;4];
```

```
% solution
GDof=numberNodes;
displacements=solution(GDof,prescribedDof,stiffness,force);
```

prescribedDof

solution.m

```
function displacements=solution(GDof,prescribedDof,stiffness,
force)
% function to find solution in terms of global displacements
activeDof=setdiff([1:GDof]',[prescribedDof]);
U=stiffness(activeDof,activeDof)\force(activeDof);
displacements=zeros(GDof,1);
displacements(activeDof)=U;
```

activeDof

setdiff

[1:GDof]' prescribedDof activeDof

displacements

```
% output displacements/reactions
outputDisplacementsReactions(displacements,stiffness, ...
   numberNodes,prescribedDof,force)
```

outputDisplacementsReactions

fprintf

```
function outputDisplacementsReactions...
    (displacements,stiffness,GDof,prescribedDof,force)

% output of displacements and reactions in
% tabular form

% GDof: total number of degrees of freedom of
% the problem

disp('Displacements')
fprintf('node    displacements\n')
% displacements
for jj=1:GDof
    fprintf('%2.0f:       %10.4e\n', jj, displacements(jj))
end

% reactions
F=stiffness*displacements;
reactions=F(prescribedDof)-force(prescribedDof);
disp('Reactions')
fprintf('node    reactions\n')
for jj=1:size(prescribedDof,1)
    fprintf('%2.0f:       %10.4e\n', prescribedDof(jj),
reactions(jj))
end
```

---

problem1dSimple.zip

problem1dSimple.m solution.m

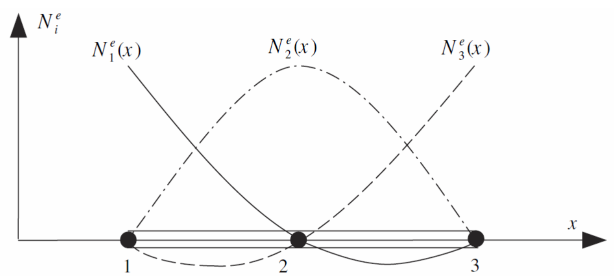
outputDisplacementsReactions.m

$$N_I^e(x_J^e) = \delta_{IJ}$$

δ_{IJ}

$$\delta_{IJ} = \begin{cases} 1 & \text{if} \quad I = J \\ 0 & \text{if} \quad I \neq J \end{cases}$$

$N^e \; x$



$N^e \; x$ x

x

$N^e \; x \quad \dfrac{x \quad a \quad x \quad b}{c} \qquad a \;\; b \qquad c$

$N^e \quad x$ $\qquad x_2^e \qquad x^e$

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{c}$$

$N^e \quad x$ $\qquad x_1^e$

$$= \frac{- \quad -}{- \quad -}$$

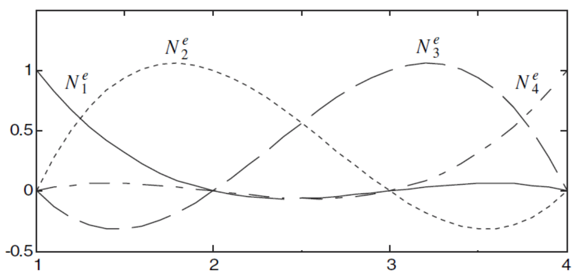
$N_2^e(x)$ $\qquad N_3^e(x)$

$N_2^e(x)$

$$u^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 + \alpha_3^e x^3$$

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)(x - x_4^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)(x_1^e - x_4^e)}$$

$N_2^e(x)$



$$N_i^e(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \equiv \frac{(x - x_1^e)}{(x_i^e - x_1^e)} \times \frac{(x - x_2^e)}{(x_i^e - x_2^e)} \times \cdots \times \frac{(x - x_{i-1}^e)}{(x_i^e - x_{i-1}^e)} \times \frac{(x - x_{i+1}^e)}{(x_i^e - x_{i+1}^e)} \times \cdots \times \frac{(x - x_n^e)}{(x_i^e - x_n^e)}$$

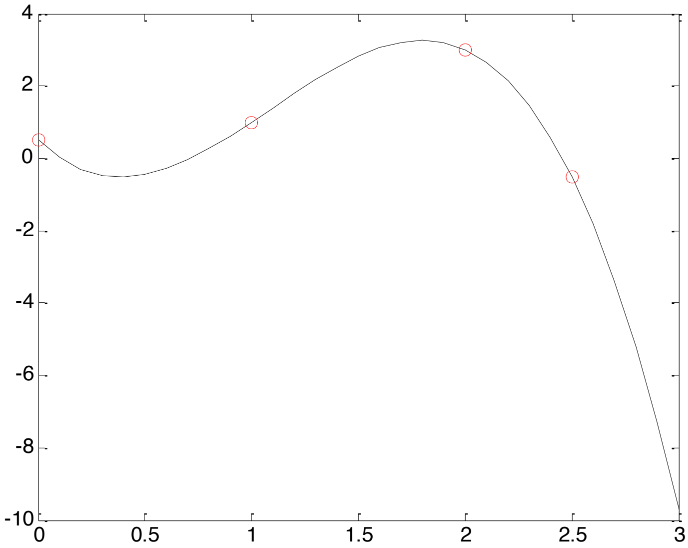
Π i

i $n-1$

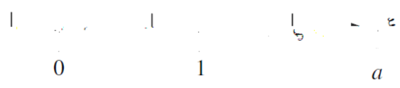
$n-1$ n

$u^e(x)$ $u^e(x)$

```
x=0:0.1:3.0;
u=-27/10.*x.^3+177/20.*x.^2-113/20.*x+1/2;
xdot=[0 1 2 5/2];
udot=[1/2 1 3 -1/2];
plot(x,u,'k-',xdot, udot, 'ro');
```



$[-1, \ +1]$



$p \le 2n_{gp} - 1$

$$\int_{-1}^{1} f(\xi) d\xi = \sum_{i=1}^{n_{gp}} W_i f(\xi_i)$$

W_i ξ_i

座標 n_{gp}

$$f(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \ldots + \alpha_{2n_{gp}-1} \xi^{2n_{gp}-1}$$

α_i

n_{gp} n_{gp}

n_{gp} n_{gp}

$(2n_{gp} - 1)$

W_i 座標 ξ_i

$$\alpha_0 \int_{-1}^{1} d\xi + \alpha_1 \int_{-1}^{1} \xi d\xi + \alpha_2 \int_{-1}^{1} \xi^2 d\xi + \ldots + \alpha_{2n_{gp}-1} \int_{-1}^{1} \xi^{2n_{gp}-1} d\xi = W_1 f(\xi_1) + W_2 f(\xi_2) + \ldots + W_{n_{gp}} f(\xi_{n_{gp}})$$
$$= \alpha_0 (W_1 + W_2 + \ldots + W_{n_{gp}}) + \alpha_1 (W_1 \xi_1 + W_2 \xi_2 + \ldots + W_{n_{gp}} \xi_{n_{gp}}) + \ldots + \alpha_{2n_{gp}-1} (W_1 \xi_1^{2n_{gp}-1} + W_2 \xi_2^{2n_{gp}-1} + \ldots + W_{n_{gp}} \xi_{n_{gp}}^{2n_{gp}-1})$$

$$\int_{-} \xi^m d\xi = \frac{}{m+} \qquad m = \quad \ldots \quad n_{gp} - \qquad\qquad \int_{-1}^{1} \xi^m d\xi = 0 \qquad m = 1,3,5,\ldots,2n_{gp}-1$$

α_i

$$\int_{-} \xi^m d\xi = \frac{}{m+} = \sum_{i=}^{n_{gp}} W_i \xi_i^m \qquad m = \quad \ldots \quad n_{gp} -$$

$$\int_{-1}^{1} \xi^m d\xi = 0 = \sum_{i=1}^{n_{gp}} W_i \xi_i^m \qquad m = 1,3,5,\ldots,2n_{gp}-1$$

$m = 0$

$m = 1$

$$\begin{aligned}
2 &= W_1 + W_2 + \ldots + W_{n_{gp}} \\
0 &= W_1\xi_1 + W_2\xi_2 + \ldots + W_{n_{gp}}\xi_{n_{gp}} \\
2/3 &= W_1\xi_1^2 + W_2\xi_2^2 + \ldots + W_{n_{gp}}\xi_{n_{gp}}^2 \\
&\ldots \\
0 &= W_1\xi_1^{2n_{gp}-1} + W_2\xi_2^{2n_{gp}-1} + \ldots + W_{n_{gp}}\xi_{n_{gp}}^{2n_{gp}-1}
\end{aligned}$$

W_i ξ_i!

W_i ξ_i

ξ_i n_{gp}

$k = \quad \ldots \; n_{gp}$

$$P_0(\xi) = 1$$
$$P_1(\xi) = \xi$$
$$\dots$$
$$P_k(\xi) = \frac{2k-1}{k}\xi P_{k-1}(\xi) - \frac{k-1}{k}P_{k-2}(\xi)$$

W_i

$$W_i = \frac{2(1-\xi_i^2)}{(n_{gp}P_{n_{gp}-1}(\xi_i))^2}$$

$n_{gp} = 2$

ξ_i

n_{gp}	Location, ξ_i	Weights, W_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	± 0.7745966692	0.555 555 5556
	0.0	0.888 888 8889
4	± 0.8611363116	0.347 854 8451
	± 0.3399810436	0.652 145 1549
5	± 0.9061798459	0.236 926 8851
	± 0.5384693101	0.478 628 6705
	0.0	0.568 888 8889
6	± 0.9324695142	0.171 324 4924
	± 0.6612093865	0.360 761 5730
	± 0.2386191861	0.467 913 9346

W_i ξ_i

n_{gp} p

$$p \leq 2n_{gp} - 1$$

p

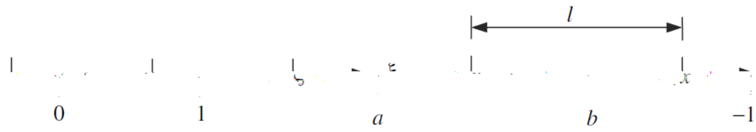
$$n_{gp} \geq \frac{p+1}{2}$$

$$\int_{-1}^{1} (\xi^2 + \sin(\xi / 2)) d\xi$$

$a \quad b$

$a \quad b$

$$I = \int_a^b f \;\; x \; dx$$



$[-1,\ 1]$

$[-1,\ 1]$

$a \quad b$

$x \qquad \xi$

$$dx = \frac{1}{2}(b \quad a)d \quad = \frac{l}{2}d \quad = Jd$$

$J \qquad = \frac{1}{2}(\; - \;)$

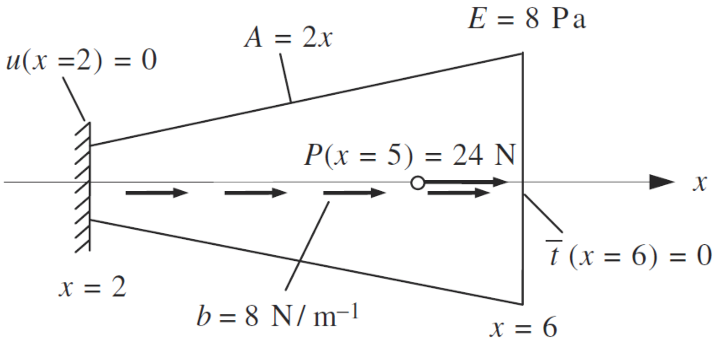
$$I = J\int_{-1}^{1} f(\xi)d\xi = J\hat{I} \qquad \text{where} \qquad \hat{I} = \int_{-1}^{1} f(\xi)d\xi$$

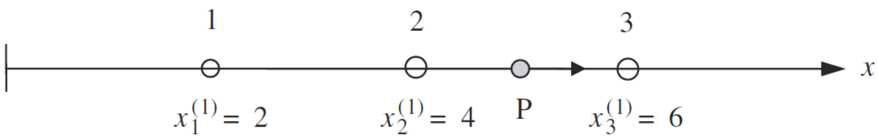
$\hat{I}$

$W_i \qquad \xi_i$

$$I = W f \;\; \xi \;\; + W \; f \;\; \xi \;\; + \ldots = \sum_{i=}^{n_{gp}} W_i f \;\; \xi_i$$

$$\int_{3}^{7} \frac{1}{1.1+x}\,dx$$





$$N_1^{(1)} = \frac{(x - x_2^{(1)})(x - x_3^{(1)})}{(x_1^{(1)} - x_2^{(1)})(x_1^{(1)} - x_3^{(1)})} = \frac{(x-4)(x-6)}{(-2)(-4)} = \frac{1}{8}(x-4)(x-6),$$

$$N_2^{(1)} = \frac{(x - x_1^{(1)})(x - x_3^{(1)})}{(x_2^{(1)} - x_1^{(1)})(x_2^{(1)} - x_3^{(1)})} = \frac{(x-2)(x-6)}{(2)(-2)} = -\frac{1}{4}(x-2)(x-6),$$

$$N_3^{(1)} = \frac{(x - x_1^{(1)})(x - x_2^{(1)})}{(x_3^{(1)} - x_1^{(1)})(x_3^{(1)} - x_2^{(1)})} = \frac{(x-2)(x-4)}{(4)(2)} = \frac{1}{8}(x-2)(x-4),$$

$$B_1^{(1)} = \frac{\mathrm{d}N_1^{(1)}}{\mathrm{d}x} = \frac{1}{4}(x-5), \qquad B_2^{(1)} = \frac{\mathrm{d}N_2^{(1)}}{\mathrm{d}x} = \frac{1}{2}(4-x), \qquad B_3^{(1)} = \frac{\mathrm{d}N_3^{(1)}}{\mathrm{d}x} = \frac{1}{4}(x-3),$$

$$\mathbf{B}^{(1)} = \frac{1}{4}[(x-5)\ \ (8-2x)\ \ (x-3)].$$

Stiffness matrix

$$\mathbf{K}^{(1)} = \mathbf{K} = \int_{x_1}^{x_3} \mathbf{B}^{(1)\mathrm{T}} A^{(1)} E^{(1)} \mathbf{B}^{(1)}\, \mathrm{d}x = \int_2^6 \frac{1}{4} \begin{bmatrix} (x-5) \\ (8-2x) \\ (x-3) \end{bmatrix} (2x)(8) \frac{1}{4} [(x-5)\ \ (8-2x)\ \ (x-3)]\, \mathrm{d}x$$

$$= \int_2^6 \begin{bmatrix} x(x-5)^2 & x(x-5)(8-2x) & x(x-5)(x-3) \\ x(8-2x)(x-5) & x(8-2x)^2 & x(8-2x)(x-3) \\ x(x-3)(x-5) & x(x-3)(8-2x) & x(x-3)^2 \end{bmatrix} \mathrm{d}x.$$

p

$$\mathbf{K} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ & 85.33 & -53.33 \\ \mathrm{sym} & & 48 \end{bmatrix} = \begin{bmatrix} 26.67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5.33 & -53.33 & 48 \end{bmatrix}.$$

K_{11}

External force matrix

b *P*

$$\mathbf{f}_{\Omega}^{(1)} = \mathbf{f}_{\Omega} = \int_{2}^{6} \mathbf{N}^{eT} b \, dx + \int_{2}^{6} \mathbf{N}^{eT} P \; \delta(x-5) \, dx$$

$$= \int_{2}^{6} \mathbf{N}^{eT} b \, dx + (\mathbf{N}^{eT} P)\big|_{x=5}$$

&

$$\int_{x_1}^{x_2} g(x)\delta(x-a)\mathrm{d}x = \begin{cases} g(a) & \text{if} \quad x_1 < a < x_2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{f}_{\Omega} = \int_{2}^{6} \begin{bmatrix} 0.125(x-4)(x-6) \\ -0.25(x-2)(x-6) \\ 0.125(x-2)(x-4) \end{bmatrix} \times 8\,\mathrm{d}x + \begin{bmatrix} 0.125(x-4)(x-6) \\ -0.25(x-2)(x-6) \\ 0.125(x-2)(x-4) \end{bmatrix}_{x=5} \times 24.$$

$$\int_{2}^{6} 0.125(x-4)(x-6) \times 8 dx$$

$$= \begin{bmatrix} 2((2.8453-4)(2.8453-6)+(5.1547-4)(5.1547-6)) \\ -4((2.8453-2)(2.8453-6)+(5.1547-2)(5.1547-6)) \\ 2((2.8453-2)(2.8453-4)+(5.1547-2)(5.1547-4)) \end{bmatrix} + \underbrace{\begin{bmatrix} -3 \\ 18 \\ 9 \end{bmatrix}}_{\text{sum} = 24}$$

$$= \underbrace{\begin{bmatrix} 5.33 \\ 21.33 \\ 5.33 \end{bmatrix}}_{8 \cdot 4} + \underbrace{\begin{bmatrix} -3 \\ 18 \\ 9 \end{bmatrix}}_{24} = \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}.$$

$$\begin{bmatrix} 26.67 & -32 & 5.33 \\ & 85.33 & -53.33 \\ \text{sym} & & \text{[illegible]} \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ \text{[illegible]} \end{bmatrix} = \begin{bmatrix} r_1 + 2.33 \\ 39.33 \\ \text{[illegible]} \end{bmatrix}$$

$$\begin{bmatrix} 85.33 & -53.33 \\ -53.33 & 48 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 39.33 \\ 14.33 \end{bmatrix} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2.1193 \\ 2.6534 \end{bmatrix}$$

Postprocessing

$$u = N_1^{(1)} u_1 + N_2^{(1)} u_2 + N_3^{(1)} u_3, \qquad \mathbf{d} = \mathbf{d}^{(1)} = \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix}.$$

$$u(x) = \frac{1}{8}(x-4)(x-6)(0) + \frac{-1}{4}(x-2)(x-6)(2.1193) + \frac{1}{8}(x-2)(x-4)(2.6534)$$
$$= -0.198\,15x^2 + 2.248\,55x - 3.7045.$$

$$\sigma(x) = E\frac{\mathrm{d}u}{\mathrm{d}x} = E\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{N}^{(1)}\mathbf{d}^{(1)}) = E\mathbf{B}^{(1)}\mathbf{d}^{(1)}$$

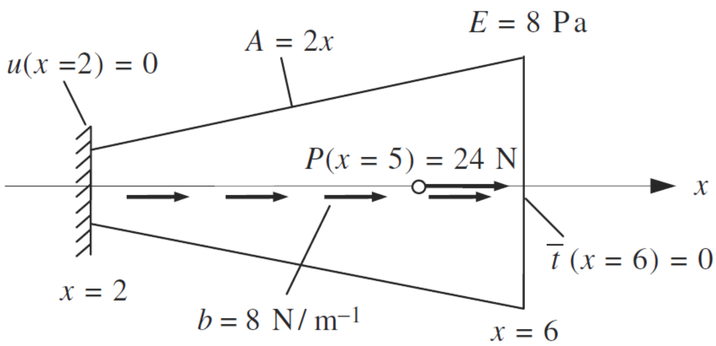
$$= 8\frac{1}{4}[(x-5)\ \ (8-2x)\ \ (x-3)] \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix} = -3.17x + 17.99.$$

p $n_{gp} \geq \frac{p+1}{2}$

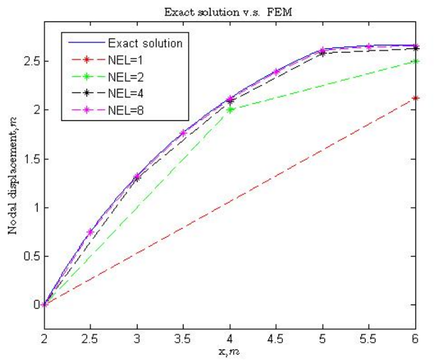
$$\frac{du^e(x)}{dx}$$

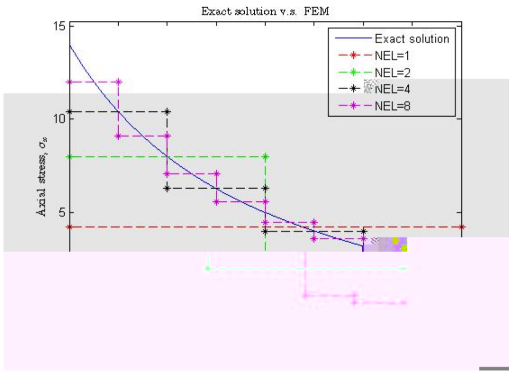
$u^e(x)$

$u^e(x)$



$$\frac{du^e(x)}{dx}$$





$$\frac{du^e(x)}{dx}$$

$\frac{du^e(x)}{dx}$ 1