05/27/2015

Chapter 6: Beam

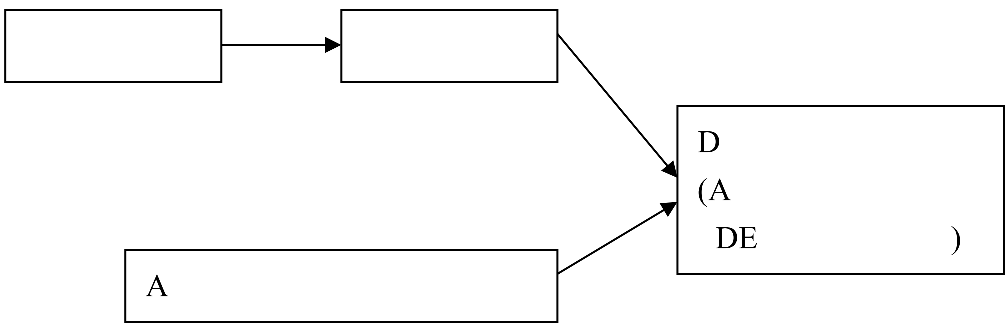
A **beam, plate** **shell** .
thin .

,
the distribution of strain through the thickness takes a very simple form. B ,
:

1. **Beams**, , **1D element**;
2. **Shells**, , **2D element**;
3. **Plates**, .

E ABA ,
.

,
. A
Advanced Structural Analysis,
.



6.1 EULER-BERNOULLI BEAM

6.1.1 Kinematics of Beam

.

Finite Element Method

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: **Euler–Bernoulli** ,

, **Timoshenko** .

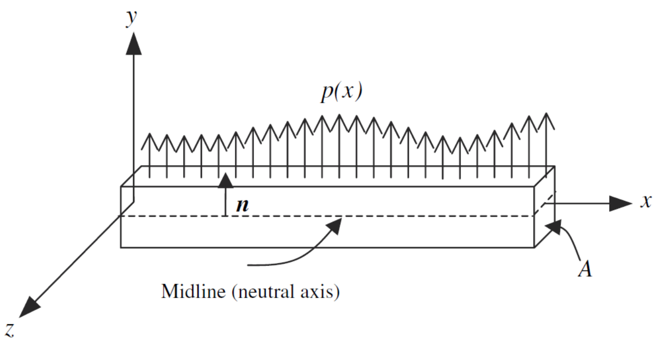
E B .

A . x-

A; **midline** (

neutral axis). x

x.

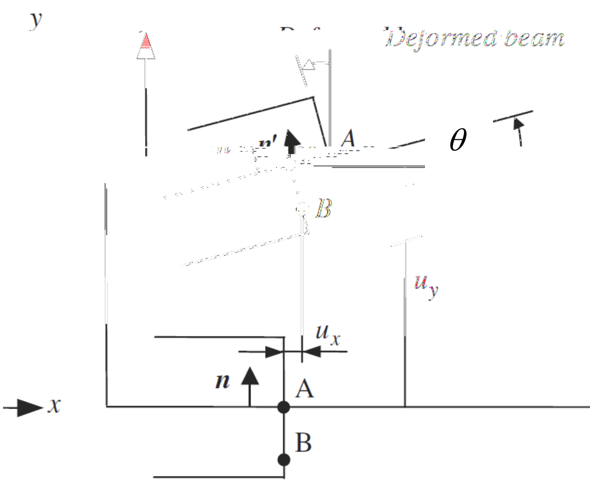


key assumption E B **normals**

straight and normal (

; ,

).



,

x- .

Q: x- u_x ?

A:

(6.1) $u_x \quad -y\sin\theta$

θ ( ) x. y B (A).

θ , $\sin\theta \approx \theta$ **the slope of midline**:

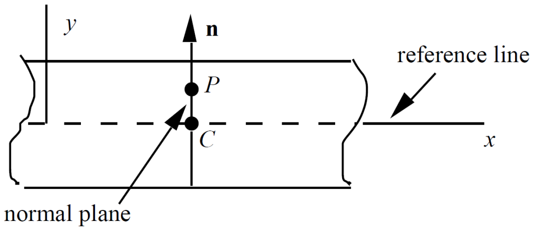
(6.2) $u_x = -y\theta = -y\dfrac{du_y}{dx}$

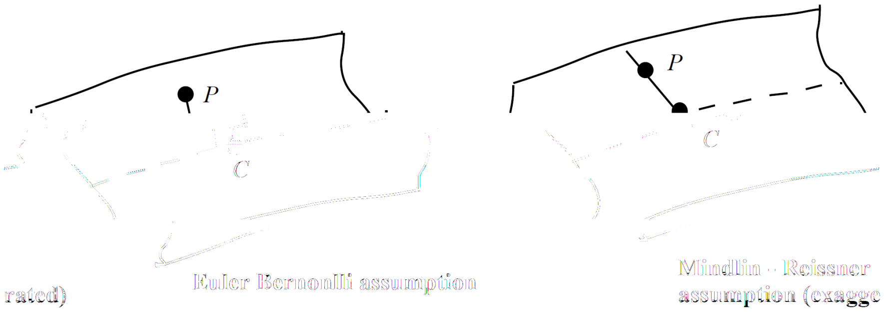
A ε_{xx} :

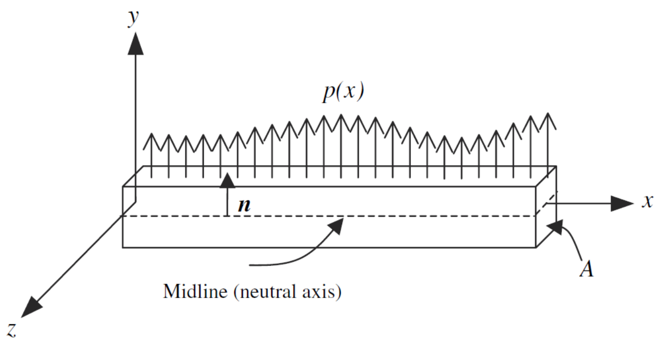
(6.3) $\varepsilon_{xx} = \dfrac{du_x}{dx} = -y\dfrac{d^2u_y}{dx^2}$

Remarks:

1. E . (6.3) E B : ε_{xx} **the strain along the axis of the beam varies linearly through the thickness of the beam**. (______ ____, ______ ________). , , .
2. E -B , ; / - / .
3. , (;); / - / .







E -B .

.

, , :

$$(6.4) \qquad u_x = u_x^M - y\frac{du_y}{dx}$$

u_x^M . :

$$\varepsilon_{xx} = \frac{du_x^M}{dx} - y\frac{d^2u_y}{dx^2}$$

$$(6.5) \qquad \varepsilon_{yy} = 0 \quad (\because u_y \text{ i f c i f } x\text{; e e hi g i he bea he i a f c i f } x)$$

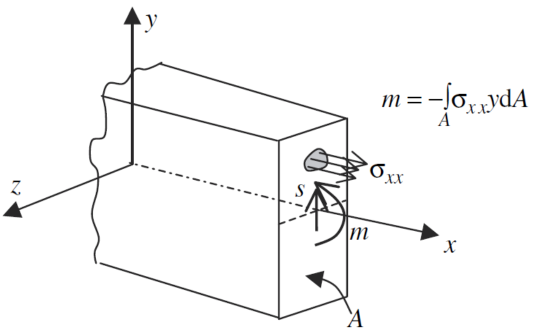
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{du_y}{dx} + \frac{du_y}{dx} = 0$$

Remarks: the strains

1. ,
(6.4); **bending or flexural strain**.
2. - - ε_{yy} .
3. γ_{xy} .
4. , (6.2) (6.3) _____ (6.4) (6.5) _____-_______ .

,
. , **flexural**
.

6.1.2 Generalized Stress-Strain Law



(**thin** *y*-),
:

(6.5) $\sigma_{xx} = E\varepsilon_{xx} = E\left(-y\dfrac{d^2u_y}{dx^2}\right)$

:

(6.6) $m = -\int_A y\sigma_{xx} dA$

(6.5) (6.6) :

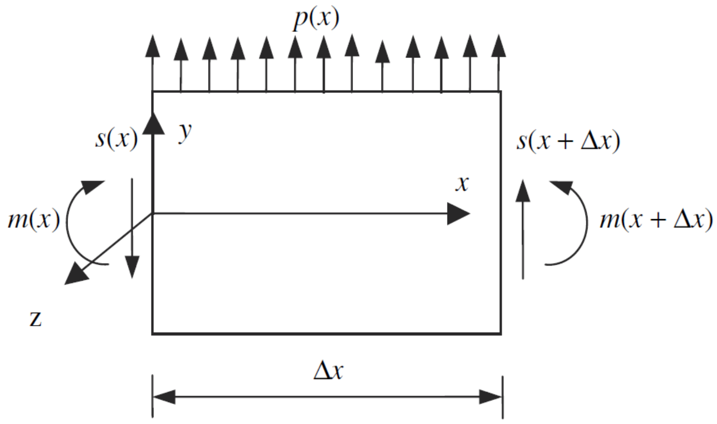
(6.7) $$m = -\int_A yE\left(-y\frac{d^2u_y}{dx^2}\right)dA = \int_A Ey^2\frac{d^2u_y}{dx^2}dA = E\frac{d^2u_y}{dx^2}\int_A y^2dA = EI\kappa$$

I $I = \int_A y^2dA$ κ $\kappa = \dfrac{d^2u_y}{dx^2}$.

Remark:

1. E . (6.7) $m = EI\kappa$:
( ) , κ .

6.1.3 Equilibrium and Governing Equations



,

.

Q: (y)
?

A:

(6.8) $$s(x+\Delta x) - s(x) + p\left(x+\frac{\Delta x}{2}\right)\Delta x = 0 \qquad \frac{\mathrm{d}s}{\mathrm{d}x} + p = 0.$$

Q: (x=y=0)?

A:

(6.9) $$m(x+\Delta x) - m(x) + \Delta x s(x+\Delta x) + \frac{1}{2}\Delta x^2 p\left(x+\frac{\Delta x}{2}\right) = 0 \qquad \frac{\mathrm{d}m}{\mathrm{d}x} + s = 0.$$

C (6.8) (6.9),

(6.10) $$\frac{d^2 m}{dx^2} - p = 0$$

E . (6.7) (6.10) ( ).

:

(6.11) $$EI\frac{d^4 u_y}{dx^4} - p = 0$$

Remark:

1. E . (6.11) E -B .
   fourth-order
   , u_y .

   , **some important changes in the boundary conditions and the development of the weak form**.

6.1.4 Boundary Conditions

:

E B C

(6.12) $u_y = \bar{u}_y$ on Γ_u

(6.12) $\text{——} = -\bar{\theta}$ on Γ_θ

B C

(6.13) $mn = EI\dfrac{d^2 u_y}{dx^2} n = \bar{m}$ on Γ_m

(6.13) $sn = -EI\dfrac{d^3 u_y}{dx^3} n = \bar{s}$ on Γ_s

Remarks:

Finite Element Method

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1. (6.13) (6.13), *n* .
(6.9)
(6.8) ,
y , .

2.
.

Q: ( )?

A:

Q: ?

A:

Q: ?

A:

3.
- .

4.
, . A

(6.14) $sn = -EI\frac{d^3u_y}{dx^3}n = \bar{s}$ on Γ_s $\Rightarrow$ $w(sn-\bar{s})\big|_{\Gamma_s} = 0$

(6.14) $mn \quad EI\frac{d^2u_y}{dx^2}n \quad \bar{m}$ on Γ_m $\Rightarrow$ $\frac{dw}{dx}(mn-\bar{m})\Big|_{\Gamma_m} \quad 0$

Remark: (6.14) $s\delta u_y$ $m\delta\theta$

. , (1)

(2)

.

,

integration by parts CE (6.14) .

<u>**first time** (6.14) :</u>

$$
\begin{aligned}
\int_\Omega w\frac{d^2m}{dx^2}dx &= \int_\Omega \frac{d}{dx}(w\frac{dm}{dx})dx - \int_\Omega \frac{dw}{dx}\frac{dm}{dx}dx \\
&= (w\frac{dm}{dx})\Big|_\Gamma - \int_\Omega \frac{dw}{dx}\frac{dm}{dx}dx \\
&= (-wsn)\big|_\Gamma - \int_\Omega \frac{dw}{dx}\frac{dm}{dx}dx \\
&= (-w\bar{s})\big|_{\Gamma_s} - \int_\Omega \frac{dw}{dx}\frac{dm}{dx}dx \quad (\because (6.14b))
\end{aligned}
\tag{6.15}
$$

<u>**second time** (6.15) :</u>

$$
\begin{aligned}
\frac{dw}{dx}\frac{dm}{dx}dx &= \quad \frac{d}{dx}(\frac{dw}{dx}m)dx \quad \frac{d^2w}{dx^2}mdx \\
&= (\frac{dw}{dx}mn)\Big| \quad \frac{d^2w}{dx^2}mdx \\
&= (\frac{dw}{dx}\bar{m})\Big|_{\ \ m} \quad \frac{d^2w}{dx^2}mdx \quad (\because (6.14c))
\end{aligned}
\tag{6.16}
$$

(6.15) (6.16) (6.14) :

(6.17) $$\int_{\Omega} \frac{d^2 w}{dx^2} m dx = \int_{\Omega} wp dx + (\frac{dw}{dx}\bar{m})\Big|_{\Gamma_m} + (w\bar{s})\Big|_{\Gamma_s} \quad \text{for} \quad \forall w \in U_0$$

- (6.7), :

(6.18) $$\int_{\Omega} \frac{d^2 w}{dx^2} EI \frac{d^2 u_y}{dx^2} dx = \int_{\Omega} wp dx + (\frac{dw}{dx}\bar{m})\Big|_{\Gamma_m} + (w\bar{s})\Big|_{\Gamma_s} \quad \text{for} \quad \forall w \in U_0$$

Remarks:

1. C^1.

2. w u_y, .

C^1

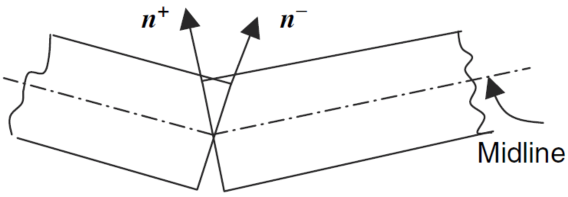
the planes normal to the midline are assumed to remain <u>plane</u> and <u>normal</u>.

C^0

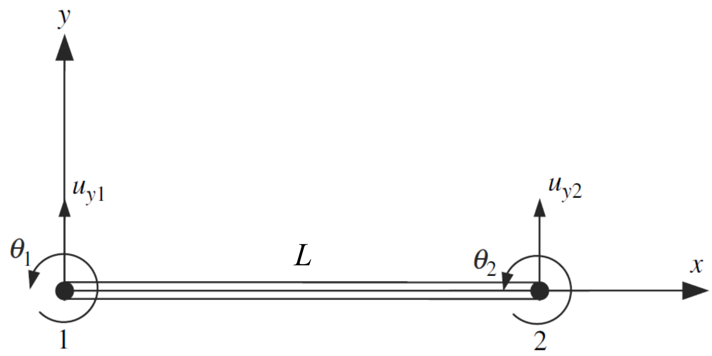
, u_y. A , ,

. ,

C^0 .



6.1.6 Finite Element Discretization



C^1.

C^1 , **displacements** **derivatives of the displacement** .

.

:

$$\text{(6.19)} \qquad \mathbf{d}^e = \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

$$\text{(6.20)} \qquad u_y(x) = \mathbf{N}^e(x)\mathbf{d}^e = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

$$= N_1(x)u_{y1} + N_2(x)\theta_1 + N_3(x)u_{y2} + N_4(x)\theta_2$$

A :

$$\mathbf{f}^e = \begin{bmatrix} f_{y1} \\ m_1 \\ f_{y2} \\ m_2 \end{bmatrix}$$

______________________________ $\mathbf{N}(x)$

A : $u_y(x) = a + bx + cx^2 + dx^3$

B :

$$\begin{aligned} (0) &= \quad {}_1 \\ '(0) &= \theta_1 \\ (\ ) &= \quad {}_2 \\ '(\ ) &= \theta_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} \qquad \mathbf{Aa} = \mathbf{d}^e$$

$$\Rightarrow \mathbf{a} = \mathbf{A}^{-1}\mathbf{d}^e$$

$$a = u_{y1}$$

$$b = \theta_1$$

$$c = \frac{3}{L^2}(\ \ u_{y1} + u_{y2}) \quad \frac{1}{2L}(4 \ {}_1 + 2 \ {}_2)$$

$$d = \frac{2}{L^3}(u_{y1} - u_{y2}) + \frac{1}{L^2}(\theta_1 + \theta_2)$$

(6.20), $u_y(x) = u_{y1}N_1(x) + \theta_1 N_2(x) + u_{y2}N_3(x) + \theta_2 N_4(x)$

:

$$\begin{cases} N_1(x) = 2(\frac{x}{L})^3 - 3(\frac{x}{L})^2 + 1 \\ N_2(x) = \frac{x^3}{L^2} - 2\frac{x^2}{L} + x \\ N_3(x) = -2(\frac{x}{L})^3 + 3(\frac{x}{L})^2 \\ N_4(x) = \frac{x^3}{L^2} - \frac{x^2}{L} \end{cases} \qquad (6.21)$$

Hermite .

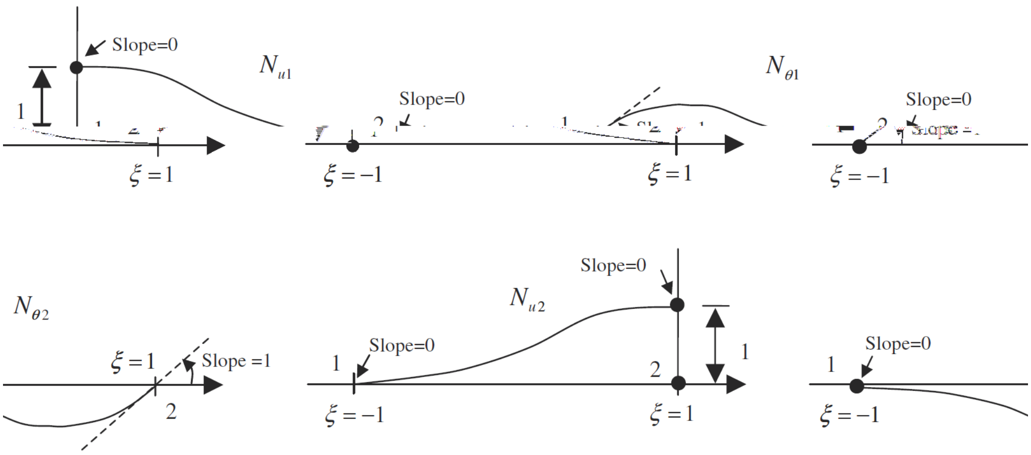
:

$$\begin{cases} x = 0,\ N_1(x) = 1 \quad N_2(x) = N_3(x) = N_4(x) = 0 \\ x = L,\ N_3(x) = 1 \quad N_1(x) = N_2(x) = N_4(x) = 0 \end{cases}$$

$$\begin{cases} x=0, & \dfrac{dN_2(x)}{dx}=1, \quad \dfrac{dN_1(x)}{dx}=\dfrac{dN_3(x)}{dx}=\dfrac{dN_4(x)}{dx}=0 \\ x=L, & \dfrac{dN_4(x)}{dx}=1, \quad \dfrac{dN_1(x)}{dx}=\dfrac{dN_2(x)}{dx}=\dfrac{dN_3(x)}{dx}=0 \end{cases}$$

Q: C ?

A:



,

$u_y = \mathbf{N}^e\mathbf{d}^e$ w $\mathbf{N}^e\mathbf{w}^e$

Remark:

ξ $(-1<\xi<1$ $\xi=\dfrac{2x}{L}-1)$. , :

$$\begin{cases} N_1(\xi)=\dfrac{1}{4}(1-\xi)^2(2+\xi) \\ N_2(\xi)=\dfrac{L}{8}(1-\xi)^2(1+\xi) \\ N_3(\xi)=\dfrac{1}{4}(1+\xi)^2(2-\xi) \\ N_4(\xi)=\dfrac{L}{8}(1+\xi)^2(\xi-1) \end{cases}$$

\-

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Example: E B

p.

(Answer)

$$\mathbf{f}_{\Omega}^{e}=\int_{0}^{L}\left[N_{1}(x)\quad N_{2}(x)\quad N_{3}(x)\quad N_{4}(x)\right]^{T}pdx=\frac{pL}{2}\begin{bmatrix}1\\ \frac{L}{6}\\ 1\\ -\frac{L}{6}\end{bmatrix}$$

A ( ) *Advanced Structural Analysis*.

.

6.1.7 Lagrange and Hermite Interpolation Functions

(6.22) u_y

θ ($\frac{du_y}{dx}$,) . **Hermite family of interpolation function**.

,

NOT , . , __________,

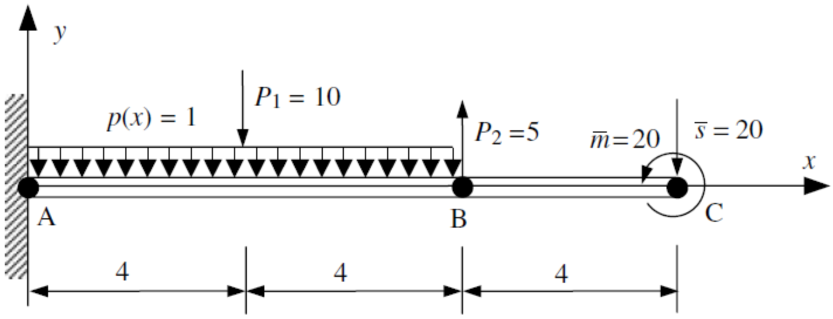
, , .

E -B

C^1, u_y **NOT ADMISSIBLE**

.

6.1.8 Numerical Example (Euler-Bernoulli Beam)

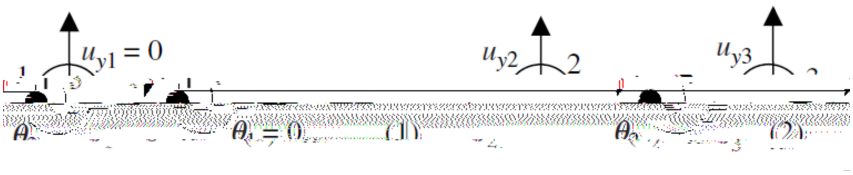


C . ABC

. , N

$\frac{N}{m}$. $EI = 10^4\ Nm^2$.

x $12\ m$ $\bar{s} = -20\ N$ $\bar{m} = 20\ Nm$.



__________________ .

$\mathbf{d}^T = [u_{y1}, \theta_1, u_{y2}, \theta_2, u_{y3}, \theta_3]$.

Element stiffness matrices:

B (6.23) , 1 ($EI = 10^4\ Nm^2$, $L = 8\ m$)

$$\mathbf{K}^e = \frac{EI}{L}\begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix} = 10^3 \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 \\ 0.94 & 5.00 & -0.94 & 2.50 \\ -0.23 & -0.94 & 0.23 & -0.94 \\ 0.94 & 2.50 & -0.94 & 5.0 \end{bmatrix} \begin{matrix} [1] \\ \\ [2] \end{matrix}$$

$$[1] \qquad [2]$$

2 ($EI = 10^4\ Nm^2$, $L = 4\ m$)

$$\mathbf{K}^e \quad \frac{EI}{L}\begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix} \quad 10^3\begin{bmatrix} 1.88 & 3.75 & -1.88 & 3.75 \\ 3.75 & 10.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix}\begin{matrix}[2] \\ \\ [3]\end{matrix}$$

$$\qquad\qquad\qquad [2] \qquad\qquad [3]$$

Global stiffness matrix:

:

$$\mathbf{K} \quad 10^3\begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix}\begin{matrix}[1] \\ \\ [2] \\ \\ [3]\end{matrix}$$

$$[1] \qquad\qquad [2] \qquad\qquad [3]$$

Boundary force matrix:

$$\mathbf{f}_\Gamma^e = (\mathbf{N}^{eT}\bar{s})\Big|_{\Gamma_s} + \left(\frac{d\mathbf{N}^{eT}}{dx}\bar{m}\right)\Big|_{\Gamma_m}$$

1, $\mathbf{f}_\Gamma^{(1)} = [0 \quad 0 \quad 0 \quad 0]^T$ Γ_s Γ_m.

2, $\mathbf{f}_\Gamma^{(2)} = [0 \quad 0 \quad 0 \quad 1]^T \bar{m} + [0 \quad 0 \quad 1 \quad 0]^T \bar{s} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ 20 \end{bmatrix}\begin{matrix}[2] \\ \\ [3]\end{matrix}$

:

$$\mathbf{f}_\Gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 20 \end{bmatrix}\begin{matrix}[1] \\ \\ [2] \\ \\ [3]\end{matrix}$$

Body force matrix:

$$\mathbf{f}_\Omega^e = \int_{x_1^e}^{x_{n_{en}}^e} \mathbf{N}^{eT} p dx$$

1D , $-1 < \xi_A \le 1$,

$$\mathbf{f}_\Omega^e = \mathbf{N}^{eT}(\xi_A) P_A$$

For element 1: 1

$p(x) = -1$ P_1 -10

$\xi = 0$,

$$\mathbf{f}_\Omega = \int_{x^e}^{x_{n_{en}}^e} \begin{bmatrix} N_u \\ N_\theta \\ N_u \\ N_\theta \end{bmatrix} pdx + P \begin{bmatrix} N_u \\ N_\theta \\ N_u \\ N_\theta \end{bmatrix}\Bigg|_{\xi=} = \begin{bmatrix} - \\ - \\ - \\ \ \end{bmatrix}$$

For element 2: 2, $P_2 = 5$, $\xi = -1$,

$$\mathbf{f}_\Omega^{(2)} = \begin{bmatrix} N_{u1} \\ N_{\theta 1} \\ N_{u2} \\ N_{\theta 2} \end{bmatrix}\Bigg|_{\xi=-1} P_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} [2] \\ \\ [3] \\ \ \end{matrix}$$

$$\mathbf{f}_\Omega = \begin{bmatrix} -9 \\ -15.3 \\ -4 \\ 15.3 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} [1] \\ \\ [2] \\ \\ [3] \\ \\ \end{matrix}$$

Boundary conditions and solution:

$$\left[\begin{array}{cc|cccc} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ \hline -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{array}\right] \left[\begin{array}{c} u_{y1} = 0 \\ \theta_1 = 0 \\ \hline u_{y2} \\ \theta_2 \\ u_{y3} \\ \theta_3 \end{array}\right] = \left[\begin{array}{c} -9 + r_{u1} \\ -15.3 + r_{\theta 1} \\ \hline -4 \\ 15.3 \\ -20 \\ 20 \end{array}\right]$$

r_{u1} r_θ

1, .

:

$$\begin{bmatrix} u_{y2} \\ \theta_2 \\ u_{y3} \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.55 \\ -0.11 \\ -1.03 \\ -0.12 \end{bmatrix}$$

B -

:

$$\begin{bmatrix} r_{u1} \\ r_{\theta 1} \end{bmatrix} = \begin{bmatrix} 33 \\ 252 \end{bmatrix}$$

Postprocessing:

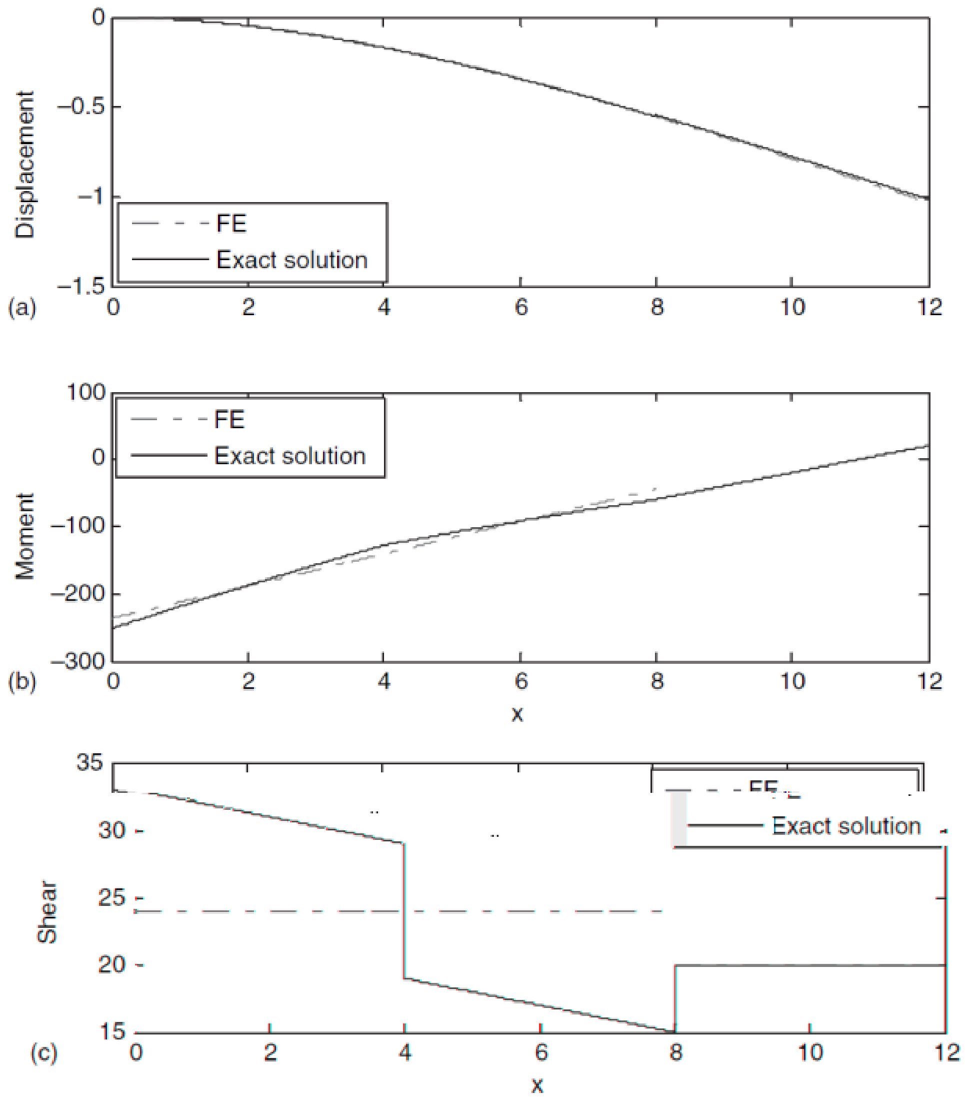
:

$$m^{(1)} = EI\frac{d^2u^{(1)}}{dx^2} = EI\begin{bmatrix} \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{\theta1}}{dx^2} & \frac{d^2N_{u2}}{dx^2} & \frac{d^2N_{\theta2}}{dx^2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ u_{y2} \\ \theta_2 \end{bmatrix} = -240.64 + 25.785x,$$

$$s^{(1)} = EI\frac{d^3u^{(1)}}{dx^3} = EI\begin{bmatrix} \frac{d^3N_{u1}}{dx^3} & \frac{d^3N_{\theta1}}{dx^3} & \frac{d^3N_{u2}}{dx^3} & \frac{d^3N_{\theta2}}{dx^3} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ u_{y2} \\ \theta_2 \end{bmatrix} = -25.785$$

$$m^{(2)} = EI\frac{d^2u^{(2)}}{dx^2} = EI\begin{bmatrix} \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{\theta1}}{dx^2} & \frac{d^2N_{u2}}{dx^2} & \frac{d^2N_{\theta2}}{dx^2} \end{bmatrix}\begin{bmatrix} u_{y2} \\ \theta_2 \\ u_{y3} \\ \theta_3 \end{bmatrix} = -104.5 + 39.75x$$

$$s^{(2)} = EI\frac{d^3u^{(2)}}{dx^3} = EI\begin{bmatrix} \frac{d^3N_{u1}}{dx^3} & \frac{d^3N_{\theta1}}{dx^3} & \frac{d^3N_{u2}}{dx^3} & \frac{d^3N_{\theta2}}{dx^3} \end{bmatrix}\begin{bmatrix} u_{y2} \\ \theta_2 \\ u_{y3} \\ \theta_3 \end{bmatrix} = -39.75$$



6.1.9 MATLAB Implementation of Euler-Bernoulli Beam

:

```
% Simple MATLAB codes for Euler Bernoulli Beam

% clear memory
clear all

% EI; bending stiffness
% L: length of beam
EI=1E4;
L = [8 4];

% nodal coordinates and element connectivities
numberElements=2;
numberNodes=3;
nodeCoordinates=[0 8 12];
elementNodes=[1 2; 2 3];
```

```
% for structure:
   % displacements: displacement vector
   % force : force vector
   % stiffness: stiffness matrix
   % GDof: global number of degrees of freedom
GDof=2*numberNodes;
U=zeros(GDof,1);
force=zeros(GDof,1);

% stiffess matrix and force vector
stiffness=...
   formStiffnessEulerBernoulliBeam(GDof,numberElements,...
   elementNodes,EI,L);
force = [0 0 0 0 -20 20]' + [-9 -15.3 -4 15.3 0 0]';

% boundary conditions
% clamped at x=0
prescribedDof=[1;2];

% solution
displacements=solution(GDof,prescribedDof,stiffness,force);

% output displacements/reactions
outputDisplacementsReactions(displacements,stiffness,...
   GDof,prescribedDof,force);
```

formStiffnessEulerBernoulliBeam.m

```
function  stiffness =...
   formStiffnessEulerBernoulliBeam(GDof,numberElements,...
   elementNodes,EI,L);

stiffness=zeros(GDof);
% calculation of the system stiffness matrix
for e=1:numberElements;
  % elementDof: element degrees of freedom (Dof)
  indice=elementNodes(e,:);
  elementDof=[ 2*(indice(1)-1)+1 2*(indice(2)-1)...
     2*(indice(2)-1)+1 2*(indice(2)-1)+2];
  % length of element
  LElem=L(e) ;
  ke=EI/(LElem)^3*[12   6*LElem -12 6*LElem;
    6*LElem 4*LElem^2 -6*LElem 2*LElem^2;
    -12 -6*LElem 12 -6*LElem ;
    6*LElem 2*LElem^2 -6*LElem 4*LElem^2];
  % stiffness matrix
  stiffness(elementDof,elementDof)=...
     stiffness(elementDof,elementDof)+ke;
end
```

solution.m . outputDisplacementsReactions.m

.

```
function outputDisplacementsReactions...
    (displacements,stiffness,GDof,prescribedDof,force)

% output of displacements and reactions in
% tabular form

% GDof: total number of degrees of freedom of
% the problem

% displacements
disp('Displacements')
%displacements=displacements1;
jj=1:GDof; format
[jj' displacements]

% reactions
F=stiffness*displacements;
reactions=F(prescribedDof)-force(prescribedDof);
disp('reactions')
[prescribedDof reactions]
```

, :

```
Displacements

ans =

    1.0000         0
    2.0000         0
    3.0000   -0.5526
    4.0000   -0.1126
    5.0000   -1.0295
    6.0000   -0.1206

reactions

ans =

    1.0000   33.0000
    2.0000  252.0000
```

B !

6.2 TIMOSHENKO BEAM

.

$(\frac{L}{h} < 10)$

transverse shear deformation .

,

composite laminated beams,

(, 2, B , , C 3 4).

, C^0

, ,

E -B . , - **shear locking** .

, . ,

2- .

selective reduced integration .

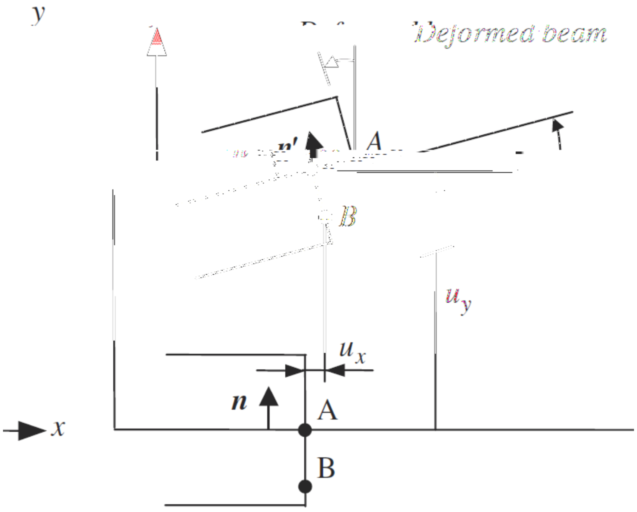
6.2.1 Kinematics

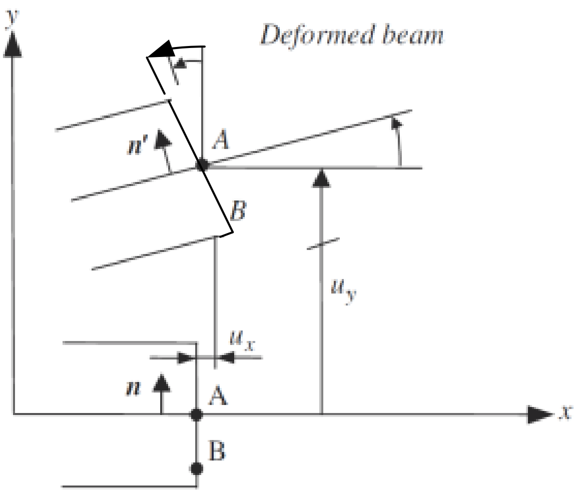
key assumption **normals** **straight but NOT normal** .

.

(x)

ϕ. :





(6.25) $\theta = \frac{du_y}{dx} + \phi$

Q: *x*- (

$\sin\theta \approx \theta$)?

A:

(6.26) $u_x = -y\theta$

θ ( ) *x*. *y*

B (A).

Q: ε_{xx}, ε_{yy}, γ_{xy}?

A: :

$$\varepsilon_{xx} = -y\frac{d\theta}{dx}$$

(6.27) $\varepsilon_{yy} = 0$ ($\because u_y$ is function of x; everything in the beam theory is a function of x)

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\theta + \frac{du_y}{dx} (\Rightarrow -\frac{du_y}{dx} - \phi + \frac{du_y}{dx} = -\phi)$$

6.2.2 Stress-Strain Relationship

, . :

$$\sigma_{xx} = E \quad {}_{xx} = E\left(\quad y\frac{d\theta}{dx}\right)$$

(6.28) $\sigma_{yy} = 0$

$$\tau_{xy} = kG \quad {}_{xy} = kG(\frac{du_y}{dx} \quad \theta)$$

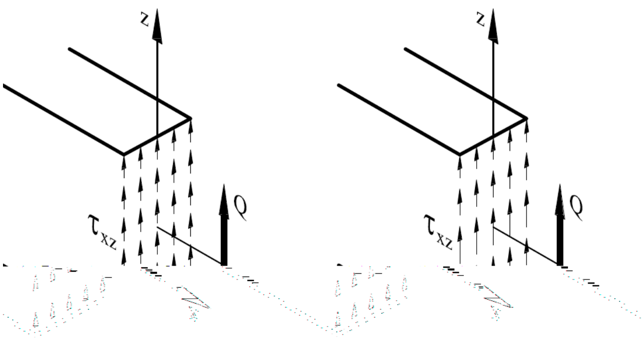
G : $G = \frac{E}{\quad + v}$ v . ***k* is a shear correction factor** .

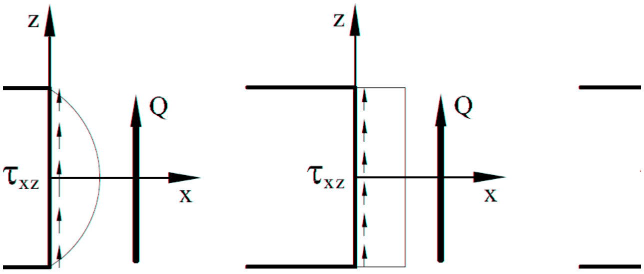
B .

(2013) -

- .

Assumed distribution Exact distribution





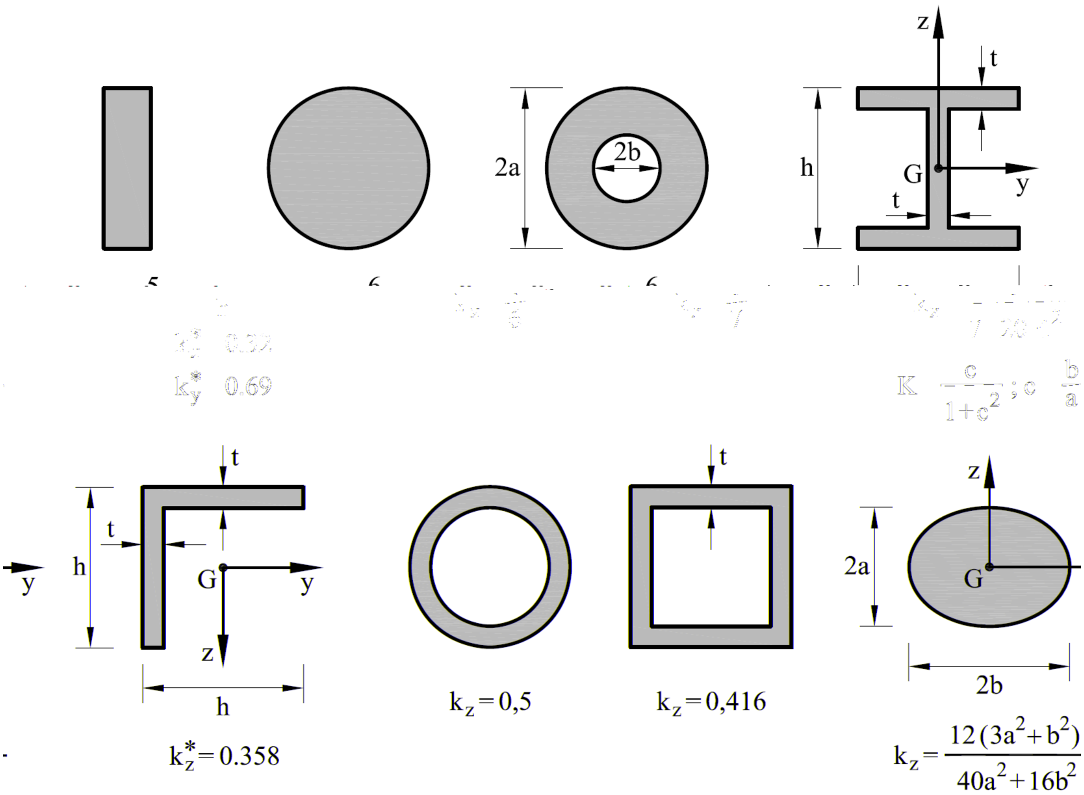


Fig. 2.3 Shear correction parameter k_z for some cross sections. Asterisk denotes values computed with the FEM [BD5,Bo,Co6]

-

E -B ,

:

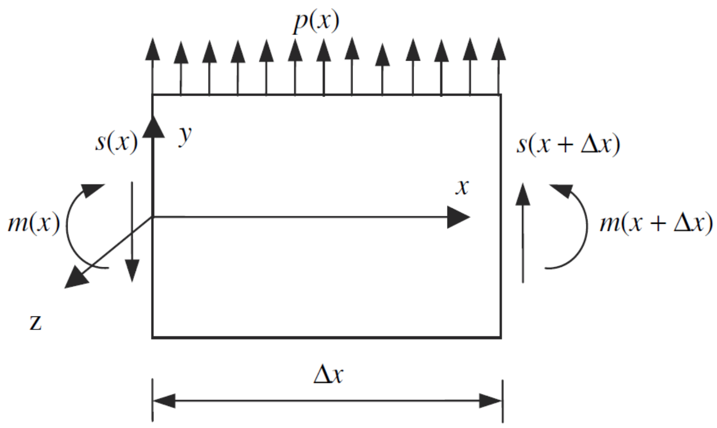
(6.29) $$m = -\int_A y\sigma_{xx} dA = \int_A Ey^2 \frac{d\theta}{dx} dA = EI\frac{d\theta}{dx}$$

I $I = \int_A y^2 dA$. ,

:

(6.30) $$s = \int_A \tau_{xy} dA = kGA(\frac{du_y}{dx} - \theta)$$

6.2.3 Equilibrium and Governing Equations



,

E -B

:

$$\begin{cases} \dfrac{ds}{dx} + p = 0 \\ \dfrac{dm}{dx} + s = 0 \end{cases}$$

:

(6.31)
$$\begin{cases} \dfrac{d}{dx}(kGA(\dfrac{du_y}{dx}-\theta))+p=0 \\ \dfrac{d}{dx}(EI\dfrac{d\theta}{dx})+kGA(\dfrac{du_y}{dx}-\theta)=0 \end{cases}$$

6.2.4 Weak Form

-

- .

- , u_y θ; E . (6.31).

w β.

Step 1: (= 0), $w(x)$ $\beta(x)$,

.

Step 2: D (. .,) $u_y(x)$ $\theta(x)$.

Step 3:

$w(x)$ $\beta(x)$ **vanish at the essential boundary conditions**.

1 2

w

β:

1 :

(6.32)
$$\int_0^L w\left(\frac{d}{dx}(kGA(\frac{du_y}{dx}-\theta))+p\right)dx=0$$

2 :

(hint: $\int_0^L w\frac{df}{dx}\,dx=(wf)\big|_0^L-\int_0^L f\frac{dw}{dx}\,dx=(wf)_{x=L}-(wf)_{x=0}-\int_0^L f\frac{dw}{dx}\,dx$ **)**

A:

$$\int_0^L w\left(\frac{d}{dx}(kGA(\frac{du_y}{dx}-\theta))+p\right)dx$$

$$\Rightarrow \int_0^L w\left(\frac{d}{dx}(kGA(\frac{du_y}{dx}-\theta))\right)dx=\int_0^L -\frac{dw}{dx}(kGA(\frac{du_y}{dx}-\theta))dx+w_1kGA(\frac{du_y}{dx}-\theta)\Big|_0^L$$

$$=-\int_0^L \frac{dw}{dx}(kGA(\frac{du_y}{dx}-\theta))dx+wkGA(\frac{du_y}{dx}-\theta)\Big|_0^L$$

(6.33) $$\underline{\underline{-\int_0^L \frac{dw}{dx}(kGA(\frac{du_y}{dx}-\theta))dx}}+\int_0^L wpdx+wkGA(\frac{du_y}{dx}-\theta)\Big|_0^L=0$$

1 :

$$\int_0^L \beta\left(\frac{d}{dx}(EI\frac{d\theta}{dx})+kGA(\frac{du_y}{dx}-\theta)\right)dx=0$$

2 :

A:

$$_0^L \quad \frac{d}{dx}(EI\frac{d}{dx}) \quad dx$$

$$_0^L \frac{d}{dx}(EI\frac{d}{dx})dx+ \quad EI\frac{d}{dx}\Big|_0^L$$

(6.34) $$\underline{\underline{-\int_0^L \frac{d\beta}{dx}(EI\frac{d\theta}{dx})dx}}+\underline{\int_0^L \beta kGA(\frac{du_y}{dx}-\theta)dx}+\beta EI\frac{d\theta}{dx}\Big|_0^L=0$$

3:

,

:

(6.35)

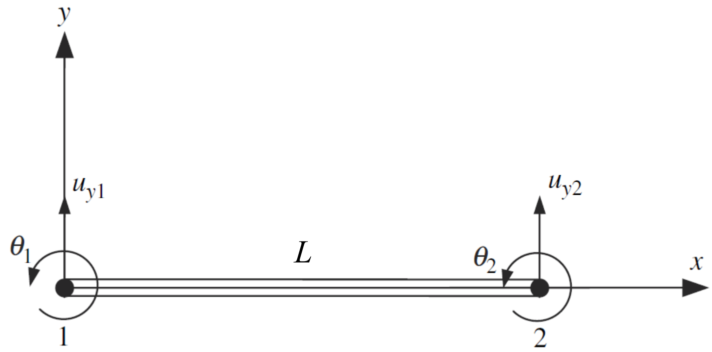
$u_y(x)$ $\theta(x)$ $u_y=\bar{u}_y$ on Γ_{u_y}

$\theta=\bar{\theta}$ on Γ_θ :

$$\underline{\underline{\int_0 —(\quad (——-\theta))}} = \int_0 \quad + \quad (——-\theta)\Big|_0 \quad =0 \text{ on } \Gamma$$

$$\underline{\underline{\int_0^L \frac{d\beta}{dx}(EI\frac{d\theta}{dx})dx}}-\underline{\int_0^L \beta kGA(\frac{du_y}{dx}-\theta)dx}=\beta EI\frac{d\theta}{dx}\Big|_0^L \quad \beta=0 \text{ on } \Gamma_\theta$$

6.2.5 Finite Element Discretization



C - *EI* *kGA*

. ,

:

$$u_y(x) = \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} \tag{6.36}$$

$$\theta(x) = \begin{bmatrix} 0 & N_1(x) & 0 & N_2(x) \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} \tag{6.37}$$

$N_1(x) = 1 - \dfrac{x}{L}$ $N_2(x) = \dfrac{x}{L}$. (6.36) (6.37) ,

:

$$\begin{bmatrix} u_y(x) \\ \theta(x) \end{bmatrix} = \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \\ 0 & N_1(x) & 0 & N_2(x) \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} = \mathbf{N}^e(x)\mathbf{d}^e \tag{6.38}$$

A , :

(6.39) $$\begin{bmatrix} w(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \\ 0 & N_1(x) & 0 & N_2(x) \end{bmatrix} \begin{bmatrix} w_1 \\ \beta_1 \\ w_2 \\ \beta_2 \end{bmatrix} = \mathbf{N}^e(x)\mathbf{w}^e$$

E

(6.38) (6.39) (6.35).

- ,

p:

(6.40) $$\begin{bmatrix} \frac{kGA}{L} & \frac{kGA}{2} & -\frac{kGA}{L} & \frac{kGA}{2} \\ \frac{kGA}{2} & \frac{EI}{L}+\frac{kGAL}{3} & -\frac{kGA}{2} & \frac{kGAL}{6}-\frac{EI}{L} \\ -\frac{kGA}{L} & -\frac{kGA}{2} & \frac{kGA}{L} & -\frac{kGA}{2} \\ \frac{kGA}{2} & \frac{kGAL}{6}-\frac{EI}{L} & -\frac{kGA}{2} & \frac{EI}{L}+\frac{kGAL}{3} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{pL}{2} \\ 0 \\ \frac{pL}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

-

$$\alpha = \frac{EI}{L\,kGA}:$$

(6.41) $$\frac{EI}{\alpha L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(\alpha+4) & -6L & -L^2(\alpha-2) \\ -12 & -6L & 12 & -6L \\ 6L & -L^2(\alpha-2) & -6L & L^2(\alpha+4) \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{pL}{2} \\ 0 \\ \frac{pL}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

Example: C

. A t 1, h, $k = -$.

A $G = \frac{E}{3}$, $E = 1$, $L = 1$. E -B

.

Answer:

, $I = \frac{h^3}{12}$ $\alpha = \frac{18h^2}{5}$. ,

, :

$$\begin{bmatrix} \frac{5h}{18} & -\frac{5h}{36} \\ -\frac{5h}{36} & \frac{h^3}{12} + \frac{5h}{54} \end{bmatrix} \begin{bmatrix} u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

, :

$$u_{y2} = \frac{36(9h^2 + 10)}{5h(18h^2 + 5)}$$

E -B , ,

:

$$\begin{bmatrix} h^3 & -\frac{h^3}{2} \\ -\frac{h^3}{2} & \frac{h^3}{3} \end{bmatrix} \begin{bmatrix} u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

, :

$$u_{y2} = \frac{4}{h^3}$$

exact

$u_{y2} = \frac{PL^3}{3EI} = \frac{4}{h^3}$. E -B

.

C , (. .,

). . A

( ),
E -B . A
stiffer E -B
. 2-
E -B . **shear locking**.

6.2.6 Shear Locking

____
_________________________. ,
- :

(6.42)

Q:

Q: ε_s ?

A:

(6.43) $\varepsilon_s = \dfrac{du_y}{dx} - \theta = \dfrac{x(\theta_2 - \theta_1)}{L} + \dfrac{-u_{y1} + u_{y2} - L\theta_1}{L}$

C

. .

.

(6.43), x :

$$\frac{x(\theta_2 - \theta_1)}{L} + \frac{-u_{y1} + u_{y2} - L\theta_1}{L} = 0 \quad \forall x$$

$$\Rightarrow \theta_1 = \theta_2 \text{ and } \theta_1 = \frac{-u_{y1} + u_{y2}}{L}$$

$\theta_1 = \theta_2$ (6.42).

.

.

6.2.7 Remedies for Shear Locking

A **selective reduced integration**

. ,

- ξ

$(-1 \le \xi \le +1)$.

-

:

(6.35) $\int_0 \text{—(} \quad (\text{——} - \theta)) \quad = \int_0 \quad + \quad (\text{——} - \theta)\Big|_0 \quad = 0 \text{ on } \Gamma$

(6.35) $\displaystyle\int_0^L \frac{d\beta}{dx}(EI\frac{d\theta}{dx})dx - \int_0^L \beta kGA(\frac{du_y}{dx} - \theta)dx = \beta EI\frac{d\theta}{dx}\Big|_0^L \quad \beta = 0 \text{ on } \Gamma_\theta$

, $(\frac{du_y}{dx}-\theta)$ ( ) $\frac{\theta}{}$ (

). :

(6.44) $\frac{d\theta}{dx}=\mathbf{B}_b\mathbf{d}^e$, $\frac{du_y}{dx}-\theta=\mathbf{B}_s\mathbf{d}^e$

(6.45) $\mathbf{B}_b=\begin{bmatrix}0 & \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx}\end{bmatrix}=\begin{bmatrix}0 & \frac{2}{L}\frac{dN_1(\xi)}{d\xi} & 0 & \frac{2}{L}\frac{dN_2(\xi)}{d\xi}\end{bmatrix}$

(6.46) $\mathbf{B}_s=\begin{bmatrix}\frac{dN_1}{dx} & -N_1 & \frac{dN_2}{dx} & -N_2\end{bmatrix}=\begin{bmatrix}\frac{2}{L}\frac{dN_1(\xi)}{d\xi} & -N_1(\xi) & \frac{2}{L}\frac{dN_2(\xi)}{d\xi} & -N_2(\xi)\end{bmatrix}$

E

:

(6.47) $\mathbf{K}^e=\mathbf{K}_b^e+\mathbf{K}_s^e$

:

(6.48) $\mathbf{K} \;=\int_0 \;\mathbf{B}\;\mathbf{B} \;=\int_{-1}^{+1} \;\mathbf{B}\;\mathbf{B}\;\frac{}{2}\;\xi$

(6.49) $\mathbf{K}_s^e=\int_0^L kGA\mathbf{B}_s^T\mathbf{B}_s dx=\int_{-1}^{+1} kGA\mathbf{B}_s^T\mathbf{B}_s\frac{L}{2}\,d\xi$

1D .

Q: (6.48) ?

A:

$n\geq\frac{p+1}{2}=\frac{1}{2}$. .

Q: (6.49) ?

A:

$n\geq\frac{p+1}{2}=\frac{2+1}{2}$. .

____

. A

- $\mathbf{K}_s^e$ ______________________________

______________ (- **reduced integration**). $\mathbf{K}_b^e$

(**selective**). -

,

p:

(6.50)
$$\begin{bmatrix} \frac{kGA}{L} & \frac{kGA}{2} & -\frac{kGA}{L} & \frac{kGA}{2} \\ \frac{kGA}{2} & \frac{EI}{L}+\frac{kGAL}{4} & -\frac{kGA}{2} & \frac{kGAL}{4}-\frac{EI}{L} \\ -\frac{kGA}{L} & -\frac{kGA}{2} & \frac{kGA}{L} & -\frac{kGA}{2} \\ \frac{kGA}{2} & \frac{kGAL}{4}-\frac{EI}{L} & -\frac{kGA}{2} & \frac{EI}{L}+\frac{kGAL}{4} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{pL}{2} \\ 0 \\ \frac{pL}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

- (6.40).

Example: C . ,

$I = \frac{h^3}{12}$ $kGA = \frac{5h}{18}$.

, , :

$$\begin{bmatrix} \frac{5h}{18} & -\frac{5h}{36} \\ -\frac{5h}{36} & \frac{h^3}{12}+\frac{5h}{72} \end{bmatrix} \begin{bmatrix} u_{y2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

, :

$$u_{y2} = \frac{3(6h^2+5)}{5h^3}$$

A , h .