Chapter 5: Error Estimation and Mesh Adaptation

5.1 Types of Errors

The errors introduced into the finite element solution of a given differential equation can be attributed to three basic sources:

1. <u>Domain approximation error</u>, which is due to approximation of the domain.

In 1D problems, the domains considered are straight lines. Thus no approximation of the domain is necessary. In 2D problems involving nonrectangular domains, domain approximation errors are introduced into finite element solutions. Nevertheless, as we refine the mesh, the domain is more accurately represented.

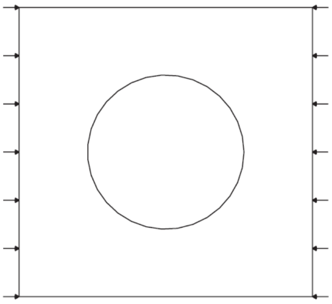
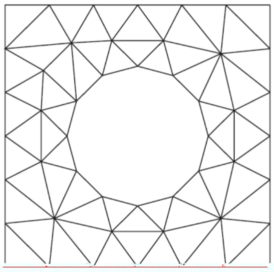
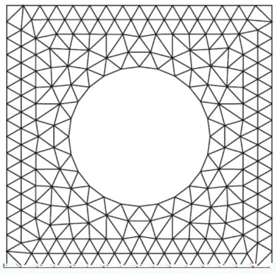


Plate with a Hole

Triangular Finite Element



Finite Element Model



Refined Finite Element Model

2. <u>Quadrature and finite arithmetic errors</u>, which are due to the numerical evaluation of integrals and the numerical computation on a computer.

3. <u>Approximation error (aka, discretization error)</u>, which is due to the approximation of the exact solution: $u_h \approx u$ in which $u(x)$ is the exact solution and u_h is finite element approximation.

The discretization error combines the errors introduced by the interpolation and mesh chosen and is the main source of error in the finite element solution. We wish to know: (1) how to establish a meaningful measure of the error $u - u_h$ and (2) how this meaningful error measure behaves as the number of elements in the mesh is increased.

5.2 Measures of Errors

In the assessment of solution quality for various types of elements, a better measure of element performance is needed than just eyeball the a few pointwise difference between an exact solution and the finite element solution as we did in the lab and homework.

The error in a finite element solution $u_h(x)$ is quantified by **norms of function** defined by:

(5.1) $$\|f(x)\|_{L_2} = \left(\int_{x_1}^{x_2} f^2(x)dx \right)^{\frac{1}{2}}$$

You might notice that the definition of the norm of a function is very similar to the way we define the norm of a vector. The norm defined by (5.1) is called L_2 norm.

Using this definition of a norm, we can define the error in a finite element solution by:

(5.2) $$\|e\|_{L_2} \quad \|u(x) - u_h(x)\|_{L_2} \quad \left(\int_{x_1}^{x_2} (u(x) - u_h(x))^2 dx \right)^{\frac{1}{2}}$$

in which $u(x)$ is the exact solution and u_h is finite element approximation.

Remarks:

1. If we think of norms as measures of distance between two functions, then the above is a **measure of the distance between the exact and the finite element displacement solution**.

2. **The error at any point in the interval contributes to this measure of error** because the integrand is the square of the error at any point. The above can be considered a root-mean-square measure of the error. Thus, the above provides a measure of error that is not affected by a good luck in unexpected absence of error at a few points.

Although the L_2 error in the displacement is useful, we are more interested in the error in the stress, which is proportional to error in the strain. Thus, a **frequently used approach is to compute the error in energy defined by:**

(5.3) $$\left(\frac{1}{2} \int_{x_1}^{x_2} E(\varepsilon(x) - \varepsilon_h(x))^2 \, dx \right)^{\frac{1}{2}}$$

in which $\varepsilon(x)$ $\frac{du}{dx}$ is the exact solution of strains and ε_h is finite element approximation.

We now come to an important conclusion for **convergence studies** without proofing:

If a finite element contains the **complete polynomial** of order *p*, then the error in the L_2 norm of the displacement varies according to

$$\boxed{(5.4) \qquad \|e\|_{L_2} = c_1 h^{p+1}}$$

in which the constant c_1 depends on the problem and the mesh. For convergence study the value of c_1 is not important. *h* stands for the element size.

The **energy norm** for an element that is complete up to order *p* is one order lower:

$$\boxed{(5.5) \qquad \|e\|_{en} = \left(\int_{1}^{2} (\varepsilon(\ ) - \varepsilon_h(\ ))^2 \, d \ \right)^{\frac{1}{2}} = c_2 h^p}$$

Again in which the constant c_2 depends on the problem and the mesh. For convergence study the value of c_2 is not important. h stands for the element size.

The constants $p+1$ and p in Eqs. (5.4) and (5.5) are called the **rate** of convergence. Note that the convergence depends on h as well as on p. We often rewrite (5.4) and (5.5) in their logarithm forms to facilitate convergence studies.

(5.6) $$\log \|e\|_{L_2} = \log c_1 + (p+1)\log h$$

(5.7) $$\log \|e\|_{en} = \log c_2 + (p)\log h$$

Q: For a given problem discretized with equally-spacing linear elements, what happens if we double the total number of elements (that is, if we reduce the element size by half)?
A:

The error of L_2 norm of the displacement decreases by a factor of 4.

The error of energy norm decreases by a factor of 2.

Q: For a given problem discretized with equally-spacing quadratic elements, what happens if we double the total number of elements (that is, if we reduce the element size by half)?
A:

The error of L_2 norm of the displacement decreases by a factor of 8.

The error of energy norm decreases by a factor of 4.

Remarks:

1. Quadratic elements give you more accuracy in comparison with linear elements and come at little cost. In linear analysis, **quadratic elements are almost always preferred.**
2. The conditioning of the linear system equations deteriorates for higher order Lagrange elements. The best tradeoff between accuracy and complexity for Lagrange interpolants is offered by quadratic elements.

Example

Here we consider a 1D example to verify the error estimates in (5.4) and (5.5). Consider the differential equation

Finite Element Method

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$$-\frac{d^2u}{dx^2} = 2 \quad for \quad 0 < x < 1 \qquad \text{with} \quad u(0) = u(1) = 0$$

The exact solution is

$$u(x) = x(1 \quad x)$$

while the finite element solutions are, for N=2

$$u_h = \begin{cases} h^2(x/h) & for\, 0 \le x \le h \\ h^2(2 - x/h) & for\, h \le x \le 2h \end{cases}$$

For N=3,

$$u_h = \begin{cases} 2h^2(x/h) & for \quad 0 \le x \le h \\ 2h^2(2 - x/h) + 2h^2(x/h - 1) & for \quad h \le x \le 2h \\ 2h^2(3 - x/h) & for \quad 2h \le x \le 3h \end{cases}$$

and, for N=4.

$$u_h = \begin{cases} 3h^2(x/h) & for \quad 0 \le x \le h \\ 3h^2(2 - x/h) + 4h^2(x/h - 1) & for \quad h \le x \le 2h \\ 4h^2(3 - x/h) + 3h^2(x/h - 2) & for \quad 2h \le x \le 3h \\ 3h^2(4 - x/h) & for \quad 3h \le x \le 4h \end{cases}$$

Q: Write down the integral expression to evaluate the errors of L_2 norm and energy norm for N=2.

A:

$$\int_0^h (x - x^2 - hx)^2\, dx + \int_h^{2h} (x - x^2 - 2h^2 + xh)^2\, dx = 0.002083$$

$$\|e\|_{L_2} = \sqrt{0.002083} = 0.04563$$

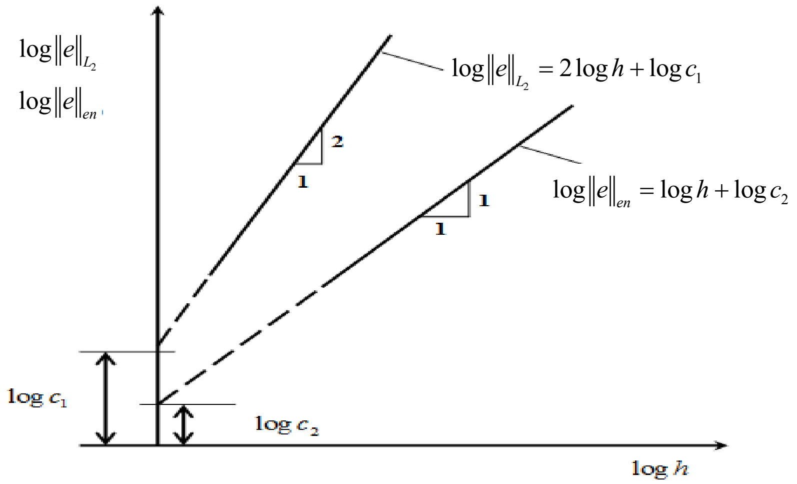
$$\left\|\frac{du}{dx}-\frac{du_h}{dx}\right\| = \int_0^h (1-2x-h)^2\,dx + \int_h^{2h} (1-2x+h)^2\,dx = 0.08333$$

$$\|e\|_{en} = \sqrt{0.0833} = 0.2887$$

Similar calculations can be performed for N=3 and N=4. Following table gives the error of L_2 norm and energy norm for N=2, 3 and 4.

h	**log₁₀h**	**||e||L2**	**log₁₀||e||L2**	**||e||en**	**log₁₀||e||en**
1/2	-0.301	0.04564	-1.341	0.2887	-0.5396
1/3	-0.477	0.02028	-1.693	0.1925	-0.7157
1/4	-0.601	0.01141	-1.943	0.1443	-0.8406

Plots of $\log\|e\|_{L_2}$ $\log\|e\|_0$ and $\log\|e\|_{en}$ versus $\log h$ show that



$$\log\|e\|_{L_2} = 2\log h + \log c_1, \quad \log\|e\|_{en} = \log h + \log c_2$$

In other words, the **rate of convergence** of the finite element solution is 2 in the L_2 norm and 1 in the energy norm, verifying the estimates in (5.4)-(5.7).

5.3 Error Estimation Techniques

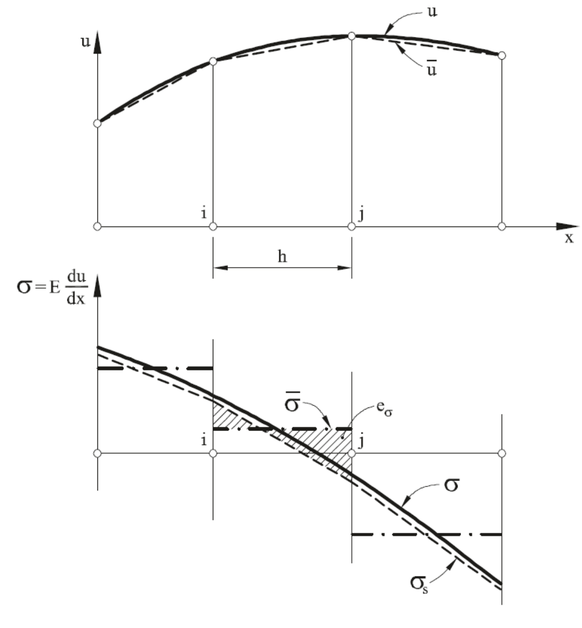
For general 2D and 3D problems, the pointwise errors of displacements, strains and stresses in the domain are often expressed in a vector form and can be written as:

$$\mathbf{e}_u = \mathbf{u} - \mathbf{u}_h,\ \ \mathbf{e}_\varepsilon = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h,\ \ \mathbf{e}_\sigma \quad \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$$

in which $\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}$ are the exact solutions and $\mathbf{u}_h,\ {}_h,\ {}_h$ are the finite element results. The key problem is that the exact solution is not known *a priori* (except for simple problems). It is therefore essential to find ways to estimate the error.

In finite element analysis, we often use a smoothed continuous stress field $\boldsymbol{\sigma}_s$ to represent the exact solution $\boldsymbol{\sigma}$. This is based on an intuitive assumption that the smoothed continuous stress field $\boldsymbol{\sigma}_s$ is a better approximation than the discontinuous distribution $\boldsymbol{\sigma}_h$ directly provided by the finite element solutions.

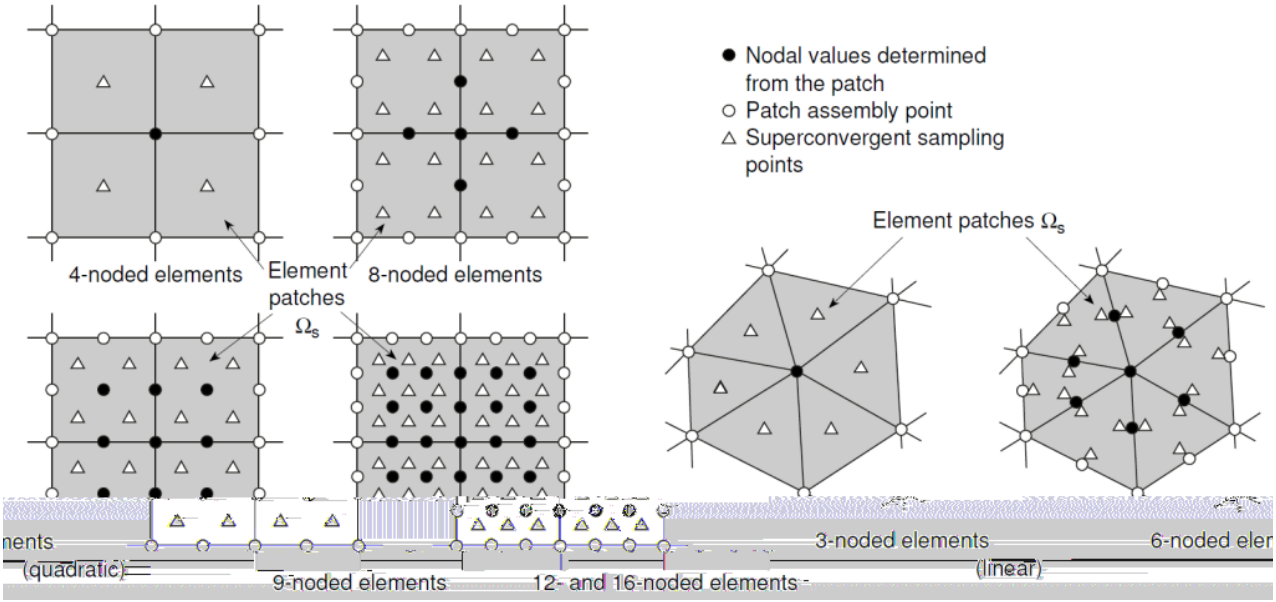
The simplest procedure to obtain the smoothed stress $\boldsymbol{\sigma}_s$ is to (1) obtain the unique nodal stresses use an extrapolation from Gauss points (often at optimal locations) followed by a nodal averaging of the stresses based on the number or the size of elements contributed to a node and (2) interpolate the smooth stress field using the shape functions with these nodal values. Below is a simple illustration of this procedure for a 1D bar under uniform axial forces analyzed with three linear elements.

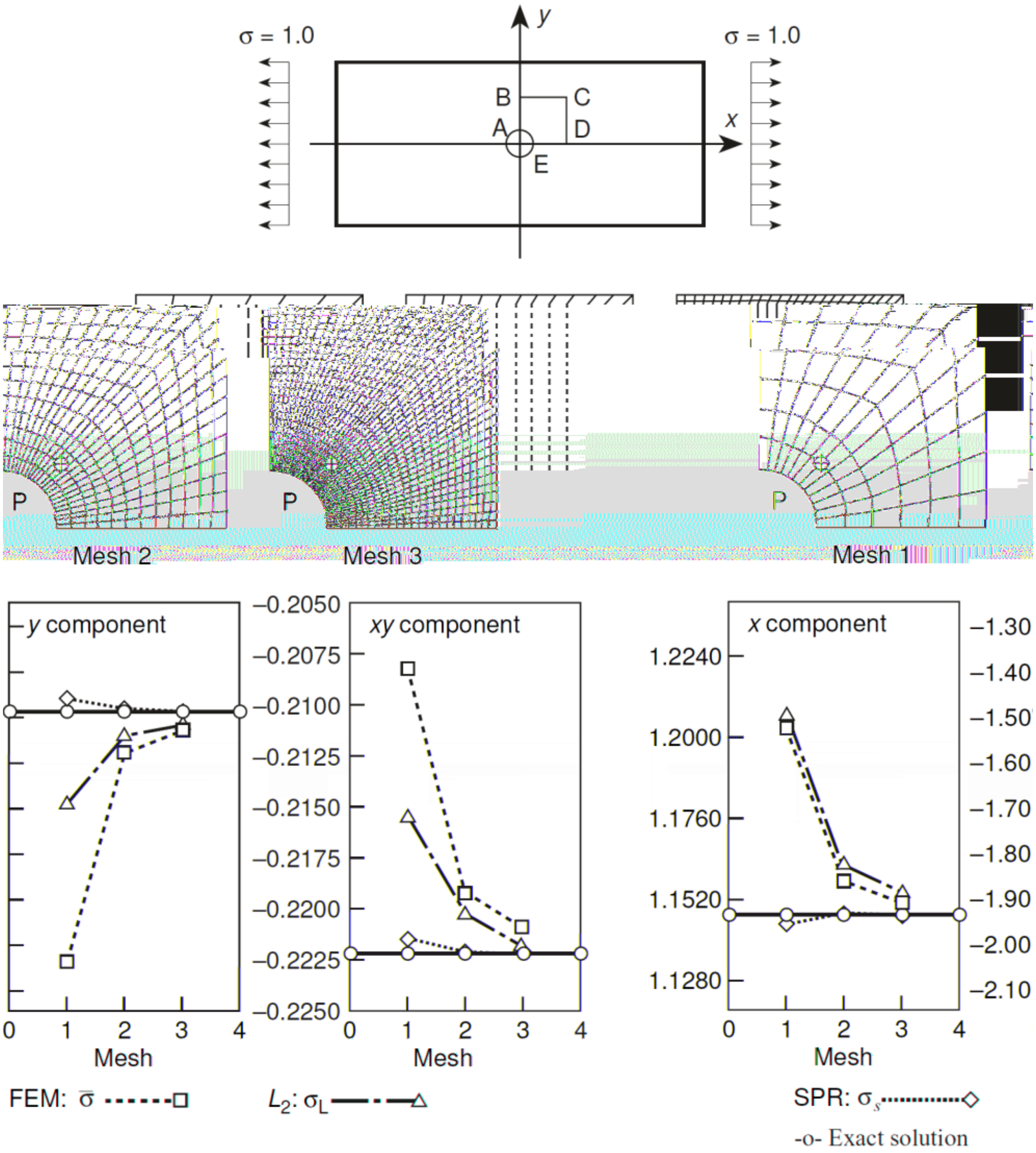


In addition to this simple averaging process, other techniques have been developed over the past 20 years to obtain the smoothed stress $\boldsymbol{\sigma}_s$. More recently for elliptical problems it has been shown that a **superconvergent patch recovery** (SPR) technique provides a way to recover accurate smoothed stress $\boldsymbol{\sigma}_s$ that can be used in an error estimator.

The name of SPR refers to the so called "superconvergence" property of the Gauss points (i.e., the stress values sampled at these points show an error which decreases more rapidly than elsewhere) Figure below shows some patches for the nodal computation of the stresses for linear and quadratic quadrilateral and triangular elements.

Figure below also shows an example of the efficiency of the SPR technique. The problem is the analysis of the stress field around a hole in a plate under uniaxial loading. The recovered SPR stresses show much improved values compared with the original FE solution and also with the standard L_2 projection, another way to obtain a smoothed stress field $\boldsymbol{\sigma}_s$.





Remarks:

1. The SPR method falls into the category of recovery-based methods: gradients of solutions obtained on a particular partition are smoothed and then compared with the gradients of the original solution to assess error.

2. Ainsworth and Oden in their book (Ainsworth, M. and Oden, T, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley, 2000 (e-access available from NTU Wiley subscription)) have carried out an extensive review of the most useful error estimation techniques.

5.4 Mesh Adaptation Strategy

A mesh adaptive strategy can be designed so that the element sizes are modified with the following two objectives:

1. To reach an optimum distribution of element sizes
2. To reduce the global error

The adaptation topic is a bit involved and please consult Onate (2009) Section 9.9 for details.