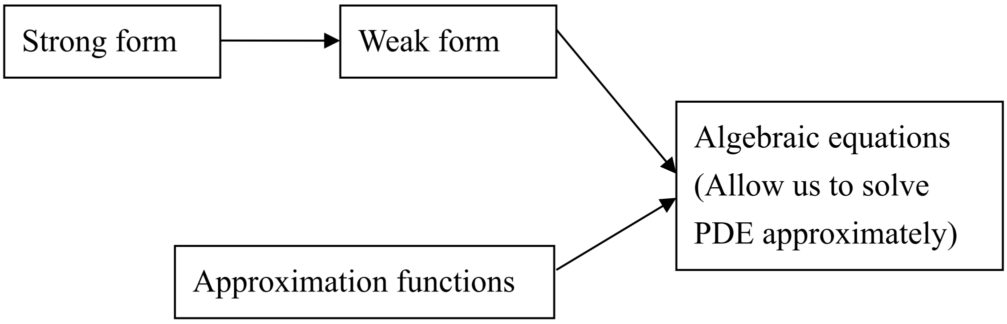
02/20/2019

1:



The finite element method (FEM) is a numerical approach by which the partial differential equations can be solved approximately via algebraic equations. A roadmap for the development of the finite element method is shown above.

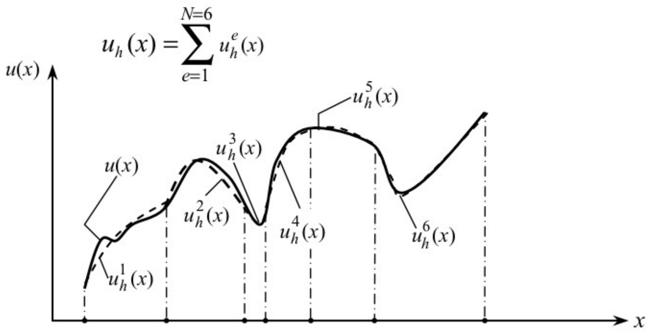
As can be seen from the roadmap, there are three distinct ingredients in FEM that are combined to arrive at the algebraic equations (for stress analysis they are called equations), which are then solved by a computer. These three ingredients are:

1. the strong form, which consists of the governing differential equations for the physical model and the boundary conditions;
2. the weak form, which converts the governing differential equations into an ;
3. the approximation functions to be combined with the weak form.

1: the weak form is the most intellectually challenging part in the development of finite elements, so you may encounter some difficulties in understanding this concept; it is probably different from anything else that you have seen before in engineering analysis. However, an understanding of these procedures and the implications of solving a weak form are crucial to understanding the character of finite element solutions. Furthermore, the **-** procedures to develop a weak form from differential equations and their boundary conditions are actually . Once it is understood for one strong form, the procedures can readily be applied to other strong forms.

2: we can classify the approximation functions to be combined with the weak form into (1) the classical way (before computer B.C.) and (2) the finite element way. The classical way from the classical methods (e.g., Galerkin, Ritz, and least square) cease to be effective because of the difficulty in constructing approximation functions for irregular domains in 2D and 3D.

3: finite element method differs from the classical methods in the manner in which are constructed. In finite element method, a given domain is viewed as a collection of subdomains (elements). Over each subdomain (element), the governing equations are approximated by the classical methods (often the Galerkin method). The main reason behind (elements) is the fact that it is easier to represent a complicated function as a collection of simple polynomials as illustrated below.



These ideas will be clearer in the sequel.

1.1

The quantities of interest in many areas of engineering and science are often to be found as , together with prescribed boundary and/or initial conditions. Examples are:

(Elliptic problems) stress analysis, heat transfer, hydrodynamics, electrostatics, magnetostatics etc.

<u>The stress analysis of a uniform axial bar</u>

$$EA\frac{d^2u(x)}{dx^2} = 0$$

(Parabolic problems) heat conduction, diffusion etc.

<u>The equation for conduction of heat in one dimension for a homogeneous body</u>

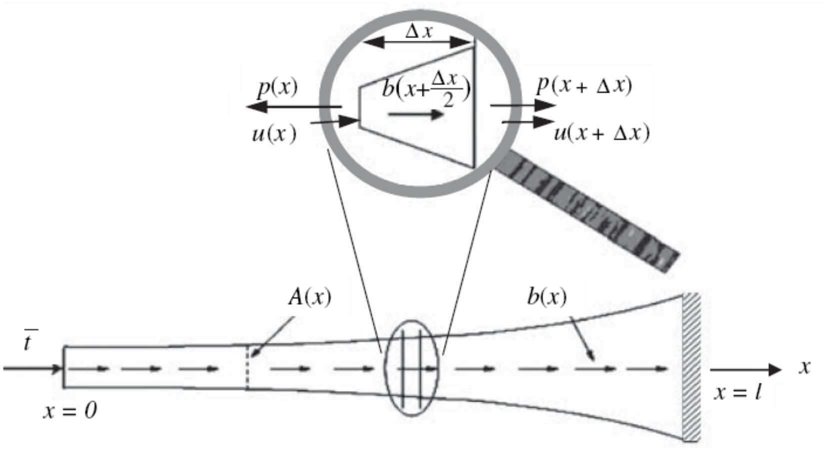
$$\alpha\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)$$

(Hyperbolic problems) sound waves, waves on strings and membrane etc.

One-dimensional longitudinal wave equation

$$\frac{\partial\ u}{\partial x}\ x\ t\ = \frac{\quad}{c}\frac{\partial\ u}{\partial t}\ x\ t$$

1.1.1 : (,)



: $u(x)$: displacement, $b(x)$: body force or distributed loading (force/length), $p(x)$: internal force, $A\ x$: the cross-sectional area, $\bar{t}$: traction (load prescribed at the end node)

: find the stress distribution $\sigma(x)$ in the bar.

: we will solve $u(x)$. From $u(x)$, we find strain $\varepsilon(x)$ and stress $\sigma(x)$.

The strong form for the problem consists of the and the .

Governing Differential Equation

The differential equation for the bar is obtained from equilibrium of internal force $p(x)$ and external force $b(x)$.

: What is the equilibrium condition in the -direction for a small element Δx

:

$$\sigma(x) = \frac{p(x)}{A(x)}$$

(1.2) $\varepsilon(x)=\lim\limits_{\Delta x\to 0}\dfrac{u(x+\Delta x)-u(x)}{\Delta x}=\dfrac{du}{dx}$

(1.3) $\sigma(x)=E(x)\varepsilon(x)$

where is the Young's modulus.

Substituting these relations into the differential equation from equilibrium, we then have a second-order differential equation in terms of $u(x)$:

(1.4) $\dfrac{d}{dx}(AE\dfrac{du}{dx})+b=0,\quad 0<x<l.$

:

The above is a - and we would like to $u\ x$.

The differential equation is a specific form of the equilibrium equation (1.1). Equation (1.1) applies to both linear and nonlinear materials whereas (1.4) assumes linearity in the definition of the strain (1.2) and the stress–strain law (1.3).

We need to prescribe boundary conditions at the two ends of the bar to solve the differential equations. For example, we can prescribe traction (force/area) and displacement boundary conditions at the ends.

$$\sigma(0)=\left(E\frac{du}{dx}\right)_{x=0}=\frac{p(0)}{A(0)}\equiv-\bar{t}$$

$$u(l)=\bar{u}$$

:

The superposed bars (e.g., $\bar{t}$ and $\bar{u}$) denote the prescribed boundary values.

The traction $\bar{t}$ has the same units as stress (force/area), but its sign is positive when it acts in the positive -direction regardless of which face it is acting on, whereas the stress is positive in tension and negative in compression, so that on a negative face (normal $n=-1$ at $x=0$), a positive stress corresponds to a negative traction.

In summary, the strong form consists of the governing differential equation and the boundary conditions. For this example, the strong form is:

(1.5a) $\frac{d}{dx}(AE\frac{du}{dx})+b=0, \quad 0<x<l.$

(1.5b) $\sigma(x=0)=\left(E\frac{du}{dx}\right)_{x=0}=\frac{p(0)}{A(0)}\equiv-\bar{t}$

(1.5c) $u(x=l)=\bar{u}$

1.2

To develop the finite element equations, the partial differential equations must be restated in an integral form called the . A weak form of the differential equations is equivalent to the governing equation and boundary conditions, i.e. the . In many disciplines, the weak form has specific names; for example, it is called the in stress analysis.

1.2.1 : (,)

There are to develop the weak form for any differential equations.

1: Move all the terms of the differential equation to one side (so it reads … = 0), multiple the entire equation with a function $w(x)$ (often called a function), and integrate over the domain.

(1.6) $\int_0^l \quad \frac{d}{dx}(\quad \frac{du}{dx}) \qquad 0$

: (1.6) is the of ( ). In general, it is difficult to obtain the exact solution that satisfies the strong form (1D is an exception). We therefore seek an approximate solution that satisfies the strong form in sense over an integral domain: We hope that is a good approximation of the exact solution.

2: Do integration by parts to reduce (i.e., weaken) continuity of .

One could use (1.6) to develop a finite element method. But because of the second derivative of in the expression, we need very smooth solutions. Such smooth solutions would be difficult to construct in 2D and 3D. Furthermore, the resulting stiffness matrix would not be

Let’s now apply integration by parts to (1.7) (hint: let $f=(AE\dfrac{du}{dx})$)

$$\int_0^l w\frac{d}{dx}(AE\frac{du}{dx})dx=(wAE\frac{du}{dx})\Big|_0^l-\int_0^l\frac{dw}{dx}AE\frac{du}{dx}dx$$

Thus, (1.7) can be written as follows:

$$\text{(1.8a)}\quad (wAE\frac{du}{dx})\Big|_0^l-\int_0^l\frac{dw}{dx}AE\frac{du}{dx}dx+\int_0^l wb\;dx=0$$

$$\text{(1.8b)}\quad (wA\sigma)\Big|_0^l-\int_0^l\frac{dw}{dx}AE\frac{du}{dx}dx+\int_0^l wb\;dx=0$$

$$\text{(1.8c)}\quad (wA\sigma)_{x=l}-(wA\sigma)_{x=0}-\int_0^l\frac{dw}{dx}AE\frac{du}{dx}dx+\int_0^l wb\;dx=0$$

3: Impose the actual boundary conditions under consideration. It is here that we require the weight function $w(x)$ to (in our case, prescribed displacement boundaries and $w(l)=0$).

Thus, the first term in (1.8c) vanishes! This is why it is convenient to construct weight functions that vanish on prescribed displacement boundaries.

We now come to the most important part of this weak form development:
( ) $w(x)$
$w(l)=0$. In other word, the solution is obtained as follows:

> Find $u(x)$ among the smooth functions that satisfy $u\;l\;=\bar{u}$ such that
>
> $$\text{(1.9)}\quad \int_0^l\frac{dw}{dx}AE\frac{du}{dx}dx=(wA\bar{t})_{x=0}+\int_0^l wb\;dx\qquad \forall w \text{ with } w(l)=0$$

The above is called the weak form. The name originates from the fact that solutions $u(x)$ to the weak form , i.e.
for the solution $u(x)$.

:

- We skip the proof of the statement. That is, the trial solution that satisfies weak form is the solution of the strong form. Please consult F&B (Chapter 3) for details.
- It is important to remember that the solutions $u(x)$ must satisfy the prescribed displacement boundary conditions (1.5c). Satisfying the displacement boundary condition is essential for the solutions, so these boundary conditions are often called (Dirichlet boundary conditions).
- We see that the prescribed traction boundary conditions emanate naturally from the weak form, so solutions need not be constructed to satisfy the traction boundary conditions. Therefore, these boundary conditions are called (Neumann boundary conditions).
- A trial solution that is smooth and satisfies the essential boundary conditions is called . Similarly, a weight function that is smooth and vanishes on essential boundaries is . When weak forms are used to solve a problem, the trial solutions and weight functions must be .
- Note that in (1.9), the integral is symmetric in and . This will lead to .
- The way we obtained the weak form is called the weighted integral method. It also called the ( ) as we seek a solution where the residual between the approximate solution and exact solution is minimized. There are many MWR criteria and four of the most popular MWR criteria are (1) the collocation method (2) the subdomain method (3) the least-square method (4) the Galerkin method. We will primarily focus on the Galerkin method.

1: develop the weak form for the strong form

(a) $\frac{d}{dx}\ AE\frac{du}{dx}\ = !\quad Ax \qquad < x <$

(b) $u_{x=}\ = u \quad = \ ^{-}$

(c) $\sigma_{x=2} = \left(E\frac{du}{dx}\right)_{x=2} = 10$

( )

1.2.2 : 1

We will now consider a more general 1D situation, where instead of specifying a stress boundary condition at $=0$ and a displacement boundary condition at $x=l$, displacement and stress boundary conditions can be prescribed at either end. For this purpose, we will need more general notations for the boundaries.

Let Ω indicate the domain. The boundary of the one-dimensional domain, which consists of two end points, is denoted by Γ. The portion of the boundary where the displacements are prescribed is denoted by Γ_u; the boundary where the traction is prescribed is denoted by Γ_t.

Notice that in this course, we will limit our discussion on the weak form formulation that the trial solution $u\ x$ to satisfy the essential boundary conditions and the weigh function $w(x)$ to vanish at the essential boundary conditions.

Thus we can write the weak form for 1D stress analysis:

(1.10)

Find $u(x)$ among the smooth functions that satisfy $u=\bar{u}$!"#! Γ_u such that

$$\int_\Omega \frac{dw}{dx} AE \frac{du}{dx} dx = (wA\bar{t})\big|_{\Gamma_t} + \int_\Omega wb\ dx \quad w=0 \text{ on } \Gamma_u$$

1.2.3 :

A trial solution $u(x)$ is often constructed in a form of a finite sum of functions:

$$(1.10) \quad u(x)=\sum_{i=0}^{n} a_i \phi_i(x)$$

The functions $\phi_i(x)$ are functions and sometimes called and the coefficients of a_i are <u>undetermined parameters to be solved</u>.

:

- When the <u>basis functions</u> are used not only for <u>trial solutions</u> but also for <u>weight functions</u>, we have the weight residual method.

- In choosing basis functions $\phi_i(x)$, an important practical consideration is to use functions that are easy to work with because we frequently calculate derivatives and integrals of $\phi_i(x)$. Thus, a logical choice would be the first few terms of a .

2( ): obtain the solution to the weak form in **1** by using trial solution and weight function of a linear polynomial form:

$u(x) = \alpha_0 + \alpha_1 x$

$w(x) = \beta_0 + \beta_1 x$

where α_0 and α_1 are unknown parameters and β and $_1$ are arbitrary parameters. Assume that is constant and $E = 10^5$.

:

2( ): now obtain the solution to the weak form in **1** by using trial solution and weight function of a quadratic form:

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

$$w\ x\ = \beta\ + \beta x + \beta\ x$$

Compare the results from (a) and (b) with the exact solution given by:

$$u^{exact}(x) = 10^{-4}(1 + 3x - \frac{x^3}{6}) \text{ and } \sigma^{exact}(x) = 10(3 - \frac{x^2}{2})$$

( )

Now we can plot and compare the approximate solutions with the exact solution. A sample MATLAB code follows:

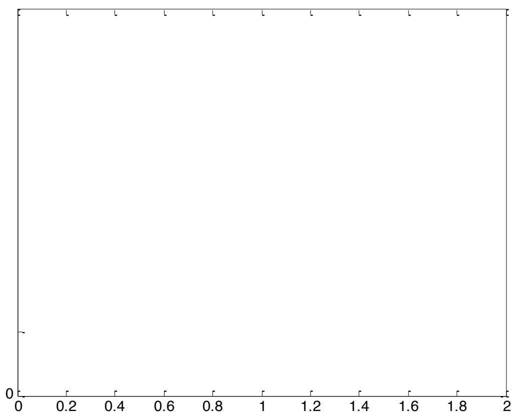
```
x=0:0.01:2;
u1=(1+7/3*x)/10000;
u2=(1+10/3*x-0.5*x.^2)/10000;
ue=(1+3*x-1/6*x.^3)/10000;

s1=70/3;
s2=10*(10/3-x);
se=(3-0.5*x.^2)*10;

plot(x,u1,'-.');
hold on;
plot(x,u2,'--');
plot(x,ue,'-');
legend('u linear','u quadratic','u exact',2);
xlabel('x');ylabel('displacement, u');

plot(x,s1,'-.');
hold on;
plot(x,s2,'--');
plot(x,se,'-');
legend('stress linear','stress quadratic','stress exact',1)
xlabel('x');ylabel('stress');
```

And then the resulting plots:



: What do you observe?

:

3 (   ):

(a) What are the requirements for the admissible solutions and admissible weight functions when developing a weak form for a strong form in the classical Galerkin method?

(   )

(b) Consider the differential equation in the range , with essential boundary conditions and . Write down an admissible polynomial solution with only <u>one</u> parameter left to be solved.

(   )

,

1. We have studied the major ingredients that are of immediate interest in the study of the finite element method. A three-step procedure for developing the weak form of differential equations was presented and methods for obtaining algebraic equations in the approximate solution were developed.

2. , i.e. for the solution .