Name: ____________________________________

Student ID: ________________________________

National Taiwan University Department of Civil Engineering	**Finite Element Method** Instructor: C.-S. CHEN

Midterm Examination, April 25, 2019
Open Printed Books and Notes, **Time: 180 minutes**
Return the Exam Sheet (試卷隨答案卷繳回)

Total 100%

1. (8%) Consider a tapered bar fixed at one end and subjected to a static point load at the other end as shown in Figure 1. The bar is also subjected to a varying body load $b(x) = x^4$. The Young's modulus is constant and the area varies linearly from A_0 to A_L where

$$A(x) = \frac{(L + (-1 + r)x)A_0}{L} \qquad \text{in which} \quad r = \frac{A_L}{A_0}$$

(a) (4%) If we use the quadratic element to mesh the domain, what is the minimum number of Gauss points we need to numerically integrate the element stiffness matrix $\mathbf{K}^e$ exactly?

$$\mathbf{K}^{\mathbf{e}} = \int \mathbf{B}^{\mathrm{T}} \mathrm{AE} \mathbf{B} \mathrm{dx}$$

$$\mathrm{n}_{gp} \geq \frac{(1+1+1)+1}{2} = 2 \;\rightarrow\; \text{Min.} \quad \mathrm{n_{gp}} = 2$$

(b) (4%) If we use the cubic element to mesh the domain, what is the minimum number of Gauss points we need to numerically integrate the element external force matrix $\mathbf{f}^e$ exactly?

$$\mathbf{f}^{\mathbf{e}} = \int \mathbf{N}^{\mathbf{T}} \mathrm{b(x)dx}$$

$$\mathrm{n_{gp}} \geq \frac{(3+4)+1}{2} = 4 \;\rightarrow\; \text{Min.} \quad \mathrm{n_{gp}} = 4$$

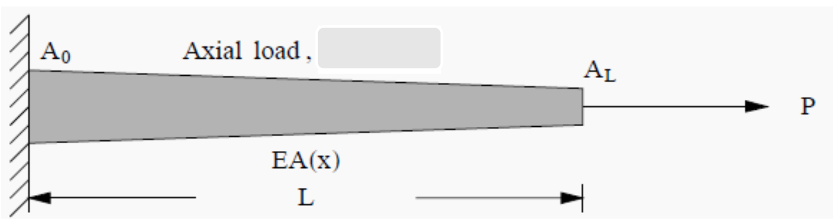


Figure 1

2. (12%) For a viscous fluid, the velocities at various depths are measured as follows:

Velocity (m/s)	3.75	4	7.5
Depth (m)	0	1.5	3.0

Use the Lagrange interpolation to obtain an expression for fluid velocity as a function of depth (d) and obtain the velocity when $d = 6$.

$$\mathrm{v(d)} = 3.75 \times \frac{(d-1.5)(d-3)}{(-1.5)\times(-3)} + 4 \times \frac{d(d-3)}{1.5\times(-1.5)} + 7.5 \times \frac{d(d-1.5)}{3\times 1.5} \quad (8\%)$$

$$\mathrm{v(6)} = 24.25 \ (3\%) \ \ \mathrm{m/s} \ (1\%)$$

3. (10%) Evaluate the integral

$$I = \int_{-2}^{3} \frac{x^2-1}{(x+3)^2}\,dx$$

using Gaussian integration with two Gauss points.

$$\xi_1 = -\frac{1}{\sqrt{3}} \quad , \quad \xi_2 = \frac{1}{\sqrt{3}}$$

$$\mathrm{w}_1 = 1 \quad , \quad \mathrm{w}_2 = 1$$

$$\mathrm{J} = \frac{1}{2}\big(3-(-2)\big) = \frac{5}{2} (2\%) \quad , \quad x = \frac{1}{2} + \frac{5}{2}\xi (2\%)$$

$$\mathrm{I} = \frac{5}{2}\left(1 \times \frac{\left[\frac{1}{2}+\frac{5}{2}\left(-\frac{1}{\sqrt{3}}\right)\right]^2 - 1}{\left[\left(\frac{1}{2}+\frac{5}{2}\left(-\frac{1}{\sqrt{3}}\right)\right)+3\right]^2} + 1 \times \frac{\left[\frac{1}{2}+\frac{5}{2}\left(\frac{1}{\sqrt{3}}\right)\right]^2 - 1}{\left[\left(\frac{1}{2}+\frac{5}{2}\left(\frac{1}{\sqrt{3}}\right)\right)+3\right]^2}\right) = 0.2190 \ (5\%)$$

4. (16%) Consider a quadratic element in Figure 2. (a) Compute the Jacobian using the isoparametric formulation (note: ξ is now ranging from -1 to 1) and (b) derive the element strain ε_x.

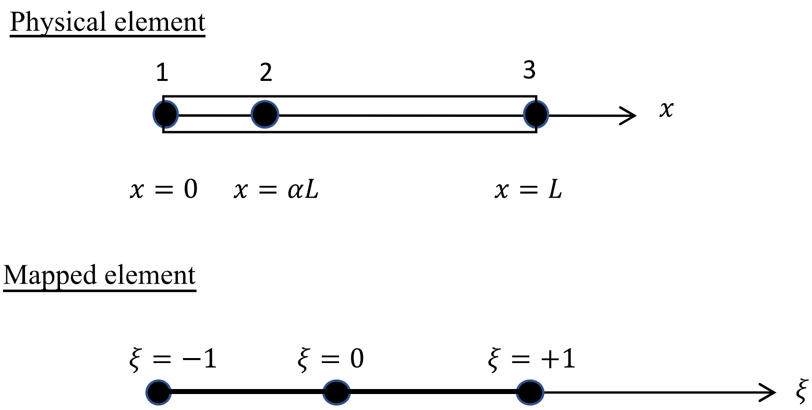


Figure 2

$$\mathrm{J}=\frac{\mathrm{dx}}{\mathrm{d\xi}}=\frac{dN(\xi)}{d\xi}\begin{bmatrix}0\\ \alpha L\\ L\end{bmatrix}\;,N(\xi)=\begin{bmatrix}\frac{1}{2}(\xi^2-\xi) & 1-\xi^2 & \frac{1}{2}(\xi^2+\xi)\end{bmatrix}$$

(a)

$$\mathrm{J}=\begin{bmatrix}\xi-\frac{1}{2} & -2\xi & \xi+\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\ \alpha L\\ L\end{bmatrix}(4\%)=-2\alpha\xi L+\xi L+\frac{1}{2}L(4\%)$$

(b)

$$\varepsilon=\frac{\mathrm{dN}}{\mathrm{dx}}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\frac{dN}{d\xi}\times\frac{d\xi}{dx}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}(4\%)$$

$$=\begin{bmatrix}\xi-\frac{1}{2} & -2\xi & \xi+\frac{1}{2}\end{bmatrix}\times\frac{1}{-2\alpha\xi L+\xi L+\frac{1}{2}L}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}(4\%)$$

5. (24%) Consider problem of finding axial displacement of a truncated solid cone made of concrete, hanging under its own weight and subjected to a downward load $F=100$ kN at the tip, as illustrated in Figure 3 below. Let the length $L=5$ m, the diameter at the top is $d_0=1$ m and changes linearly to $d_L=\frac{1}{4}$ m at the bottom. The concrete weighs $24\ \mathrm{kN/m^3}$ and its modulus is $E=2\times10^7\ \mathrm{kN/m^2}$.
 (a) (5%) State the strong form (governing equation and boundary conditions).
 (b) (10%) Construct the element stiffness matrix and element external force matrix using two linear elements equally distributed as shown below. The linear element needs to take the linear cross section into consideration. Assemble these two elements to obtain the global stiffness matrix and global external force matrix.
 (c) (5%) Find the nodal displacements.
 (d) (4%) Find the element nodal forces and draw the free body diagram at node 2. Is the

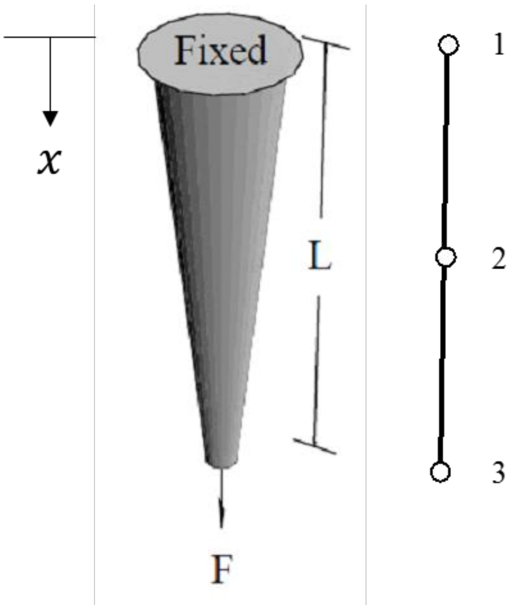


Figure 3

(a)

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(EA\frac{du}{dx}\right)+b(x)=0 \quad (1\%)$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[E\left(-\frac{3}{40}x+\frac{1}{2}\right)^2\pi\frac{du}{dx}\right]+24\left(-\frac{3}{40}x+\frac{1}{2}\right)^2\pi=0 \quad (1\%)$$

$0<\mathrm{x}<5$ (1%)

$$\sigma(5)=\frac{100}{\left(\frac{1}{8}\right)^2\pi}=2037.18\ \frac{kN}{m^2} \quad (1\%)$$

$u(0)=0\ m$ (1%)

(b)

$\mathbf{K}=\int \mathrm{EA}\mathbf{B}^{\mathrm{T}}\mathbf{B}$

Element 1

$$\mathbf{N}^1(\mathbf{x})=\frac{2}{5}[2.5-x \quad x] \quad , \quad \mathbf{B}^1(\mathbf{x})=\frac{2}{5}[-1 \quad 1]$$

$$\mathbf{K}^1=\int_0^{2.5}2\times10^7\left(-\frac{3}{40}x+\frac{1}{2}\right)^2\pi\times\frac{4}{25}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx=\begin{bmatrix}0.422 & -0.422\\ -0.422 & 0.422\end{bmatrix}\times10^7$$

Element 2

$$\mathbf{N}^2(\mathbf{x})=\frac{2}{5}[5-x \quad -2.5+x] \quad , \quad \mathbf{B}^2(\mathbf{x})=\frac{2}{5}[-1 \quad 1]$$

$$\mathbf{K}^2=\int_{2.5}^{5}2\times10^7\left(-\frac{3}{40}x+\frac{1}{2}\right)^2\pi\times\frac{4}{25}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx=\begin{bmatrix}0.1276 & -0.1276\\ -0.1276 & 0.1276\end{bmatrix}\times10^7$$

$$\mathbf{K}=\begin{bmatrix}0.422 & -0.422 & 0\\ -0.422 & 0.5496 & -0.1276\\ 0 & -0.1276 & 0.1276\end{bmatrix}\times10^7\ \frac{kN}{m} \quad (5\%)$$

$\boldsymbol{f} = \int \mathrm{N}^{\mathrm{T}} b(x) dx$

Element 1

$$f^1 = \int_0^{2.5} \frac{2}{5}\begin{bmatrix} 2.5 - x \\ x \end{bmatrix} \times 24 \times \left(-\frac{3}{40}x + \frac{1}{2}\right)^2 \pi \ dx = \begin{bmatrix} 18.22 \\ 13.44 \end{bmatrix} \ kN$$

Element 2

$$f^2 = \int_{2.5}^{5} \frac{2}{5}\begin{bmatrix} 5 - x \\ -2.5 + x \end{bmatrix} \times 24 \times \left(-\frac{3}{40}x + \frac{1}{2}\right)^2 \pi \ dx = \begin{bmatrix} 6.07 \\ 3.50 \end{bmatrix} \ kN$$

$$f = \begin{bmatrix} 18.22 \\ 13.44 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 6.07 \\ 3.50 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 18.22 \\ 19.51 \\ 103.50 \end{bmatrix} \ kN \quad (5\%)$$

(c)

$\mathbf{D} = \mathbf{K}^{-1}\mathbf{F}$

$$\begin{bmatrix} 0.422 & -0.422 & 0 \\ -0.422 & 0.5496 & -0.1276 \\ 0 & -0.1276 & 0.1276 \end{bmatrix} \times 10^7 \times \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 18.22 + r_1 \\ 19.51 \\ 103.50 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix}^{-1} \begin{bmatrix} 19.512 \\ 103.500 \end{bmatrix} = \begin{bmatrix} 2.914 \times 10^{-5} \\ 1.102 \times 10^{-4} \end{bmatrix} m \quad (3\%)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.914 \times 10^{-5} \\ 1.102 \times 10^{-4} \end{bmatrix} \quad (2\%)$$

$\mathrm{r}_1 = -141.24 \ \mathrm{kN}$

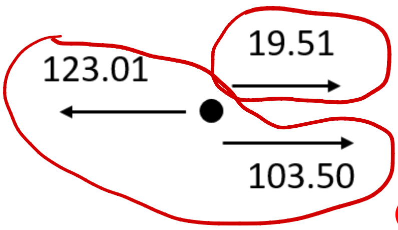
(d)

Element force 1

$$\begin{bmatrix} 4221515.13 & -4221515.13 \\ -4221515.13 & 4221515.13 \end{bmatrix} \begin{bmatrix} 0 \\ 2.914 \times 10^{-5} \end{bmatrix} = \begin{bmatrix} -123.01 \\ 123.01 \end{bmatrix} \ kN \quad (1\%)$$

Element force 2

$$\begin{bmatrix} 1276272.02 & -1276272.02 \\ -1276272.02 & 1276272.02 \end{bmatrix} \begin{bmatrix} 2.914 \times 10^{-5} \\ 1.102 \times 10^{-5} \end{bmatrix} = \begin{bmatrix} -103.50 \\ 103.50 \end{bmatrix} \ kN \quad (1\%)$$



(2%)

6. (16%) A uniform bar is subjected to uniform axial load q along its length. The bar is fixed at the left end but there is a gap of g between the support and the right end before the load is applied as shown in Figure 4. Assuming that the applied load is large enough to close the gap.
(a) (3%) Write down the admissible solution $u(x)$ in terms of a quadratic polynomial.

(b) (3%) Write down the admissible weight function $w(x)$ in terms of a quadratic polynomial.

(c) (10%) Use trial solution and weight function of a quadratic polynomial and obtain the approximate solution using the Galerkin method.

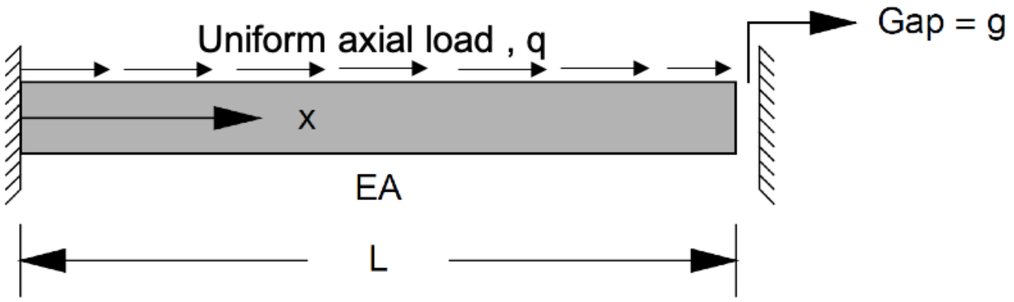


Figure 4

(a)

$u(0) = 0 \quad , \quad u(L) = g$

$u(x) = \alpha_1 x^2 + \alpha_2 x$

$u(L) = \alpha_1 L^2 + \alpha_2 L = g \quad , \quad \alpha_2 = \dfrac{g - \alpha_1 L^2}{L}$

$u(x) = \alpha_1 x^2 + \left(\dfrac{g}{L} - \alpha_1 L\right) x$ (3%)

(b)

$w(0) = 0 \quad , \quad w(L) = 0$

$w(x) = \beta_1 x^2 + \beta_2 x$

$w(L) = \beta_1 L^2 + \beta_2 L = 0 \quad , \beta_2 = -\beta_1 L$

$w(x) = \beta_1 x^2 - \beta_1 L x$ (3%)

(c)

strong form

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + q = 0 \quad , \quad 0 < x < L \quad , \quad \sigma(L) = \frac{Eg}{L}$$

weak form

$$wA\frac{Eg}{L}\bigg|_{x=L} - \int_0^L \frac{dw}{dx} AE \frac{du}{dx} dx + \int_0^L wq\, dx = 0$$

$$(B_1 L^2 - \beta_1 L^2)A\frac{Eg}{L} - \int_0^L (2\beta_1 x - \beta_1 L)AE\left(2\alpha_1 x + \left(\frac{g}{L} - \alpha_1 L\right)\right)dx + \int_0^L (\beta_1 x^2 - \beta_1 L x) q dx = 0$$

$$\alpha_1 = -\frac{q}{2AE}$$

$$u(x) = -\frac{q}{2AE}x^2 + \left(\frac{g}{L} + \frac{qL}{2AE}\right)x \quad (10\%)$$

7. (14%) Consider a general structural member subjected to a torsion T acting about the x axis shown in Figure 5a. An arbitrary point located on a cross section at position x is shown in

Figure 5b. If the cross section twists through angle θ, the point moves through arc ds and the displacement components in the y and z directions are:

$$v = -z\theta, w = y\theta$$

Owing to the noncircular cross section, plane sections do not remain plane; instead, there is warping, hence displacement, in the x direction described by:

$$u = u(y, z)$$

If we further introduce the angle of twist per unit length ϕ such that the rotation of any cross section can be expressed as $\theta = \phi x$. The displacement components are then expressed as:

(1) $u = u(y, z), v = -z\phi x, w = y\phi x$

(a) (8%) Compute the strain components based on Eq. (1).

(b) (6%) Assume there exists a scalar function ψ such that

$$\tau_{xy} = \frac{\partial \psi}{\partial z} \text{and} \tau_{xz} = -\frac{\partial \psi}{\partial y}$$

Derive the governing equation in terms of ψ, G and ϕ.

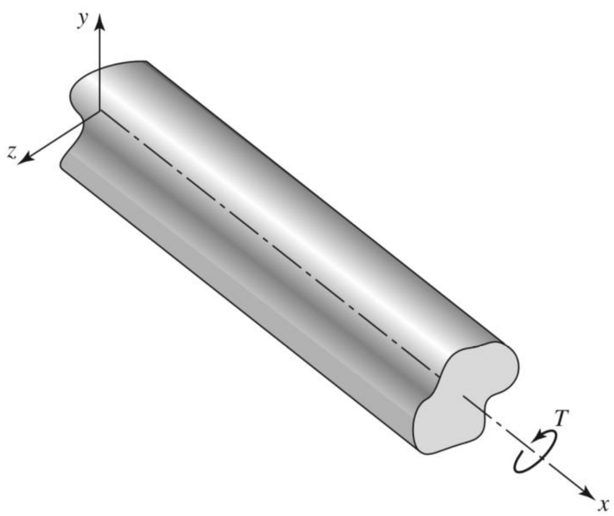


Figure 5a: a general structural member with a non-circular section in torsion

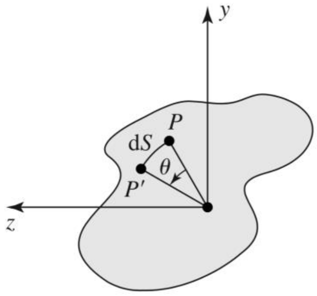


Figure 5b: motion of a point from P to as a result of cross section rotation

(a) (8%)

$u = u(y,z), v = -z\phi x, w = y\phi x$

$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \quad , \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \quad , \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$

$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - z\phi\right)$

$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(-\phi x + \phi x) = 0$

$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial z} + y\phi\right)$

(b)

$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$

$\tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz}$

$$\frac{\partial \psi}{\partial z} = 2G\left[\frac{1}{2}\left(\frac{\partial u}{\partial y} - z\phi\right)\right]$$

$$-\frac{\partial \psi}{\partial y} = 2G\left[\frac{1}{2}\left(\frac{\partial u}{\partial z} + y\phi\right)\right]$$

(4%)

$$\frac{\partial^2 \psi}{\partial z^2} = G\left(\frac{\partial^2 u}{\partial y \partial z} - \phi\right)$$

$$-\frac{\partial^2 \psi}{\partial y^2} = G\left(\frac{\partial^2 u}{\partial y \partial z} + \phi\right)$$

$$\therefore \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2G\phi$$ (2%)