National Taiwan University | **Finite Element Method**
Department of Civil Engineering | Instructor: C.-S. CHEN

Homework 1, 03/15/2018 Due: 03/21/2018

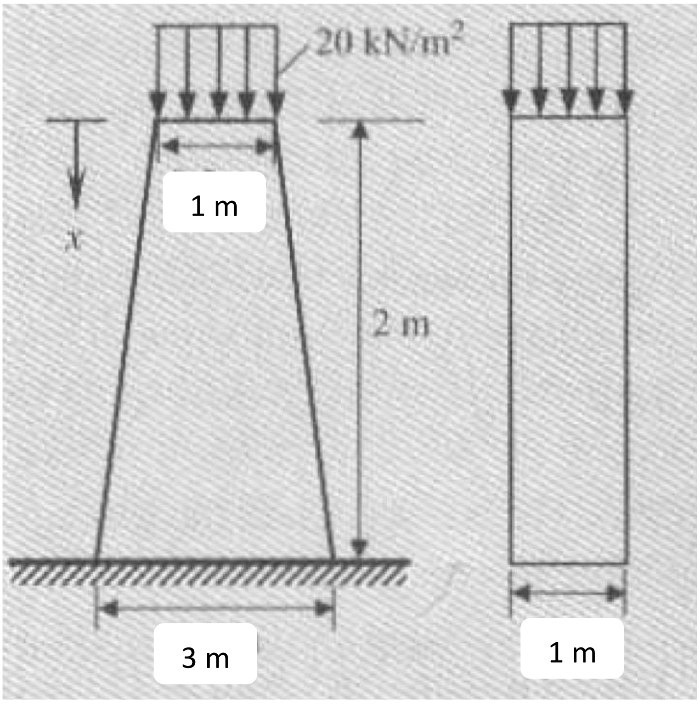
A4 professional format, collecting at the BEGINNING of class (09:09 am)

(late submission within 24 hours: score*0.9; late submission before post of solution: score*0.8

(the solution will be posted usually within a week))

Total 80%

1. **(50%)** A bridge is supported by several concrete piers, and the geometry and loads of a typical pier are shown in the figure below. The load $20\ \text{kN/m}^2$ represent the weight of bridge and a distribution of the traffic on the bridge. The concrete weighs $24\ \text{kN/m}^3$ and its modulus is $E = 2\times10^7\ \text{kN/m}^2$. We wish to analyze the pier for displacements and stresses.
   (a) (10%) Derive the strong form (governing equation and boundary conditions).
   (b) (10%) Derive the weak form with admissible trial solutions and weight functions.
   (c) (10%) Obtain the displacements and stresses to the weak form by using trial solution and weight function of a linear polynomial.
   (d) (10%) Obtain the displacements and stresses to the weak form by using trial solution and weight function of a quadratic polynomial.
   (e) (10%) Obtain the exact solutions and plot the comparisons with approximate displacements and stresses from (c) and (d).



(a) (10%)

$A(x) = 1 + x \ (\mathrm{m}^2)$
$b(x) = 24(1+x) \ (\mathrm{kN/m})$
$E = 2 \times 10^7 \ (\mathrm{kN/m}^2)$

$G.E.$

$$\frac{d}{dx}(A(x)E\frac{du}{dx}) + b(x) = 0 \Rightarrow \frac{d}{dx}(2\times 10^7(1+x)\frac{du}{dx}) + 24(1+x) = 0 \ (3\%)$$

$0 < x < 2$ (2%)
$B.C.$

$$\sigma(0) = E\frac{du}{dx}\bigg|_{x=0} = -20 \ (2\%) \ \mathrm{kN/m}^2 \ (1\%)$$

$u(2) = 0$ (2%) m

(b) (10%)

$$\int_0^2 w\frac{d}{dx}[2\times 10^7(1+x)\frac{du}{dx} + 24(1+x)]dx = 0$$

$$2\times 10^7(1+x)w\frac{du}{dx}\bigg|_0^2 - \int_0^2 2\times 10^7(1+x)\frac{du}{dx}\frac{dw}{dx}dx + \int_0^2 w[24(1+x)]dx = 0$$

Find $u(x)$ among the smooth functions that satisfy $u(2)=0$ such that

$$20w(0) - \int_0^2 2\times 10^7(1+x)\frac{du}{dx}\frac{dw}{dx}dx + \int_0^2 w[24(1+x)]dx = 0 (9\%) \quad \forall w \text{ with } w(2) = 0 \ (1\%)$$

(c) (10%)

$$\begin{cases} u = a_0 + a_1 x \\ w = b_0 + b_1 x \end{cases} (1\%) \Rightarrow \text{ to be admissible} \begin{cases} u(2) = 0 \rightarrow u = a_1(x-2) \\ w(2) = 0 \rightarrow w = b_1(x-2) \end{cases} (1\%)$$

$$\int_0^2 2\times 10^7(1+x)a_1 b_1 dx + 40b_1 - \int_0^2 24b_1(x-2)(x+1)dx = 0 (2\%)$$

$\Rightarrow b_1$ arbitrary, $a_1 = -1.5\times 10^{-6}$

$$\begin{cases} u(x) = -1.5\times 10^{-6}(x-2) \ (2\%) \ \mathrm{m} \ (1\%) \\ \sigma(x) = E\frac{du}{dx} = -30 \ (2\%) \ \mathrm{kN/m}^2 \ (1\%) \end{cases}$$

(d) (10%)

$$\begin{cases} u = a_0 + a_1 x + a_2 x^2 \\ w = b_0 + b_1 x + b_2 x^2 \end{cases} \text{(1\%)} \Rightarrow \text{ to be admissible} \begin{cases} u(2)=0 \to u = a_1(x-2) + a_2(x^2-4) \\ w(2)=0 \to w = b_1(x-2) + b_2(x^2-4) \end{cases} \text{(1\%)}$$

$$40b_1 + 80b_2 + \int_0^2 2\times10^7(1+x)(a_1+2a_2x)(b_1+2b_2x) - 12[b_1(x-2)+b_2(x^2-4)]dx = 0 \text{(2\%)}$$

$$\Rightarrow b_1\ ,\ b_2 \text{ arbitrary, } a_1 = -9.28\times10^{-7},\ a_2 = -2.46\times10^{-7}$$

$$\begin{cases} u(x) = 10^{-7}(28.3 - 9.27x - 2.45x^2) \text{ (2\%)} \quad \text{m (1\%)} \\ \sigma(x) = E\dfrac{du}{dx} = -18.55 - 9.82x \text{ (2\%)} \quad \text{kN/m}^2 \text{ (1\%)} \end{cases}$$

(e) (10%)

$$\frac{d}{dx}[(1+x)E\frac{du}{dx}] + 24(1+x) = 0 \Rightarrow (1+x)E\frac{du}{dx} = -12(1+x)^2 + c_1$$

$$\because \sigma(0) = E\frac{du}{dx}\Big|_{x=0} = -20 \quad \therefore \text{let } x=0,\ c_1 = -8$$

$$\frac{du}{dx} = \frac{-1}{E}[12(1+x) + \frac{8}{1+x}] \Rightarrow u(x) = \frac{1}{E}[-12x - 6x^2 - 8\ln(1+x) + c_2] \quad \text{(1\%)}$$

$$\because u(2) = 0 \therefore \text{let } x=2,\ c_2 = 48 + 8\ln 3$$

$$\begin{cases} u_{exact}(x) = \dfrac{1}{E}[-12x - 6x^2 - 8\ln(1+x) + 48 + 8\ln 3] \text{ (1\%)} \quad \text{m (1\%)} \\ \sigma_{exact}(x) = E\dfrac{du}{dx} = -12 - 12x - \dfrac{8}{1+x} \text{ (1\%)} \quad \text{kN/m}^2 \text{ (1\%)} \end{cases}$$

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%FEM HW1 ( 5%: code3%, figure2% )
clc;clear;close all;

% displacement
x=0:0.01:2;
u1=-15*10^-7*(-2+x); %linear
u2=(56.74-18.55.*x-4.91.*x.^2)/(2*10^7); %quadratic
u_exact=(48+8*log(3)-12.*x-6.*x.^2-8.*log(1+x))/(2*10^7); %exact
figure(1);
plot(x,u1,'-.','LineWidth',2);hold on;
plot(x,u2,'k--','LineWidth',2);
plot(x,u_exact,'-','LineWidth',1);
title('Displacement');
xlabel('x (m)')
ylabel('u(x) (m)')
legend('u linear','u quadratic','u exact',2);

%stress
s1=-30;
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s2=-18.55-9.82.*x;
s_exact=-12-12.*x-8./(1+x);
figure(2);
plot([0 2],[s1 s1],'-.','LineWidth',2);hold on;
plot(x,s2,'k--','LineWidth',2);
plot(x,s_exact,'-','LineWidth',1);
title('Stress');
legend('stress linear','stress quadratic','stress exact',1)
xlabel('x (m)');
ylabel('stress (kN/m^2)');
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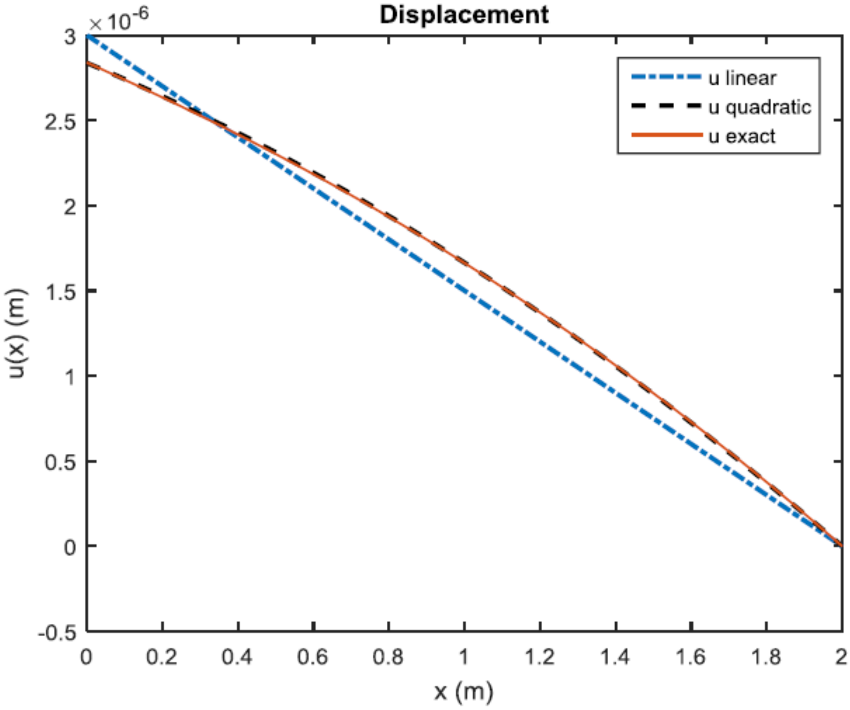

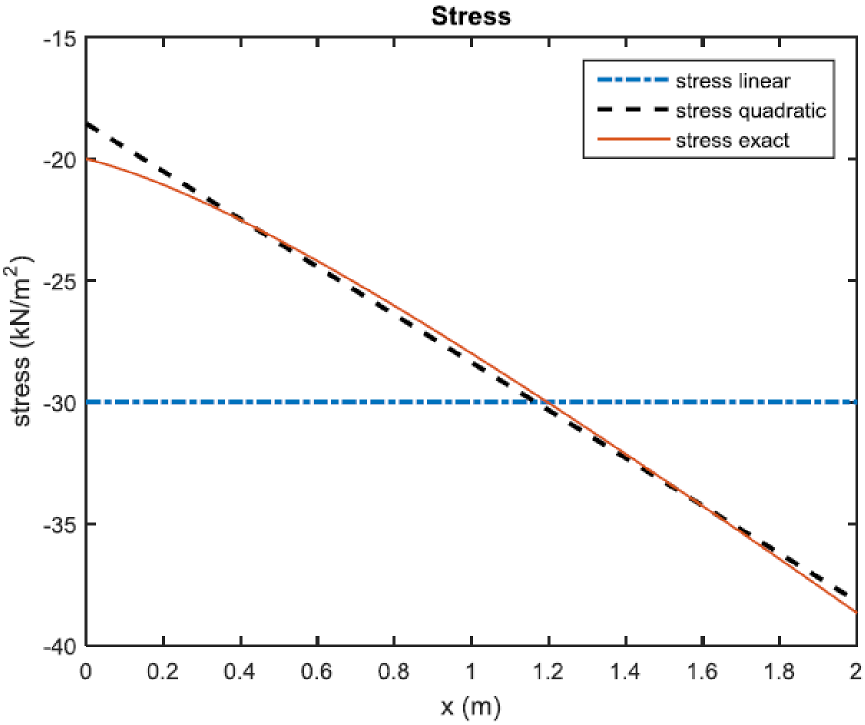

2. **(30%)** An engineering analysis problem is formulated in terms of the following ordinary differential equation:

$$-\frac{d^2u}{dx^2} = x^2, \quad 0 < x < 1.$$
$$u(0) = u(1) = 0.$$

(a) (15%) Develop the corresponding weak form from the strong form.

(b) (15%) Obtain the solution to the weak form by using trial solution and weight function of a cubic form:

$$u(x) = x(1-x)(\alpha_0 + \alpha_1 x)$$
$$w(x) = x(1-x)(\beta_0 + \beta_1 x)$$

(a) (15%)

$$\frac{d^2u(x)}{dx^2} + x^2 = 0$$

$$\Rightarrow \int_0^1 w(x)\frac{d^2u(x)}{dx^2}dx + \int_0^1 w(x)x^2 dx \Rightarrow w(x)\frac{du}{dx}\Big|_0^1 - \int_0^1 \frac{dw(x)}{dx}\frac{du(x)}{dx}dx + \int_0^1 w(x)x^2 dx = 0 \text{ (5\%)}$$

$$\Rightarrow w(1)\frac{du}{dx} - w(0)\frac{du}{dx} - \int_0^1 \frac{dw(x)}{dx}\frac{du(x)}{dx}dx + \int_0^1 w(x)x^2 dx = 0 \text{ (5\%)}$$

$$\because u(0) = u(1) = 0 \qquad \therefore w(0) = w(1) = 0$$

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx = \int_0^1 w(x)x^2 dx \text{(3\%)} \quad \forall w \text{ with } w(0) = w(1) = 0 \text{(2\%)}$$

(b) (15%)

$$\begin{cases} u(x) = x(1-x)(\alpha_0 + \alpha_1 x) \\ w(x) = x(1-x)(\beta_0 + \beta_1 x) \end{cases} \Rightarrow \text{ to be admissible} \begin{cases} u(0) = u(1) = 0 \\ w(0) = w(1) = 0 \end{cases}$$

$$\frac{du}{dx} = \alpha_0 + 2(\alpha_1 - \alpha_0)x - 3\alpha_1 x^2 \text{ (2\%)}$$

$$\frac{dw}{dx} = \beta_0 + 2(\beta_1 - \beta_0)x - 3\beta_1 x^2 \text{ (2\%)}$$

$$\Rightarrow \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx = \int_0^1 w(x)x^2 dx$$

$$\Rightarrow \beta_0\left(\frac{\alpha_0}{3} + \frac{\alpha_1}{6} - \frac{1}{20}\right) + \beta_1\left(\frac{\alpha_0}{6} + \frac{2\alpha_1}{15} - \frac{1}{30}\right) = 0 \text{ (3\%)}$$

$$\Rightarrow \beta_0, \beta_1 \text{ arbitrary, } \alpha_0 = \frac{1}{15}\text{(3\%)}, \ \alpha_1 = \frac{1}{6}\text{(3\%)}$$

$$\Rightarrow u(x) = x(1-x)\left(\frac{1}{15} + \frac{1}{6}x\right) \text{ (2\%)}$$