APPENDIX C

Equations from Elasticity Theory

Introduction

In this appendix, we will develop the basic equations of the theory of elasticity. These equations should be referred to frequently throughout the structural mechanics portions of this text.

There are three basic sets of equations included in theory of elasticity. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. These sets of equations are (1) the differential equations of equilibrium formulated here in terms of the stresses acting on a body, (2) the strain/displacement and compatibility differential equations, and (3) the stress/strain or material constitutive laws.

C.1 Differential Equations of Equilibrium

For simplicity, we initially consider the equilibrium of a plane element subjected to normal stresses σ_x and σ_y, in-plane shear stress τ_{xy} (in units of force per unit area), and body forces X_b and Y_b (in units of force per unit volume), as shown in Figure C–1. The stresses are assumed to be constant as they act on the width of each face. However, the stresses are assumed to vary from one face to the opposite. For example, we have σ_x acting on the left vertical face, whereas $\sigma_x + (\partial \sigma_x / \partial x)\, dx$ acts on the right vertical face. The element is assumed to have unit thickness.

Summing forces in the x direction, we have

$$\sum F_x = 0 = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy(1) - \sigma_x\, dy(1) + X_b\, dx\, dy(1) + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx(1) - \tau_{yx}\, dx(1) = 0 \qquad \text{(C.1.1)}$$

After simplifying and canceling terms in Eq. (C.1.1), we obtain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X_b = 0 \qquad \text{(C.1.2)}$$

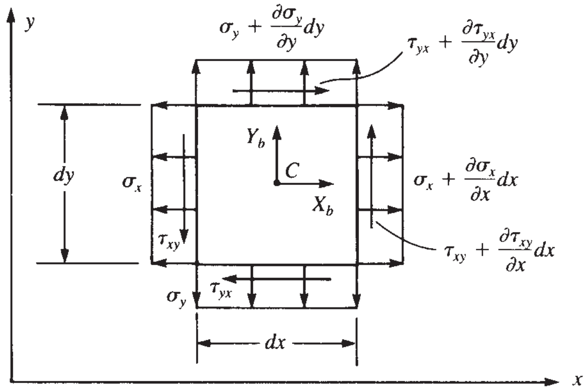

Figure C–1 Plane differential element subjected to stresses

Similarly, summing forces in the y direction, we obtain

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y_b = 0 \tag{C.1.3}$$

Because we are considering only the planar element, three equilibrium equations must be satisfied. The third equation is equilibrium of moments about an axis normal to the x-y plane; that is, taking moments about point C in Figure C–1, we have

$$\sum M_z = 0 = \tau_{xy}\, dy(1)\frac{dx}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right)\frac{dx}{2}$$

$$- \tau_{yx}\, dx(1)\frac{dy}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right)\frac{dy}{2} = 0 \tag{C.1.4}$$

Simplifying Eq. (C.1.4) and neglecting higher-order terms yields

$$\tau_{xy} = \tau_{yx} \tag{C.1.5}$$

We now consider the three-dimensional state of stress shown in Figure C–2, which shows the additional stresses σ_z, τ_{xz}, and τ_{yz}. For clarity, we show only the stresses on three mutually perpendicular planes. With a straightforward procedure, we can extend the two-dimensional equations (C.1.2), (C.1.3), and (C.1.5) to three dimensions. The resulting total set of equilibrium equations is

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b &= 0 \end{aligned} \tag{C.1.6}$$

and

$$\tau_{xy} = \tau_{yx} \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy} \tag{C.1.7}$$

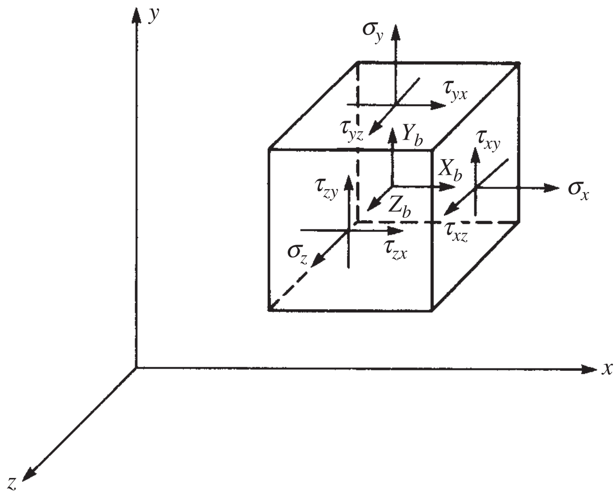

Figure C–2 Three-dimensional stress element

C.2 Strain/Displacement and Compatibility Equations

We first obtain the strain/displacement or kinematic differential relationships for the two-dimensional case. We begin by considering the differential element shown in Figure C–3, where the undeformed state is represented by the dashed lines and the deformed shape (after straining takes place) is represented by the solid lines.

Considering line element AB in the x direction, we can see that it becomes $A'B'$ after deformation, where u and v represent the displacements in the x and y directions. By the definition of engineering normal strain (that is, the change in length divided by

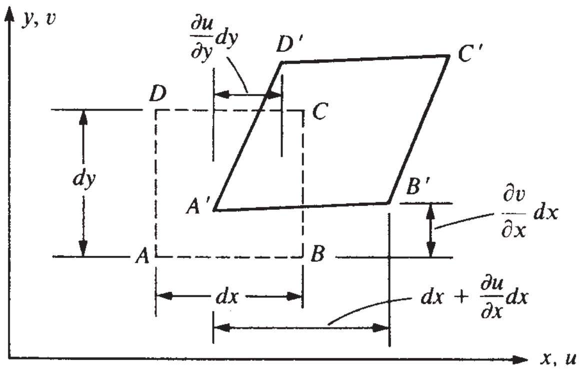

Figure C–3 Differential element before and after deformation the original length of a line), we have

$$\varepsilon_x = \frac{A'B' - AB}{AB} \tag{C.2.1}$$

Now

$$AB = dx \tag{C.2.2}$$

and

$$(A'B')^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2 \tag{C.2.3}$$

Therefore, evaluating $A'B'$ using the binomial theorem and neglecting the higher-order terms $(\partial u/\partial x)^2$ and $(\partial v/\partial x)^2$ (an approach consistent with the assumption of small strains), we have

$$A'B' = dx + \frac{\partial u}{\partial x}dx \tag{C.2.4}$$

Using Eqs. (C.2.2) and (C.2.4) in Eq. (C.2.1), we obtain

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{C.2.5}$$

Similarly, considering line element AD in the y direction, we have

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{C.2.6}$$

The shear strain γ_{xy} is defined to be the change in the angle between two lines, such as AB and AD, that originally formed a right angle. Hence, from Figure C–3, we can see that γ_{xy} is the sum of two angles and is given by

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{C.2.7}$$

Equations (C.2.5) through (C.2.7) represent the strain/displacement relationships for in-plane behavior.

For three-dimensional situations, we have a displacement w in the z direction. It then becomes straightforward to extend the two-dimensional derivations to the three-dimensional case to obtain the additional strain/displacement equations as

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{C.2.8}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{C.2.9}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{C.2.10}$$

Along with the strain/displacement equations, we need compatibility equations to ensure that the displacement components u, v, and w are single-valued continuous functions so that tearing or overlap of elements does not occur. For the planar-elastic case, we obtain the compatibility equation by differentiating γ_{xy} with respect to both x and y and then using the definitions for ε_x and ε_y given by Eqs. (C.2.5) and (C.2.6). Hence,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\frac{\partial u}{\partial y} + \frac{\partial^2}{\partial x \partial y}\frac{\partial v}{\partial x} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \tag{C.2.11}$$

where the second equation in terms of the strains on the right side is obtained by noting that single-valued continuity of displacements requires that the partial differentiations with respect to x and y be interchangeable in order. Therefore, we have $\partial^2/\partial x \partial y = \partial^2/\partial y \partial x$. Equation (C.2.11) is called the *condition of compatibility*, and it must be satisfied by the strain components in order for us to obtain unique expressions for u and v. Equations (C.2.5), (C.2.6), (C.2.7), and (C.2.11) together are then sufficient to obtain unique single-valued functions for u and v.

In three dimensions, we obtain five additional compatibility equations by differentiating γ_{xz} and γ_{yz} in a manner similar to that described above for γ_{xy}. We need not list these equations here; details of their derivation can be found in Reference [1].

In addition to the compatibility conditions that ensure single-valued continuous functions within the body, we must also satisfy displacement or kinematic boundary conditions. This simply means that the displacement functions must also satisfy prescribed or given displacements on the surface of the body. These conditions often occur as support conditions from rollers and/or pins. In general, we might have

$$u = u_0 \qquad v = v_0 \qquad w = w_0 \tag{C.2.12}$$

at specified surface locations on the body. We may also have conditions other than displacements prescribed (for example, prescribed rotations).

C.3 Stress-Strain Relationships

We will now develop the three-dimensional stress-strain relationships for an isotropic body only. This is done by considering the response of a body to imposed stresses. We subject the body to the stresses σ_x, σ_y, and σ_z independently as shown in Figure C–4.

We first consider the change in length of the element in the x direction due to the independent stresses σ_x, σ_y, and σ_z. We assume the principle of superposition to hold; that is, we assume that the resultant strain in a system due to several forces is the algebraic sum of their individual effects.

Considering Figure C–4(b), the stress in the x direction produces a positive strain

$$\varepsilon'_x = \frac{\sigma_x}{E} \tag{C.3.1}$$

where Hooke's law, $\sigma = E\varepsilon$, has been used in writing Eq. (C.3.1), and E is defined as the *modulus of elasticity*. Considering Figure C–4(c), the positive stress in the

y direction produces a negative strain in the x direction as a result of Poisson's effect given by

$$\varepsilon_x'' = -\frac{\nu\sigma_y}{E} \tag{C.3.2}$$

where ν is Poisson's ratio. Similarly, considering Figure C–4(d), the stress in the z direction produces a negative strain in the x direction given by

$$\varepsilon_x''' = -\frac{\nu\sigma_z}{E} \tag{C.3.3}$$

Using superposition of Eqs. (C.3.1) through (C.3.3), we obtain

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{C.3.4}$$

The strains in the y and z directions can be determined in a manner similar to that used to obtain Eq. (C.3.4) for the x direction. They are

$$\begin{aligned} \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned} \tag{C.3.5}$$

Solving Eqs. (C.3.4) and (C.3.5) for the normal stresses, we obtain

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x(1-\nu) + \nu\varepsilon_y + \nu\varepsilon_z]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z] \qquad \text{(C.3.6)}$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z]$$

The Hooke's law relationship, $\sigma = E\varepsilon$, used for normal stress also applies for shear stress and strain; that is,

$$\tau = G\gamma \qquad \text{(C.3.7)}$$

where G is the *shear modulus*. Hence, the expressions for the three different sets of shear strains are

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} \qquad \text{(C.3.8)}$$

Solving Eqs. (C.3.8) for the stresses, we have

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \qquad \text{(C.3.9)}$$

In matrix form, we can express the stresses in Eqs. (C.3.6) and (C.3.9) as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}$$
$$\times \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \text{(C.3.10)}$$

where we note that the relationship

$$G = \frac{E}{2(1+\nu)}$$

has been used in Eq. (C.3.10). The square matrix on the right side of Eq. (C.3.10) is called the *stress-strain* or *constitutive matrix* and is defined by $[D]$, where $[D]$ is

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \dfrac{1-2\nu}{2} & 0 & 0 \\ & & & & \dfrac{1-2\nu}{2} & 0 \\ \text{Symmetry} & & & & & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{C.3.11}$$

Reference

[1] Timoshenko, S., and Goodier, J., *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970.