RIEMANN-ROCH THEOREM FOR SURFACES.

1 Arithmetic genus and irregularity of a surface.

Let $k = \bar{k}$ be a field, $R = k[T_0, \dots, T_n]$ and $\mathbb{P}^n = \mathrm{Proj}(R)$. Let $X \subseteq \mathbb{P}^n$ be a closed subset and let $R(X) = R/I(X)$ be its homogeneous coordinate ring, which is a finitely generated graded R-module.

Fact (cf. Thm. I.7.5)**.** There is a unique polynomial $P_X(T) \in \mathbb{Q}[T]$ such that

$$P_X(d) = \dim_k R(X)_d$$

for all big enough d. We call it the *Hilbert polynomial* of X.

Example. Let us first compute the Hilbert polynomial of projective space. Some combinatoric argument shows that

$$P_{\mathbb{P}^n}(T) = \binom{n+T}{T} \in \mathbb{Q}[T]$$

Let us now compute the Hilbert polynomial of a degree d hypersurface $X \subseteq \mathbb{P}^n$, defined as the zero locus of some homogeneous polynomial $f \in R_d$. We have a short exact sequence $0 \to R(-d) \to R \to R/(f) \to 0$ of graded R-modules. Looking at the degree T part, we have a short exact sequence of k-vector spaces. By additivity of the dimension (vector spaces $\Rightarrow$ sequence splits) we obtain

$$P_X(T) = \binom{n+T}{T} - \binom{n+T-d}{T-d} \in \mathbb{Q}[T]$$

Definition. Let X be a variety of dimension r in $\mathbb{P}^n$ and $P_X \in \mathbb{Q}[T]$ be its Hilbert polynomial. We define the *arithmetic genus* of X as

$$p_a(X) = (-1)^r (P_X(0) - 1)$$

Note that the Hilbert polynomial depends on the projective embedding of X. For example if X is a degree d hypersurface in $\mathbb{P}^n$, then $(n-1)!$ times the leading coefficient is d. But d depends on the embedding, e.g. $\mathbb{P}^1$ can be embedded in $\mathbb{P}^2$ as a linear subspace or as a conic. One could think that the arithmetic genus depends on the embedding, as the Hilbert polynomial does. But this is not the case, as the following formula shows

Fact (cf. Exercise III.5.2)**.** The Hilbert polynomial can also be characterized by

$$P_X(d) = \chi(\mathscr{O}_X(d))$$

where recall that $\chi(\mathscr{F}) = \sum_i (-1)^i h^i(\mathscr{F})$ and $h^i(\mathscr{F}) = \dim_k H^i(X, \mathscr{F})$.

In particular, we may write the arithmetic genus of our X as

$$p_a(X) = (-1)^r (\chi(\mathscr{O}_X) - 1)$$

This shows that the arithmetic genus is independent of the projective embedding of X and also allows us to generalize this definition to any projective scheme of dimension r over a field k.

Suppose first that $X = C$ is a smooth projective curve. Then the previous formula reads

$$p_a(C) = -(h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) - 1) = h^1(\mathcal{O}_C)$$

By Serre duality, $H^1(C, \mathcal{O}_C)$ is dual to $H^0(C, \omega_C)$ and in particular we have

$$p_a(C) = h^1(\mathcal{O}_C) = h^0(\omega_C) = p_g(C)$$

Suppose now that $X = S$ is a smooth projective surface. Then the formula reads

$$p_a(S) = -(h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) - 1) = h^2(\mathcal{O}_S) - h^1(\mathcal{O}_S)$$

Again by Serre duality we get

$$p_a(S) = h^0(\omega_S) - h^1(\mathcal{O}_S) = p_g(S) - h^1(\mathcal{O}_S) \leqslant p_g(S)$$

This motivates the following

Definition. Let S be a smooth projective surface over some field k. Its *irregularity* is defined as

$$q(S) = p_g(S) - p_a(S) = h^1(\mathcal{O}_S) = \dim_k H^1(S, \mathcal{O}_S)$$

Proposition. Irregularity, arithmetic genus and geometric genus are birational invariants of smooth projective surfaces.

Proof:

We know that the geometric genus is a birational invariant (cf. Thm. II.8.19). It suffices to show that the arithmetic genus is also a birational invariant.

Recall from João's talk that we can factor a birational equivalence into a finite sequence of blow ups at certain points (cf. Thm. V.5.5). So it suffices to show that the arithmetic genus is invariant under blowing up a point. But in fact we have a much stronger statement:

$$H^i(S, \mathcal{O}_S) \cong H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$$

for all $i \geqslant 0$ (cf. Prop. V.3.4).

□

Example. Let $S \subseteq \mathbb{P}^3$ be a smooth projective surface over $k = \bar{k}$. Say $S = V(f)$ with $\deg(f) = d$. From the short exact sequence $0 \to \mathcal{O}(-d) \to \mathcal{O} \to i_*\mathcal{O}_S \to 0$, the corresponding long exact sequence $\cdots \to H^1(\mathbb{P}^3, \mathcal{O}) \to H^1(\mathbb{P}^3, i_*\mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}(-d)) \to H^2(\mathbb{P}^3, \mathcal{O}) \to \cdots$ and the usual cohomology vanishing on $\mathbb{P}^3$ (cf. Thm. III.5.1) we deduce that $H^1(\mathbb{P}^3, i_*\mathcal{O}_S) = 0$. But closed immersions are affine, so by Exercise III.8.2 we have

$$H^1(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^3, i_*\mathcal{O}_S) = 0$$

and therefore $q(S) = 0$. Therefore surfaces in $\mathbb{P}^3$ are "regular", meaning that they have zero irregularity.

2 (Recall) Divisors and linear systems.

2.1 Divisors and line bundles.

Let now X be a smooth projective variety over $k = \bar{k}$ (for convenience/simplicity).

Definition. A *Weil divisor* is a formal $\mathbb{Z}$-linear combination of codimension 1 subvarieties $Y \subseteq X$. A divisor whose coefficients are all non-negative is called *effective*. Thanks to DVR's we can associate to every non-zero rational function $f \in K(X)^\times$ its divisor (f) of zeros and poles. Such a divisor is called *principal*, and two divisors are said to be *linearly equivalent* if their difference is principal.

Definition. A *Cartier divisor* is a global section of the sheaf $\mathscr{K}_X^\times / \mathscr{O}_X^\times$. A divisor which is a global section of the subsheaf $\mathscr{K}_X^\times \cap \mathscr{O}_X / \mathscr{O}_X^\times$ is called *effective*. A divisor which is in the image of the natural map $\mathscr{K}_X^\times \to \mathscr{K}_X^\times / \mathscr{O}_X^\times$ is called *principal*. Two divisors are said to be *linearly equivalent* if their difference is principal.

A Cartier divisor can be thought of as some vanishing/pole data on our variety: it can be represented by an open cover U_i together with some rational functions $f_i \in \mathscr{K}_X^\times$ such that $f_i f_j^{-1} \in \mathscr{O}_X^\times(U_i \cap U_j)$ is a unit, that is, such that f_i and f_j have the same zero/pole behavior on $U_i \cap U_j$. With this description, a divisor is effective if $f_i \in \mathscr{O}_X(U_i)$ is regular on U_i and principal if it can be represented by the trivial open cover and a unique rational function. And the sum of divisors corresponds to multiplication of rational functions.

These two notions are equivalent in our situation: with the DVR's we go from Cartier divisors to Weil divisors (same zero/pole behavior on intersections implies that this map is well-defined) and since X is regular every Weil divisor is locally principal, giving thus a Cartier divisor (cf. Prop. II.6.2 and Prop. II.6.11). And the notions of effective and principal divisor are preserved by this correspondence, so we may safely talk about *divisors* on X and use one description or the other depending on the situation.

We denote the group of divisors on X by $\mathrm{Div}(X)$, and the group of divisors on X modulo linear equivalence by $\mathrm{Cl}(X)$.

Definition. An *invertible sheaf* or *line bundle* on X is an $\mathscr{O}_X$-module which is locally isomorphic to the structure sheaf $\mathscr{O}_X$ (the name line bundle is justified by the correspondence with geometric line bundles as established in Exercise II.5.18).

Line bundles come by definition with some trivializing open cover U_i and isomorphisms $\varphi_i \colon \mathscr{O}_{U_i} \xrightarrow{\cong} \mathscr{L}_{U_i}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathscr{O}_{U_i \cap U_j} & \xrightarrow{\varphi_i} & \mathscr{L}_{U_i \cap U_j} \\ \downarrow \cong & & \| \\ \mathscr{O}_{U_i \cap U_j} & \xleftarrow[\varphi_j^{-1}]{} & \mathscr{L}_{U_i \cap U_j} \end{array}$$

The isomorphisms $\mathscr{O}_{U_i \cap U_j} \to \mathscr{O}_{U_i \cap U_j}$ are given by multiplication with some invertible elements $f_{ij} \in \mathscr{O}_X^\times(U_i \cap U_j)$. These are the transition functions.

This third notion is again equivalent to the two previous ones in our situation. There is an obvious relation between Cartier divisors and line bundles: if we have a line bundle, the transition functions f_{0i} from some fixed U_0 to each U_i give a Cartier divisor (every two non-empty open sets intersect), and if we have a Cartier divisor we can get an obvious line bundle by taking the $\mathscr{O}_X$-module locally generated by the f_i^{-1} (taking the inverse rational function here makes these two constructions mutually inverse). We denote the line bundle associated to the divisor D by $\mathscr{O}_X(D)$. For an open subset U of X we have

$$f \in H^0(U, \mathscr{O}_X(D)) \Leftrightarrow D|_U + (f)|_U \text{ is an effective divisor on } U.$$

So the sections of $\mathscr{O}_X(D)$ are rational functions with zeros forced by the poles of D and with poles allowed by the zeros of D.

Note that the line bundle associated to a divisor is a subsheaf of the sheaf of $\mathscr{K}_X$. This is not the case for all invertible sheaves, but for integral schemes such as X we have at least that every invertible sheaf is isomorphic to a subsheaf of $\mathscr{K}_X$.

Remark. The (Weil) divisor corresponding to a line bundle $\mathscr{L}$ as above plays an important role in intersection theory. It is called the *first Chern class* of the line bundle, denoted by $c_1(\mathscr{L})$. There are higher codimensional analogues of the first Chern class for vector bundles (locally free sheaves) of higher rank. They appear for example in the Hirzebruch Riemann Roch theorem, a generalization of the Riemann Roch theorems for curves and surfaces.

Example. Let $X = \mathbb{P}^1$ and let $\mathscr{O}(1)$ be Serre's twisting sheaf. Let $\Omega_0 = \{[t_0 : t_1] \in \mathbb{P}^1 \mid t_0 \neq 0\} \cong \operatorname{Spec}(k[T_1/T_0]) = \mathbb{A}^1$ and $\Omega_1 = \operatorname{Spec}(k[T_0/T_1])$ be the usual open cover of $\mathbb{P}^1$.

Over Ω_0 we have sections such as $T_0, T_1, T_1^2/T_0, T_1^3/T_0^2, \ldots$ Note that these aren't rational functions, so our line bundle is not the line bundle associated to any divisor. But since X is integral, the equivalence class of $\mathscr{O}(1)$ contains line bundles associated to divisors, and we may use the procedure described above to find one.

Since T_0 is a invertible over Ω_0 we have an isomorphism $\mathscr{O}|_{\Omega_0} \xrightarrow{\cong} \mathscr{O}(1)|_{\Omega_0}$ given by multiplication with T_0. Similarly, over Ω_1 we have an isomorphism $\mathscr{O}|_{\Omega_1} \xrightarrow{\cong} \mathscr{O}(1)|_{\Omega_1}$ whose inverse is given by multiplication with T_1^{-1}. If we fix Ω_0 as our favorite open set in the cover, we have transition functions 1 (from Ω_0 to Ω_0) and T_0/T_1 (from Ω_0 to Ω_1). Hence, the corresponding Cartier divisor is given by the data $\{(\Omega_0, 1), (\Omega_1, T_0/T_1)\}$ and the corresponding Weil divisor has all coefficients equal to 0 (the valuation of 1 in any of the local rings) except at most the coefficient of the point $[0 : 1] \in \mathbb{P}^1$, which is the only point not contained in the open Ω_0. We compute its coefficient as the valuation of the element $T = T_0/T_1$ in the local ring of $\mathbb{A}^1 = k[T]$ at the origin $T = 0$, that is, the valuation of T in $k[T]_{(T)}$, which is 1. Therefore we obtain the Weil divisor $[0 : 1] =: H \in \operatorname{Div}(X)$.

Now we may go backwards and consider the corresponding line bundle $\mathscr{O}(H)$, which by definition has $\mathscr{O}(H)|_{\Omega_0} = \mathscr{O}|_{\Omega_0}$ and $\mathscr{O}(H)|_{\Omega_1} = (T_1/T_0)\mathscr{O}|_{\Omega_1}$. Over Ω_0, the isomorphism of $k[T_1/T_0]$-modules $\langle T_0, T_1, T_1^2/T_0, T_1^3/T_0^2, \ldots\rangle \to k[T_1/T_0]$ given by multiplication with T_0^{-1} induces an isomorphism of sheaves of modules $\mathscr{O}(1)|_{\Omega_0} \xrightarrow{\cong} \mathscr{O}(H)|_{\Omega_0}$ (via the equivalence of categories of Corollary II.5.5). Similarly, multiplication by T_0^{-1} gives an isomorphism $\mathscr{O}(1)|_{\Omega_1} \xrightarrow{\cong} \mathscr{O}(H)|_{\Omega_1}$. Hence the map given (also on global sections) by multiplication with T_0^{-1} is an isomoprhism $\mathscr{O}(1) \cong \mathscr{O}(H)$, showing that they are the same element in $\mathrm{Pic}(\mathbb{P}^1)$.

2.2 Line bundles and maps to projective space.

Let X be a smooth projective variety over $k = \bar{k}$ as before (again, for simplicity). We saw in the previous section that Weil divisors, Cartier divisors and line bundles are all the same in this case.

Proposition (cf. Thm. II.7.1)**.** Maps $\varphi\colon X \to \mathbb{P}^n$ correspond bijectively to the data of a line bundle $\mathscr{L}$ on X and $n+1$ global sections $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$ which generate $\mathscr{L}$, up to isomorphism of line bundles which maps one tuple of global sections into the other.

Proof:

Suppose we are given a map $\varphi\colon X \to \mathbb{P}^n$. On $\mathbb{P}^n$ we have the line bundle $\mathscr{O}(1)$, which is generated by the global sections $T_0, \ldots, T_n$. Then $\varphi^*\mathscr{O}(1)$ is a line bundle on X (local triviality of $\mathscr{O}(1)$ and inverse image functor commuting with open restrictions implies that locally we are tensoring with the base ring, so nothing happens) and the global sections $\varphi^*(T_i)$ generate $\varphi^*\mathscr{O}(1)$ (since T_i generate $\mathscr{O}(1)$, we get a surjective morphism $\mathscr{O}^{\oplus n+1} \twoheadrightarrow \mathscr{O}(1)$, and since the inverse image functor is exact and tensor product is right exact we get a surjective morphism $\varphi^{-1}\mathscr{O}^{\oplus n+1} \otimes_{\varphi^{-1}\mathscr{O}} \mathscr{O}_X = (\varphi^{-1}\mathscr{O} \otimes_{\varphi^{-1}\mathscr{O}} \mathscr{O}_X)^{\oplus n+1} = \mathscr{O}_X^{\oplus n+1} \twoheadrightarrow \varphi^*\mathscr{O}(1)$). Note how this direction is easier to think of with the more general machinery of invertible sheaves than with actual geometry and divisors.

Conversely, if we are given a line bundle $\mathscr{L}$ generated by global sections $s_0, \ldots, s_n$, then there is a very explicit and geometric way to define a map $\varphi\colon X \to \mathbb{P}^n$. Let $x \in X$ be a closed point. Since the sections s_i generate $\mathscr{L}$, at least one of them is not in the maximal ideal of the stalk $\mathscr{L}_x \cong \mathscr{O}_{X,x}$, i.e. there is some $i \in \{0, \ldots, n\}$ such that $s_i(x) \neq 0$ in the residue field k. Then we can map x to the (closed) point $[s_0(x)/s_i(x) : \ldots : s_n(x)/s_i(x)] \in \mathbb{P}^n$. If $s_j(x)$ was also not zero, the corresponding coordinates are obtained from the previous ones by multiplication with $s_i(x)/s_j(x) \in k^\times$, hence represent the same point of projective space. Since the subsets on which a section s_i is not zero in the residue field are open, cover X and on the intersections the previous formula agrees, we get a well defined map $\varphi\colon X \to \mathbb{P}^n$.

□

This result can be restated in terms of representable functors and in fact one can take this proposition as the definition of $\mathbb{P}^n$ (cf. Stacks Project Tag 01ND).

Note that the (effective) divisors of zeros of the global sections s_i (defined by gluing the divisors of regular functions that they locally represent) correspond to hyperplane sections of $\varphi(X)$. Indeed, if H_i denotes the hyperplane $t_i = 0$ in $\mathbb{P}^n$ and $x \in X$ is a closed point, then $s_i(x) = 0$ if and only if $\varphi(x) \in H_i$. Moreover, we have a bijection between effective divisors linearly equivalent to these hyperplane sections and non-zero global sections $H^0(X, \mathscr{L})$ up to scalar (cf. Prop. II.7.7). This motivates the following

Definition. A complete linear system on X is the set of effective divisors linearly equivalent to some given divisor D, denoted $|D|$. The previous bijection shows that $|D|$ is in bijection with the closed points of the projective space over the (finite dimensional) k-vector space $H^0(X, \mathscr{L})$, with $\mathscr{L} \cong \mathscr{O}(D)$.

A linear system $\mathfrak{d}$ on X corresponds to a linear subspace of the projective space $|D|$, which in turn corresponds to a k-vector subspace of global sections $V \subseteq H^0(X, \mathscr{L})$.

A base point of the linear system is a point $x \in X$ which lies in the support of all divisors in the linear system (the support of a divisor being the union of its prime divisors).

Hence, a base point of the linear system corresponding to V is a point $x \in X$ such that $s(x) = 0$ in k for all $s \in V$. In particular we can reformulate the previous proposition as follows: a map $\varphi\colon X \to \mathbb{P}^n$ corresponds to a base point free linear system on X and a set $s_0, \ldots, s_n \in V$ of generators of the corresponding vector space. If we omit the generators, we understand that we are taking a basis of V. The maps induced by two bases differ only by an automorphism of $\mathbb{P}^n$, so there is no much ambiguity. If our linear system has base points, we would get at most a rational map $X \dashrightarrow \mathbb{P}^n$ defined on the open set $U \subseteq X$ (possibly empty) on which a basis of V has no common zeros.

Definition. If $\mathfrak{d}$ gives an embedding into projective space $\mathbb{P}^n$, we say that the invertible sheaf $\mathscr{L}$ is very ample (relative to $\operatorname{Spec}(k)$, as everything else we have said so far).

Note again that in this case any (effective) divisor $D \in \mathfrak{d}$ corresponds to a hyperplane section of X sitting inside $\mathbb{P}^n$. This is a more interesting case, since the embedding allows us to study the geometry of X as a subvariety (maybe only quasi-projective) of projective space.

A theorem by Serre (Thm. II.5.17) asserts that for any $\mathscr{F} \in \mathbf{Coh}(X)$ there is some $n \in \mathbb{N}$ such that $\mathscr{F}(m)$ can be generated by finitely many global sections for all $m \geqslant n$. This property turns out to be the right definition of ample invertible sheaf, since it characterizes (over nice schemes like our X) the invertible sheaves $\mathscr{L} \in \operatorname{Pic}(X)$ such that $\mathscr{L}^m$ is very ample (relative to $\operatorname{Spec}(k)$). For simplicity we state the definition directly in this nice case:

Definition. A divisor D on X is ample if some positive multiple mD with $m > 0$ is a hyperplane section in some projective embedding $X \hookrightarrow \mathbb{P}^n$.

Note that the existence of an ample divisor on a complete variety X ensures projectivity, because an immersion of a proper scheme is automatically a closed immersion. So ampleness is indeed an interesting notion. It has also some analogues in higher codimension, but the case of divisors is especially nice, because we have three different approaches that lead to the same notion of ampleness (over nice schemes like our X), namely:

- Geometric approach (our definition): some positive multiple of our divisor is a hyperplane section in some projective space.

- Cohomological approach: tensoring with high enough multiples of the associated invertible sheaf kills cohomology groups of coherent sheaves (cf. Thm. III.5.2).

- Numerical approach: the divisor and its self intersections are numerically positive (Nakai Criterion).

Back to linear systems: from our base point free linear system $\mathfrak{d}$ we get a morphism $\varphi\colon X \to \mathbb{P}^n$. When is φ a closed immersion?

Fact (cf. Prop. II.7.3 and Remark II.7.8.2)**.** The map $\varphi\colon X \to \mathbb{P}^n$ induced by the linear system $\mathfrak{d}$ is a closed immersion if and only if $\mathfrak{d}$ separates points and tangent vectors.

Separating points means that for any two different (closed) points in X, there is some $D \in \mathfrak{d}$ passing through one of them but not through the other. This ensures injectivity (and homeomorphism onto the image if X is complete): recall that the elements in our linear system corresponded to hyperplane sections of the image, so we are asking that two different points in X go to points of projective space such that there is a hyperplane passing through one but not through the other (hence different points).

Separating tangent vectors means that for any (closed) point $x \in X$ and any non-zero tangent vector $t \in T_xX$, there is some $D \in \mathfrak{d}$ passing through x but in a direction orthogonal to t, that is, such that $t \notin T_xD$. More explicitly, if f is a local equation for D around x (hence $f_x \in \mathfrak{m}_x$), we want $t(\bar{f_x}) \neq 0$, where $\bar{f_x}$ is the reduction modulo $\mathfrak{m}_x^2$. If the map induced by φ on tangent spaces at x is not injective and t is a nontrivial element of the kernel, this this doesn't happen. So this condition implies injectivity on tangent spaces. And this, after some local algebra, implies surjectivity on the stalks.

Example. Consider again $X = \mathbb{P}^1$ and the line bundle $\mathscr{O}(2) \in \mathrm{Pic}(\mathbb{P}^1)$. Take the global sections T_0^2, T_0T_1 and T_1^2. They generate $\mathscr{O}(2)$, so we get a map

$$\begin{aligned} \varphi\colon \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [t_0 : t_1] &\longmapsto [t_0^2 : t_0t_1 : t_1^2] \end{aligned}$$

Since the previous sections generate $\mathscr{O}(2)$, they never vanish simultaneously on $\mathbb{P}^1$. Hence the image of a point is never $[0:0:0]$.

The linear system that we are dealing with corresponds to the whole $H^0(\mathbb{P}^1, \mathscr{O}(2))$, and we may denote it by $|\mathscr{O}(2)|$. The (effective) divisors in this linear system the are sums of two points of $\mathbb{P}^1$.

Since φ is just the inclusion of a plane conic in $\mathbb{P}^2$ (draw picture for $t_0 \neq 0$), we know it is a closed embedding. But let us check that φ satisfies the conditions to be a closed embedding. $|\mathscr{O}(2)|$ separates points, because if $P \neq Q$ are different points in $\mathbb{P}^1$ then $2P \in |\mathscr{O}(2)|$ separates them. $|\mathscr{O}(2)|$ separates tangent vectors, because if $P \in \mathbb{P}^1$ is a point, we may take any other point $Q \in \mathbb{P}^1 \setminus P$ and then $D = P + Q \in |\mathscr{O}(2)|$ vanishes at P but only with multiplicity 1, which implies that $T_PD = 0$. Hence φ is a closed immersion.

3 The case of curves.

Let $X = C$ be a smooth projective curve over $k = \bar{k}$. Recall that a divisor on C is just a $\mathbb{Z}$-linear combination of (closed) points in our curve and in particular divisors on curves have a well defined degree, equal to the sum of the coefficients. This doesn't work in higher dimension (we can say that a reduced closed point has degree 1 without ambiguity, but not anymore for curves or higher dimensional varieties). Recall also that in this case the canonical sheaf ω_C is just the sheaf of relative differentials $\Omega_{C/k}$ and that Serre duality implies $p_a(C) = p_g(C) = h^1(\mathscr{O}_C) =: g$. Recall finally

Theorem (Riemann-Roch for curves)**.** Let $K \in \mathrm{Div}(C)$ be the canonical divisor corresponding to ω_C and let $D \in \mathrm{Div}(C)$ be any other divisor on C. Then

$$h^0(\mathscr{O}(D)) - h^0(\mathscr{O}(K - D)) = \deg(D) + 1 - g$$

We will use this theorem and the previous discussion on divisors to prove the following

Proposition. A divisor $D \in \mathrm{Div}(C)$ is ample if and only if $\deg(D) > 0$.

Proof:

Lemma. $|D|$ base point free $\Leftrightarrow \dim |D - P| = \dim |D| - 1$ for all $P \in C$.

Proof:

Consider the short exact sequence

$$0 \to \mathscr{O}(D - P) \to \mathscr{O}(D) \to \kappa(P) \to 0$$

where $\kappa(P)$ denotes here the skyscraper sheaf (check the stalks to see exactness). We take global sections to get an exact sequence

$$0 \to H^0(C, \mathscr{O}(D - P)) \to H^0(C, \mathscr{O}(D)) \to k$$

If the last map is zero, then $\dim |D - P| = \dim |D|$. If not, then it is surjective, the sequence (of finite dimensional k-vector spaces) splits and we get $\dim |D - P| = \dim |D| - 1$. Hence, it suffices to show that the map

$$\begin{aligned} \varphi \colon |D - P| &\to |D| \\ E &\mapsto E + P \end{aligned}$$

which is linear and injective (as it is induced by multiplication with some non-zero global section of $\mathscr{O}(P)$), is not surjective. But surjectivity of this map is equivalent to P being a base point of $|D|$, so we deduce the equivalence that we wanted.

□

Lemma. D very ample $\Leftrightarrow \dim |D - P - Q| = \dim |D| - 2$ for all $P, Q \in C$.

Proof:

We may assume that $|D|$ is base point free, as this is a necessary condition for both sides of the equivalence (on the right side, we saw in the proof of the previous lemma that at each step the dimension reduces at most by 1, so in order for it to reduce by 2 when removing two points it must reduce by 1 when removing the first one, so $|D|$ must be base point free).

So assume $|D|$ base point free and consider the corresponding map $\psi\colon C \to \mathbb{P}^r$. We have to check that ψ separates points and tangent vectors.

Separating points means that for all pairs of distinct points $P, Q \in C$ we have that Q is not a base point of $|D - P|$. By the previous lemma, this is equivalent to $\dim |D - P - Q| = \dim |D - P| - 1$, but since $|D|$ is base point free we also have $\dim |D - P| = \dim |D| - 1$.

Separating tangent vectors means P is not a base point of $|D - P|$ for any $P \in C$ (same argument as above). Hence, again by the previous lemma, this is equivalent to the condition on the RHS of the equivalence.

$\square$

Lemma. If $\deg D \geqslant 2g$, then $|D|$ has no base points. Moreover, if strict inequality holds, then D is very ample.

Proof:

By RR, the degree of the canonical divisor K corresponding to the canonical invertible sheaf is $2g - 2$. So if $\deg D > 2g - 2$, then $K - D$ has negative degree and thus $\mathscr{O}(K - D)$ has no global sections.

In our case, $\deg D \geqslant \deg D - P = \deg D - 1 \geqslant 2g - 1 > 2g - 2$, so we have $h^0(\mathscr{O}(K - D)) = h^0(\mathscr{O}(K - D + P)) = 0$. From RR we deduce now that $\dim |D| = h^0(\mathscr{O}(D)) - 1 = \deg D - g$ and similarly that $\dim |D - P| = \deg(D - P) - g = \deg D - 1 - g$. In particular we see that $\dim |D - P| = \deg |D| - 1$, so by the first lemma above $|D|$ is base point free. And in case of strict inequality, we also get $\deg(D - P - Q) > 2g - 2$. As before, this implies that $\dim |D - P - Q| = \dim |D| - 2$, so by the second lemma above D is very ample.

$\square$

Finally we can aim at our original goal. Suppose that D is an ample divisor on our curve C. Then there is some integer $n > 0$ such that nD is very ample, that is, nD is linearly equivalent to a hyperplane section in some projective embedding of C. But this implies that $\deg nD > 0$, hence $\deg D > 0$.

Suppose conversely that $\deg D > 0$. Then we may find some big enough integer $n > 0$ such that $\deg nD > 2g$. So by the third lemma above nD is very ample, and therefore D is ample.

$\square$

4 The case of surfaces.

Let $X = S$ be a smooth projective surface over $k = \bar{k}$. In this case, a divisor on S is a $\mathbb{Z}$-linear combination of curves (not necessarily smooth) contained in S. To simplify the language, we will use the term curve now for any non-zero effective divisor on S. Hence a curve may have several irreducible components and a non reduced structure.

4.1 (Recall) The intersection pairing.

Recall that we had our (unique) intersection pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ with the following properties:

1. If C and D are non-singular and they meet transversally, then $C.D$ is the number of points of $C \cap D$.
2. Symmetry: $C.D = D.C$.
3. Additivity: $(C_1 + C_2).D = C_1.D + C_2.D$.
4. Invariant under linear equivalence: if $C_1 \sim C_2$, then $C_1.D = C_2.D$.

Recall also that if C is a non-singular irreducible curve and D meets C transversally, then the number of intersection points is equal to the degree of the restriction of $\mathscr{O}(D)$ to C:

$$\#(C \cap D) = \deg_C(\mathscr{O}(D) \otimes \mathscr{O}_C)$$

And if C and D are curves in S without any common component, then we may compute its intersection number with some algebra:

$$C.D = \sum_{P \in C \cap D} \mathrm{mult}_P(C \cap D)$$

where $\mathrm{mult}_P(C \cap D)$ is the intersection multiplicity of C and D at P, defined as the dimension of $\mathscr{O}_P/(f,g)$ over k, where f (resp. g) is a local equation for C (resp. for D) around P.

An interesting case is the self intersection of a curve $C.C = C^2$. Here all components are in common, so we cannot sum the intersection multiplicities at each intersection point. To deal with this case we must follow the existence proof and we see that the expression

$$C^2 = \deg_C \mathscr{O}(C) \otimes \mathscr{O}_C$$

still makes sense. But the thing on the RHS is precisely the normal sheaf of C in S. Indeed, the ideal sheaf of C in S is $\mathscr{O}(-C)$, so by tensoring with $\mathscr{I} = \mathscr{O}(-C)$ in $0 \to \mathscr{I} \to \mathscr{O}_S \to \mathscr{O}_C \to 0$ we get $\mathscr{I}/\mathscr{I}^2 \cong \mathscr{O}(-C) \otimes \mathscr{O}_C$. On the other hand, the normal sheaf of C in S is defined as its dual of $(\mathscr{I}/\mathscr{I}^2)^\vee = \mathscr{H}om(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_C)$. So we have what we wanted:

$$C^2 = \deg_C \mathscr{N}_{C/S}$$

Example (cf. Prop. V.3.2)**.** Let us study as an example the intersection theory on the blow-up $\pi\colon \tilde{S} \to S$ of our surface at a point $x \in S$. Denote by E the exceptional divisor.

First note that $E^2 = -1$, because the normal sheaf of E in $\tilde{S}$ is just the tautological line bundle on $\mathbb{P}^1$ (draw a picture).

Now we claim that $\operatorname{Pic}(\tilde{S}) = \operatorname{Pic}(S) \oplus \mathbb{Z}$, where the summand $\mathbb{Z}$ is generated by E and the inclusion $\operatorname{Pic}(S) \to \operatorname{Pic}(\tilde{S})$ is given by π^*. For this, consider the exact sequence (cf. Prop. II.6.5)

$$\mathbb{Z} \to \operatorname{Pic}(\tilde{S}) \to \operatorname{Pic}(S) \to 0$$

which on the left send $1 \mapsto E$. For any $n \neq 0$, $n \mapsto nE$ and $nE \neq 0$ in $\operatorname{Pic}(\tilde{S})$, because $(nE)^2 = -n^2 \neq 0$. So the map on the left is injective and we have a short exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Pic}(\tilde{S}) \to \operatorname{Pic}(S) \to 0$$

We can find a section of the map on the right by pulling back line bundles with π^*, so the sequence splits and we get our claim.

Next, if C is any curve in S, then

$$(\pi^* C).E = 0$$

The reason is that we may find a curve linearly equivalent to C and not passing through x to compute the intersection number, because in the proof of Lemma V.1.2 we had by Bertini a whole dense open subset of choices of such divisors. So we may assume $\pi^* C$ is disjoint with E, hence their intersection number is 0.

The same argument shows that to compute the intersection $(\pi^* C).(\pi^* D)$ for $C, D \in \operatorname{Pic}(S)$ we may assume that $C, D \in \operatorname{Pic}(S \setminus \{x\})$, hence π^* (which restricts to an isomorphism there) does not affect the outcome and we get

$$(\pi^* C).(\pi^* D) = C.D$$

Finally, again by assuming that C does not go through x, we get that if $C \in \operatorname{Pic}(S)$ and $D \in \operatorname{Pic}(\tilde{S})$, then

$$(\pi^* C).D = C.(\pi_* D)$$

where $\pi_*\colon \operatorname{Pic}(\tilde{S}) \to \operatorname{Pic}(S)$ denotes the projection.

4.2 The adjunction formula.

In general, if X is a smooth projective variety of dimension n and Y is a smooth subvariety of codimension 1, we can express the canonical sheaf of Y in terms of the canonical sheaf of X as follows.

Start with the short exact sequence (Thm. II.8.17)

$$0 \to \mathscr{T}_Y \to \mathscr{T}_X \to \mathscr{N}_{Y/X} \to 0$$

or its less geometric dual

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_X \otimes \mathscr{O}_Y \to \Omega_Y \to 0$$

Now take determinants (Exercise II.5.16)

$$\omega_X \otimes \mathscr{O}_Y \cong \det(\Omega_X \otimes \mathscr{O}_Y) = \det(\mathscr{I}/\mathscr{I}^2) \otimes \det(\Omega_Y) \cong \det(\mathscr{I}/\mathscr{I}^2) \otimes \omega_Y$$

Tensor both sides by the inverse line bundle of $\det(\mathscr{I}/\mathscr{I}^2)$, which is $\det(\mathscr{N}_{Y/X}) = \mathscr{N}_{Y/X}$ as we mentioned early, to obtain

$$\omega_Y \cong \omega_X \otimes \mathscr{O}_Y \otimes \mathscr{N}_{Y/X}$$

Finally, identify as before $\mathscr{N}_{Y/X}$ with $\mathscr{O}(Y) \otimes \mathscr{O}_Y$ to obtain our general adjunction formula

$$\omega_Y \cong (\omega_X \otimes \mathscr{O}_Y) \otimes (\mathscr{O}(Y) \otimes \mathscr{O}_Y)$$

In our case, if C is a smooth curve of genus g on S and K is a canonical divisor on S, we get

$$C.(C+K) = \deg_C(\omega_S \otimes \mathscr{O}(C) \otimes \mathscr{O}_C) = \deg_C(K_C) = 2g-2$$

Note that with this formula we can easily compute the genus of a curve in a surface, for example the genus of a degree d curve in $\mathbb{P}^2$ would be

$$g = \frac{d(d-3)+2}{2} = \frac{1}{2}(d-1)(d-2)$$

(Another nice way to show this is using Hurwitz's theorem. Hint: project to one axis, reformulate ramification and use Bézout).

Example (cf. Prop. V.3.3)**.** To illustrate this formula, let us compute the canonical divisor of the blow-up $\pi\colon \tilde{S} \to S$ of our surface at a point $x \in S$. From the previous example and since the canonical sheaf on $\tilde{S} \setminus E$ is the same as the canonical sheaf on $S \setminus \{x\}$, we may write

$$K_{\tilde{S}} = \pi^* K + nE$$

for some $n \in \mathbb{Z}$. We only have to determine this n, and for this we use our adjunction formula for E. We get $E.(E + K_{\tilde{S}}) = -1 + E.K_{\tilde{S}} = -2$, hence $E.K_{\tilde{S}} = -1$. On the other hand, by the previous example, $E.K_{\tilde{S}} = E.(\pi^*K + nE) = E.nE = -n$. Therefore $n = 1$.

In particular we have $K_{\tilde{S}}^2 = K^2 - 1$.

4.3 Riemann-Roch theorem for surfaces.

Recall that the arithmetic genus of our surface S was given by $p_a = \chi(\mathscr{O}) - 1$. Recall also our sloppy notation $h^i(\mathscr{F}) = h^i(S, \mathscr{F})$ for $\dim_k H^i(S, \mathscr{F})$. Sometimes we will be even more sloppy and write $h^i(D)$ instead of $h^i(\mathscr{O}(D))$.

Theorem (Riemann-Roch)**.** For any divisor D on S we have

$$h^0(D) - h^1(D) + h^0(K-D) = \frac{1}{2}D.(D-K) + 1 + p_a$$

Proof:

First, by Serre duality, we have $h^0(K - D) = h^2(D)$. Hence, on the LHS we have just $\chi(D)$.

Both RHS and LHS depend only on the linear equivalence class of D, so by the usual Bertini trick we may assume that $D = C - E$ is a difference of smooth curves on S.

From the short exact sequences

$$0 \to \mathscr{O}(-E) \to \mathscr{O} \to \mathscr{O}_E \to 0 \quad \text{and} \quad 0 \to \mathscr{O}(-C) \to \mathscr{O} \to \mathscr{O}_C \to 0$$

we obtain by tensoring with $\mathscr{O}(C)$ the short exact sequences

$$0 \to \mathscr{O}(C - E) \to \mathscr{O}(C) \to \mathscr{O}(C) \otimes \mathscr{O}_E \to 0$$

and

$$0 \to \mathscr{O} \to \mathscr{O}(C) \to \mathscr{O}(C) \otimes \mathscr{O}_C \to 0$$

Now we use additivity of the Euler characteristic in short exact sequences to deduce

$$\chi(D) = \chi(C - E) = \chi(\mathscr{O}) + \chi(\mathscr{O}(C) \otimes \mathscr{O}_C) - \chi(\mathscr{O}(C) \otimes \mathscr{O}_E)$$

But $\chi(\mathscr{O}) = 1 + p_a$ and the other two terms can be computed with Riemann-Roch for curves:

$$\chi(\mathscr{O}(C) \otimes \mathscr{O}_C) = C^2 + 1 - g_C \quad \text{and} \quad \chi(\mathscr{O}(C) \otimes \mathscr{O}_E) = C.E + 1 - g_E$$

Using the adjunction formula we get as above we obtain

$$g_C = \frac{1}{2}C.(C + K) + 1 \quad \text{and} \quad g_E = \frac{1}{2}E.(E + K) + 1$$

And putting it all together we are done.

□

4.4 Nakai-Moishezon criterion.

In the case of curves we had that D is ample if an only if $\deg D > 0$. If we try to do something similar here, we have a problem: how to define the degree of a divisor on a surface? Such a divisor is a curve, and the degree of a curve is not intrinsic. It depends on the embedding. So instead the condition that we look for is that D and all its self intersections should be numerically positive.

Theorem (Nakai-Moishezon criterion for ampleness)**.** A divisor D on S is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in S.

This characterization also works in higher dimensions:

Theorem. Let X be a complete scheme of finite type over $k = \bar{k}$. Let D be a (Cartier) divisor on X. Then D is ample if and only if $D^s.Y > 0$ for all integral closed subschemes $Y \subseteq X$ of dimension s and all $s \leqslant \dim(X)$.