HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. We introduce the stability condition for Higgs bundles and discuss the Hitchin–Kobayashi correspondence [Hit87, §3 and §4].

CONTENTS

—parts in gray will be omitted during the talk—

1. INTRODUCTION

Let M be a compact Riemann surface of genus g. Fix a Hermitian metric on M normalized to unit volume, i.e. such that

$$\mathrm{Vol}(M) = \int_M \omega_M = 1,$$

where $\omega_M \in A^{1,1}(M)$ is the corresponding Kähler form [Voi02, Beginning of §3.3] and $\mathrm{Vol}(M)$ is the Riemannian volume [Voi02, Lemma 3.8].

Hitchin's equations [Hit87, (1.3)] were originally stated for principal G-bundles over M, but for $G = \mathrm{SO}(3)$ they reduce to the study of rank 2 complex vector bundles on M [Hit87, Beginning of §2]. So let E be a rank 2 complex vector bundle on M and fix a Hermitian metric h on E [Huy05, Proposition 4.1.4]. A solution to Hitchin's equations in this set up consists of a unitary connection $\nabla : A^0(E) \to A^1(E)$ and a traceless endomorphism valued $(1, 0)$-form $\Phi \in A^{1,0}(\mathrm{End}_0(E))$ such that

$$\begin{cases} F + [\Phi, \Phi^*] = -2\pi i \mu(E) \mathbb{1}_E \omega_M, \\ \nabla^{0,1}\Phi = 0. \end{cases} \tag{1}$$

About the notation:

- $F : A^0(E) \longrightarrow A^2(E)$ is the curvature of the connection ∇.
- Φ can be locally written as ϕdz, where ϕ is a local section of $\mathrm{End}_0(E)$ and z is a local holomorphic coordinate. Then Φ^* can be locally written as $\phi^* d\bar{z}$, where ϕ^* is the h-adjoint of ϕ; and $[\Phi, \Phi^*]$ can be locally written as $[\phi, \phi^*]dz \wedge d\bar{z}$.
- $\mu(E)$ is the slope of the vector bundle E, defined as $\frac{1}{2}\deg(\Lambda^2 E)$.
- $\mathbb{1}_E \in A^0(\mathrm{End}(E))$ denotes the identity endomorphism.
- $\nabla^{0,1} : A^0(E) \longrightarrow A^{0,1}(E)$ denotes the $(0,1)$-part of ∇.
- We followed the sign convention in [Wen16, (4.17)]. Other references, such as [Tho05, p. 38], follow the opposite sign convention.

Since $d\bar{z} \wedge d\bar{z} = 0$, the integrability condition is automatic on a Riemann surface, so $\nabla^{0,1}$ defines a holomorphic structure on E [Huy05, Theorem 2.6.26]. We can regard Φ as a smooth section of $\mathrm{End}_0(E) \otimes K$. The condition $\nabla^{0,1}\Phi = 0$ then translates into Φ being a holomorphic section of $\mathrm{End}_0(E) \otimes K$. This in turn can be regarded as a holomorphic morphism $E \longrightarrow E \otimes K$. Moreover, if (E, Φ) is a solution which does not degenerate into a direct sum of line bundles, then every line bundle $L \subseteq$ such that $\Phi(L) \subseteq L \otimes K$ verifies $\mu(L) < \mu(E)$ [Wen16, Proposition 4.2.16]. This condition is what we will call *stability* of the Higgs bundle (E, Φ).

The existence proof in [Hit87, (4.3)] is based on Donaldson's proof of the existence of connections with constant central curvature on stable vector bundles [Don83], which was discussed in Tanuj's talk. The difference between the two settings is the introduction of the Higgs field Φ, so the first goal of this talk will be to understand stability of such pairs (E, Φ). In the second part of the talk we will see how to modify Donaldson's proof to produce solutions to Hitchin's equations.

2. Stability of Higgs bundles

Notation. In this first section, all vector bundles, morphisms and sections are assumed to be holomorphic. We will often go back and forth between vector bundles E and their sheaves of sections $\mathcal{E}$. We denote the set of rank 2 vector bundles on M by $\mathbf{Vec}_2(M)$.

Definition (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \in \mathbf{Vec}_2(M)$ and $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. We call Φ a *Higgs field* on E.

Remark. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{K})$$

we may identify Φ with a morphism $\Phi : E \longrightarrow E \otimes K$.

Definition (Stability). A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ-invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Exercise A. There are no stable Higgs bundles on $\mathbb{P}^1$. [*Hints below*[1]]

Lemma 1. *For every $E \in \mathbf{Vec}_2(M)$ there is a short exact sequence*
$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow 0$$
in which the left hand side is the trivial line bundle generated by $\mathbb{1}_E$. In particular, by Hirzebruch–Riemann–Roch,
$$\chi(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) = 3g - 3.$$

Proof. Over $x \in M$ we consider the projection $\operatorname{End}(E_x) \longrightarrow \operatorname{End}_0(E_x)$ given by
$$\phi \longmapsto \phi - \frac{\operatorname{tr}(\phi)}{2} \mathbb{1}_{E_x}.$$
The endomorphisms in the kernel are precisely the multiples of the identity, and this globalizes to the desired short exact sequence. For the computation of the Euler characteristic we use [Har77, Appendix A]. We have
$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$
therefore
$$\operatorname{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\mathcal{K}) = 3 + 3c_1(\mathcal{K}).$$
We also have
$$\operatorname{td}(\mathcal{K}^\vee) = 1 - \frac{c_1(\mathcal{K})}{2},$$
so multiplying the two expressions we obtain
$$\operatorname{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \operatorname{td}(\mathcal{K}^\vee) = 3 + \frac{3}{2}c_1(\mathcal{K}).$$
Since $\deg(c_1(\mathcal{K})) = 2g - 2$, we deduce that
$$\deg(\operatorname{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \operatorname{td}(\mathcal{K}^\vee))_1 = 3g - 3.$$
The second assertion follows now from Hirzebruch–Riemann–Roch [Har77, Theorem A.4.1]. $\square$

Notation. We define the following subsets of $\mathbf{Vec}_2(M)$:
- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{for general } \Phi \text{ there is no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

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[1]Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, (3.2) (iii)]

Lemma 2. *If $g \geqslant 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

Proof. Let $\pi \colon \mathbb{P}(E) \to M$ be the projectivization of our rank 2 vector bundle and let $O(-1) \to \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v. Let $O(l) := O(-1)^{\otimes(-l)}$. If $\mathcal{F}$ is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $O(l)$. Using the relative Proj construction we can write $\mathbb{P}(E) = \mathbb{P}(\mathcal{E}^\vee) := \mathrm{Proj}_M(S(\mathcal{E}^\vee))$, so [Har77, Exercise III.8.4] implies that

$$\pi_* \mathcal{O}(l) = \begin{cases} S^l(\mathcal{E}^\vee) & \text{if } l \geqslant 0, \\ 0 & \text{if } l < 0. \end{cases}$$

Let $x \in M$. Given $\phi \in \mathrm{End}(E_x)$, we define the quadratic form $v \mapsto \phi(v) \wedge v$ with values in $\Lambda^2 E_x$, which corresponds to an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $\phi = \lambda \mathbb{1}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 1 we obtain an injective homomorphism $\mathrm{End}_0(E_x) \to S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\mathrm{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_* \mathcal{O}(2) \otimes (\Lambda^2 \mathcal{E}) \otimes \mathcal{K}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_*(\pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2))$, hence an isomorphism

$$\psi \colon H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \cong H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2)).$$

Let $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. By construction of ψ, a line bundle $L \subseteq E$ is Φ-invariant if and only if $\psi(\Phi)$ vanishes at all $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$. In other words, L is Φ-invariant if and only if $\sigma(M) \subseteq \mathrm{div}(\psi(\Phi))$, where $\mathrm{div}(-)$ denotes the divisor of zeros of a section and $\sigma \colon M = \mathbb{P}(L) \to \mathbb{P}(E)$ is the section induced by $L \subseteq E$.

Suppose that $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is not nilpotent and let $L \subseteq E$ be a Φ-invariant line bundle. Over a general point $x \in M$ the corresponding traceless endomorphism $\phi \in \mathrm{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v \in E_x \setminus L_x$ in an eigenspace other than L_x. This gives us a point $[v] \in \mathbb{P}(E) \setminus \sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\mathrm{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\mathrm{div}(\psi(\Phi))$ is irreducible, then there are no Φ-invariant line bundles $L \subseteq E$. By Lemma 1 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g - 3 \geqslant 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Iit82, Theorem 7.19] tells us that $\mathrm{div}(\psi(\Phi))$ is irreducible for a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since in our case $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is nilpotent if and only if $\Phi^2 = 0$, a general Φ is not nilpotent. Therefore $E \in \mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^*\mathcal{L}(l)$ with $\mathcal{L}$ an invertible sheaf on M and $l \in \mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st, where $t \in H^0(\mathbb{P}(E), \pi^*\mathcal{N}(2 - l))$. Since our line bundle had non-zero global sections, both $\pi^*\mathcal{L}(l)$ and $\pi^*\mathcal{N}(2 - l)$ must have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

a) $l = 0$;
b) $l = 1$;
c) $l = 2$.

We analyze each case separately to conclude that

$$E \in \begin{cases} \mathbf{A} & \text{if } l = 0, \\ \mathbf{B} & \text{if } l \in \{1, 2\}. \end{cases}$$

Let us start with case a). Let $\pi^*s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by π^*s respectively we obtain the following commutative diagram:

$$
\begin{array}{ccc}
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) & \xrightarrow{\;\cong\;} & H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2)) \\
{\scriptstyle /s}\downarrow{\scriptstyle \cong} & & {\scriptstyle /\pi^*s}\downarrow{\scriptstyle \cong} \\
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee) & \xrightarrow{\;\cong\;} & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E})(2))
\end{array}
$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case b). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \to L$. The fiber-wise kernel has then dimension

1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Let $N \subseteq E$ be the largest line subbundle of E contained in the kernel of s. If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \operatorname{div}(s) \subseteq \operatorname{div}(\psi(\Phi))$. Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\operatorname{div}(\psi(\Phi))$ for all Φ and N is Φ-invariant for all Φ. Hence $E \in \mathbf{B}$ in this case.

In case c), the fixed divisor corresponds to a non-zero global section of $\pi^* \mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^* \mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2 \mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2 \mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s-invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$-invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\operatorname{tr}(s^2) = 0$. Suppose on the contrary that $\operatorname{tr}(s^2) \neq 0$. Fix some non-zero $s_1 \in H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E})$ and consider the linear map

$$\theta : \ H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) \longrightarrow H^0(M, \mathcal{K}^2)$$
$$\Phi' \longmapsto \operatorname{tr}(s^2) s_1 \Phi'$$

Since $\operatorname{tr}(s^2) s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Lemma 1 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) \geqslant 3g - 3 = h^0(M, \mathcal{K}^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to $\mathcal{K}^2$. But $\deg(\mathcal{K}^2) = 4g - 4 \geqslant 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Hence $E \in \mathbf{B}$ as well in this case. $\square$

Exercise B. Assume $g \geqslant 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to **A** or to **B**? [*Hints below[2]*]

Lemma 3. *Let $E \in \mathbf{Vec}_2(M)$ and $L \subseteq E$ a line bundle. We have the following short exact sequences:*

a) $0 \longrightarrow \mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^{\vee} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{E} \otimes \mathcal{L}^{-1} \otimes \mathcal{K} \longrightarrow 0.$

b) $0 \longrightarrow \mathcal{E}^{\vee} \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{L}^{-2} \otimes (\Lambda^2 \mathcal{E}) \otimes \mathcal{K} \longrightarrow 0.$

Moreover, the non-zero sections of the image of $\mathcal{L} \otimes (\Lambda^2 \mathcal{E}^{\vee}) \otimes \mathcal{K}$ in a) leave only L invariant; and the sections of the image of $\mathcal{E}^{\vee} \otimes \mathcal{L} \otimes \mathcal{K}$ in b) are those which leave at least L invariant.

Proof. All the short exact sequences are the result of tensoring another short exact sequence with $\mathcal{K}$, so let us find the necessary short exact sequences without $\mathcal{K}$. Under the isomorphism $E \cong E^{\vee} \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{\vee} \otimes \Lambda^2 \mathcal{E} \longrightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^{\vee} \subseteq E^{\vee}$. Tensoring with L and composing with the inclusion $E^{\vee} \otimes L \subseteq E^{\vee} \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^{\vee} \subseteq \mathrm{End}(E)$. Choosing a basis on each fiber and chasing all the identifications we have made so far, we see that the image of $L^2 \otimes \Lambda^2 E^{\vee}$ lies actually in $\mathrm{End}_0(E)$. Indeed, let V be a two dimensional $\mathbb{C}$-vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector l, which we may assume to be e_1. The first identification we have is $V \cong \mathrm{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \mapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^{\vee} \otimes \Lambda^2 V$, where $\alpha_v \in V^{\vee}$ is the linear form sending $e_1 \mapsto v_2$ and $e_2 \mapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in $L^{\vee}$, we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$V \longrightarrow L^{\vee} \otimes \Lambda^2 V$$
$$v \longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2)$$

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[2]Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \overset{\beta}{\longmapsto} \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$V \longrightarrow V$$
$$v \longmapsto \mu \lambda_v^\beta l$$

A basis for L is e_1, a basis for $L^\vee \otimes \Lambda^2 V$ is $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$ and a basis for $L \otimes (L^\vee \otimes \Lambda^2)^\vee$ is $e_1 \otimes \beta_0$, where $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$ is such that $\lambda_{e_2}^{\beta_0} = 1$. Writing the image of the basis $e_1 \otimes \beta_0$ under the map $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \to \operatorname{End}(V)$ as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because $\overline{\alpha_{e_1}} = 0$ and therefore $\lambda_{e_1}^\beta = 0$ for any β. We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \operatorname{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only L. This is the morphism from which we obtain the short exact sequence in a).

We regard this as a homomorphism into $\operatorname{End}(V)$ for a moment and use the basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\operatorname{End}(V)$, where e_{ij} denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the ij-th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\operatorname{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of $V^\vee$ represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism $\mathrm{End}(V) \cong \mathrm{End}(V^\vee)$ given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\mathrm{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of V that leave at least L invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion $\mathrm{Hom}(V, L) \subseteq \mathrm{Hom}(V, V)$ and the projection $\mathrm{pr}_0 : \mathrm{Hom}(V, V) \to \mathrm{End}_0(V)$, which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the short exact sequence in $b)$ and the right hand side of the short exact sequence in $a)$. $\square$

Lemma 4. *If $g \geqslant 2$ and E is a stable rank 2 vector bundle, then $E \in \mathbf{A}$.*

Proof. By Lemma 2 it suffices to show that it is not in $\mathbf{B}$. So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ-invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Consider the short exact sequence $b)$ from Lemma 3

$$0 \to \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \to \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \to \mathcal{L}^{-2} \otimes (\Lambda^2 \mathcal{E}) \otimes \mathcal{K} \to 0.$$

Since all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ leave L invariant, we get an induced isomorphism on global sections $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. The left hand side of the short exact sequence $a)$ in Lemma 3 factors by construction into a short exact sequence of the form

$$0 \to \mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K} \to \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \to \mathcal{Q} \otimes \mathcal{K} \to 0,$$

where $\deg(\mathcal{Q})$ can be seen to be 0 by computing first Chern classes. Riemann–Roch says then that $h^0(\mathcal{Q} \otimes \mathcal{K}) \in \{g - 1, g\}$. Since E is stable, we have $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$, and since the complete linear system corresponding to $\mathcal{K}$ is base-point free [Har77, Lemma IV.5.1] we have $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \leqslant g - 1$ by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \leqslant 2g - 1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g - 3.$$

If we want the two dimensions to be equal we must have $g = 2$ and $h^0(M, \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$. From the same long exact sequence as before we deduce, using that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) < g = 2$, that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) = 1$. In particular, $\deg(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 0$. We have $\deg(\mathcal{K}) = 2$ and by stability we had $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$, so we must have

$$\deg(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \in \{0, 1\}.$$

We treat each case separately, showing that both of them lead to contradiction.

If it is 0, then the existence of global sections implieis that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \to \mathcal{O} \to \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \to \mathcal{K} \to 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then $\mathcal{E}$ would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of $\mathcal{E}$ than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension, hence its extension class ξ, defined as the image of $\mathbb{1}_\mathcal{K}$ under the coboundary map $\delta \colon \mathrm{Hom}(\mathcal{K}, \mathcal{K}) \to \mathrm{Ext}^1(\mathcal{K}, \mathcal{O})$, is non-zero [Har77, Exercise III.6.1]. The coboundary map $H^0(M, \mathcal{K}) \to H^1(M, \mathcal{O})$ is given by cup product with $\xi \in H^1(M, \mathcal{T}_M)$ [Bre97, Theorem II.7.1]. Hence it is non-zero, and the long exact sequence implies then that $h^0(M, \mathcal{E}^\vee \otimes \mathcal{K} \otimes \mathcal{L}) \leqslant 2$, contradicting our previous conclusion that this dimension was 3.

If it is 1, again by the existence of a non-zero global secction we deduce that it is the line bundle corresponding to some point $x \in M$. The short exact sequence becomes

$$0 \to \mathcal{O}(x) \to \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \to \mathcal{K} \to 0,$$

and the extension class $\xi \in H^1(M, \mathcal{K}^{-1}(x))$ is again non-zero. Let $s \in H^0(M, \mathcal{K}(-x))$ be a non-zero global section, which exists because subtracting a point makes the dimension of the space of global sections go down at most by one, and in this case precisely by one by base point freeness. Since $h^0(M, \mathcal{K}) = h^0(M, \mathcal{K}^2(-x))$, we can write every section in $H^0(M, \mathcal{K}^2(-x))$ as st for some $t \in H^0(M, \mathcal{K})$. By Serre duality, the cup product gives us a perfect pairing

$$H^0(M, \mathcal{K}^2(-x)) \times H^1(M, \mathcal{K}^{-1}(x)) \to H^1(M, \mathcal{K}) \cong \mathbb{C}.$$

Since $\xi \neq 0$, we can find some $st \in H^0(M, \mathcal{K}^2(-x))$ such that $st \cup \xi \neq 0$. But the cup product is associative [Bre97, Proposition II.7.3] and the product of section st is by definition their cup product, so we must have $t \cup \xi \neq 0$. Therefore the coboundary map $\delta \colon H^0(M, \mathcal{K}) \to H^1(M, \mathcal{O})$ is non-zero, because it is given by cup product with ξ [Bre97, Theorem II.7.1] and so

$\delta(t) = t \cup \xi \neq 0$. This contradicts again our previous conclusion that $h^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) = 3$. $\square$

Proposition 5. *If $g \geqslant 2$, then*

$$\mathbf{S} = \mathbf{A}.$$

Proof. By definition $\mathbf{A} \subseteq \mathbf{S}$, so let us see the other inclusion. Let $E \in \mathbf{S}$. If E is stable, then $E \in \mathbf{A}$ by Lemma 4. So assume there exists $L \subseteq E$ such that $\mu(L) \geqslant \mu(E)$. Then $\deg(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 2g-2$, so Riemann–Roch implies that $h^0(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \in \{g-1, g\}$. In particular, from the short exact sequence *a)* in Lemma 3 we deduce that there exists a non-zero Higgs field leaving only L invariant. So if $E \in \mathbf{B}$, then this L must be the only line bundle invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since $\mu(L) \geqslant \mu(E)$, this contradicts the assumption that $E \in \mathbf{S}$. $\square$

Summarizing what we have seen in this first section, if M is a compact Riemann surface of genus $g \geqslant 2$ and E is a rank 2 vector bundle on M, then there are only two mutually exclusive possibilities:

- A general section $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ leaves no line bundle $L \subseteq E$ invariant.
- There exists a line bundle $L \subseteq E$ which is invariant under all sections $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$.

Moreover, E appears in a stable pair (E, Φ) if and only if a general section $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ leaves no line bundle $L \subseteq E$ invariant.

3. Idea of the existence theorem

Notation. From now on, all vector bundles, morphisms and sections are only assumed to be smooth unless otherwise stated. As in the introduction, we follow the notation in [Kob87] for differential forms, e.g. $A^{p,q}(E)$ denotes the space of (p, q)-forms over M with values in E. If h is a Hermitian metric on E and $\bar{\partial}_E : A^0(E) \to A^{0,1}(E)$ is holomorphic structure on E, we will denote by $\nabla(h; \bar{\partial}_E)$ the corresopnding Chern connection [Huy05, Proposition 4.2.14].

The plan is to reformulate Equation (1) in terms of a moment map and then apply the following:

Meta-Lemma 6. *Let (N, ω) be a Kähler mainfold and G a real Lie group acting on N symplectically. Assume we are in the following situation:*

- *The G action admits a G-equivariant moment map*

$$\mu : N \to \mathfrak{g}^\vee.$$

- *The Lie algebra $\mathfrak{g}$ admits an adjoint-invariant inner product $\langle \cdot, \cdot \rangle$ under which we can identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$.*
- *The G action extends to a complex group G^c of holomorphic transformations.*

Consider the smooth function $\|\mu\|^2 : N \longrightarrow \mathbb{R}$ given by

$$x \longmapsto \langle \mu(x), \mu(x) \rangle.$$

Assume moreover that we can find some $x_0 \in N$ such that

    a) G^c acts freely on the orbit of x_0;
    b) and x_0 minimizes $\|\mu\|^2$ in its G^c-orbit.

Then $\mu(x_0) = 0$.

Sketch of the argument. For $\xi \in \mathfrak{g}$ we denote by $\mu_\xi : N \longrightarrow \mathbb{R}$ the smooth function

$$x \longmapsto \langle \mu(x), \xi \rangle$$

and by $\xi_N \in \mathfrak{X}(N)$ the vector field given by the infinitesimal action

$$\xi_N(p) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot p.$$

Recall from Vincent's talk that μ being a moment map means that

$$d\mu_\xi(-) = \omega(\xi_N, -).$$

But since we are on a Kähler manifold [Voi02, §3] we can also write

$$d\mu_\xi(-) = g(I\xi_N, -).$$

By definition of the gradient, this last equation can be reformulated as

$$\operatorname{grad}(\mu_\xi) = I\xi_N.$$

Using that G^c acts freely on the orbit of x_0 we have that

$$I\xi_N(x_0) = (i\xi)_N(x_0) = \frac{d}{dt}\Big|_{t=0} \exp(ti\xi) \cdot x_0 \neq 0$$

for all $0 \neq \xi \in \mathfrak{g}$, because the exponential is invertible near $0 \in \mathfrak{g}$. Since x_0 minimizes $\|\mu\|^2$ along its G^c-orbit and $\operatorname{grad}(\|\mu\|^2)$ is tangential to the G^c-orbit [AB83, (8.12)], we have

$$\operatorname{grad}(\|\mu\|^2)(x_0) = 0.$$

Next we compute this gradient. On the one hand, since we are on a real Lie group, our inner product is symmetric, so that

$$d(\langle \mu, \mu \rangle) = \langle d\mu, \mu \rangle + \langle \mu, d\mu \rangle = 2\langle \mu, d\mu \rangle.$$

On the other hand, since $\mu_{\mu(x_0)} = \langle -, \mu(x_0) \rangle \circ \mu$, the chain rule implies that

$$\langle -, \mu(x_0) \rangle \circ d\mu_{x_0} = d(\langle -, \mu(x_0) \rangle)_{\mu(x_0)} \circ d\mu_{x_0} = d(\mu_{\mu(x_0)})_{x_0}.$$

If we take now $v \in T_{x_0}N$, then

$$
\begin{aligned}
g(\mathrm{grad}(\|\mu\|^2)(x_0), v) &= d(\|\mu\|^2)_{x_0}(v) \\
&= 2\langle \mu(x_0), d\mu_{x_0}(v) \rangle \\
&= 2(\langle \mu(x_0), - \rangle \circ d\mu_{x_0})(v) \\
&= 2(\langle -, \mu(x_0) \rangle \circ d\mu_{x_0})(v) \\
&= 2d(\mu_{\mu(x_0)})_{x_0}(v) \\
&= g(2\,\mathrm{grad}(\mu_{\mu(x_0)})(x_0), v).
\end{aligned}
$$

Therefore $\mathrm{grad}(\|\mu\|^2)(x_0) = 2\,\mathrm{grad}(\mu_{\mu(x_0)})(x_0) = 0$. As pointed out above, the map $\mathrm{grad}(\mu_{(-)})(x_0) : \mathfrak{g} \longrightarrow T_{x_0}N$ is injective, so this implies that

$$
\mu(x_0) = 0.
$$

$\square$

Let us see how to apply this to solve Equation (1). Assume from now on that $g \geqslant 2$. Let $E \to M$ be a complex vector bundle of rank 2 and let h be a Hermitian metric on E. Let $\mathcal{A}_E$ denote the space of unitary connections on E. We can endow N with a Kähler structure given by

$$
\omega_N((A_1, \Phi_1), (A_2, \Phi_2)) = -\int_M \mathrm{tr}(A_1 \wedge A_2) + 2i\,\mathrm{Im}(\mathrm{tr}(\Phi_1 \Phi_2^*))
$$

for $A_1, A_2 \in A^1(\mathfrak{u}_h(E))$ and $\Phi_1, \Phi_2 \in A^{1,0}(\mathrm{End}_0(E))$, where $\mathfrak{u}_h(E)$ is the vector bundle of skew-hermitian endomorphisms of E [Huy05, Corollary 4.2.11]. The gauge group $G = A^0(\mathrm{SU}_h(E))$ of special unitary automorphisms of E acts on N by conjugation on each factor, and this action admits a moment map

$$
\mu(\nabla, \Phi) = -F - [\Phi, \Phi^*] - 2\pi i \mu(E) \mathbb{1}_E \omega_M,
$$

cf. [Gal19, Proposition III.3.2]. The Lie algebra $\mathfrak{g} = A^0(\mathfrak{su}_h(E))$ of G consists of traceless skew-hermitian endomorphisms of E, and strictly speaking we have given an element in $A^2(\mathfrak{su}_h(E))$ rather than in $\mathfrak{g}^\vee$. But we can define on $\mathfrak{g}$ an invariant inner product

$$
\langle \alpha, \beta \rangle := \int_M \mathrm{tr}(\alpha\beta)\omega_M,
$$

and, under the analogous pairing between $\mathfrak{g}$ and $A^2(\mathfrak{su}_h(E))$, we may identify $\mu(\nabla, \Phi)$ with an element in $\mathfrak{g}^\vee$. The gauge action extends to the complex gauge group $G^c = A^0(\mathrm{SL}(E))$ of automorphisms of E with determinant 1 by the formula

$$
\psi \cdot (\nabla, \Phi) = (\nabla(h; \psi \nabla^{0,1} \psi^{-1}), \psi \Phi \psi^{-1}).
$$

By [Hit87, (3.15)] from Jin's talk, the action of G^c is free on the orbit of the stable pair (∇_0, Φ_0), so we can apply Meta-Lemma 6 to our situation.

Since the condition $\nabla^{0,1}\Phi$ is preserved by the G^c action, it suffices to find a minimum of $\|\mu\|^2$ on the orbit of such a stable pair to produce a solution to Equation (1).

Let then (∇_0, Φ_0) be a stable Higgs bundle, meaning that (E, Φ_0) is a Higgs bundle with respect to the holomorphic structure $\bar{\partial}_E = \nabla^{0,1}$. Since $\|\mu\|^2 \colon N \to \mathbb{R}_{\geqslant 0}$ is bounded below, we can pick a minimizing sequence (∇_n, Φ_n) for $\|\mu\|^2$ in the orbit of (∇_0, Φ_0). After some analysis [Hit87, pp. 80-81] we can apply Uhlenbeck's theorem, which allows us to assume that (∇_n, Φ_n) is weakly convergent. We have to show that its limit (∇, Φ) is still in the G^c-orbit of (∇_0, Φ_0).

Let $\psi_n \in G^c$ map (∇_n, Φ_n) to (∇_0, Φ_0). Doing some more analysis, we may assume that the sequence ψ_n is weakly convergent to some $\psi \neq 0$ [Hit87, p. 82]. By definition we have identities

$$\nabla^{0,1}_{(n,0)}\psi_n = 0 \text{ and } \Phi_0\psi_n = \psi_n\Phi_n,$$

where $\nabla_{(n,0)}$ denotes the connection induced by ∇_n and ∇_0 on $\mathrm{End}(E)$ [Huy05, Example 4.2.6 iii)]. These remain true in the limit, so ψ is a holomorphic morphism from E with the holomorphic structure $\nabla^{0,1}$ to E with the holomorphic structure $\nabla_0^{0,1}$ such that $\Phi_0\psi = \psi\Phi$. But ψ is non-zero and its determinant is a holomorphic function $M \to \mathbb{C}$, hence constant. So we only have the following two possibilities:

- ψ is an isomorphism, yielding an element in G^c that sends (∇, Φ) to (∇_0, Φ_0). In this case (∇, Φ) is in the same G^c-orbit as (∇_0, Φ_0), so we have found a solution to Equation (1).
- The image of ψ has generically rank 1, going down to 0 only at special points by upper-semicontinuity [Ati89, Proposition 1.3.2].

Assume then that we are in the second case. Since ψ is holomorphic, the saturation of $\psi(E)$ in E defines a line bundle $L \subseteq E$ holomorphic with respect to $\nabla_0^{0,1}$. But $\Phi_0\psi = \psi\Phi$, so L is Φ_0-invariant. Indeed, it suffices to check this generically, so let $x \in M$ such that $L_x = \psi(E_x)$ and let $v = \psi(v') \in L_x$. Then $\phi_0(v) = \phi_0(\psi(v')) = \psi(\phi(v')) \in \psi(E_x)$. By Proposition 5, since (∇_0, Φ_0) is stable, this is impossible for Φ_0 on a non-empty Zariski open subset $U \subseteq H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes K)$. So for a general Φ_0 the limit ψ had to be an isomorphism.

This already shows that a stable pair produces plenty of solutions, but not necessarily on the same orbit. To strengthen the conclusion, we also have to deal with arbitrary Φ. This can be done with a similar argument, applying Uhlenbeck's theorem again and taking advantage now from what we have shown for a general Φ. Indeed, let again (∇_n, Φ_n) be a minimizing sequence in the orbit of (∇_0, Φ_0), and assume now that $\Phi_n \in U$ for all $n \in \mathbb{N}$. We may assume again that this sequence weakly converges to some (∇, Φ).

The equation $\mu = 0$ is preserved under weak limits and, so $\mu(\nabla, \Phi)$ is a solution to Equation (1), hence stable by [Hit87, (2.1)]. The same argument as above shows that we have a holomorphic morphism $\psi : E \to E$ with respect to the holomorphic structures $\nabla^{0,1}$ and $\nabla_0^{0,1}$ respectively. By [Hit87, (3.15)] from Jin's talk it must be an isomorphism, so ψ yields an element in G^c taking (∇, Φ) to (∇_0, Φ_0).

Combining the discussion above with [Hit87, (2.7)] from Jin's talk, we have the following:

Theorem 7. *Let M be a compact Riemann surface of genus $g \geqslant 2$ and let (E, Φ) be a stable Higgs bundle. Then there is an automorphism of E with determinant 1 which takes $(\nabla(h; \bar{\partial}_E), \Phi)$ to a solution of Equation (1), unique up to gauge equivalence.*

We may think of this as an example of the Kempf–Ness theorem discussed in Vincent's talk, but in an infinite dimensional set up:

Corollary 8 (Hitchin–Kobayashi correspondence)**.** *Let M be a compact Riemann surface of genus $g \geqslant 2$. Let E be a complex vector bundle of rank 2 on M and h a Hermitian metric on E. Let G be the gauge group of special unitary automorphisms of E and G^c the complex gauge group of special linear automorphisms of E. Let $\mathcal{M}^s /\!/ G^c$ be the moduli space of stable Higgs bundles, obtained as a GIT-quotient, and let $\mathcal{M} /\!/ G$ be the moduli space of solutions to Equation (1), obtained as a symplectic quotient. Then we have a bijection*

$$\mathcal{M} /\!/ G \cong \mathcal{M}^s /\!/ G^c.$$

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