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Blow-ups in Algebraic Geometry

B.Sc Thesis by

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Abstract

The blow-up construction arose originally to deal with extension problems. For example, the obvious map $\mathbb{A}^2 - O \twoheadrightarrow \mathbb{P}^1$ does not extend to $\mathbb{A}^2$, but we may regard $\mathbb{A}^2 - O$ as an open subset of its blow-up at the origin and extend the map to the blow-up. Despite this different initial motivation, the blow-up became gradually a very important tool to resolve singularities. In 1964 it was proven by Hironaka that any algebraic variety in characteristic zero can be resolved by a finite sequence of blow-ups.

We will start studying the blow-up of algebraic varieties, where this construction will be more explicit. Some examples will be computed by hand in this context. Then we will move forward to schemes, where we will lose a bit of explicitness in exchange for a greater generality. We will discuss the main properties of the blow-up and three different ways to define it, namely with the gluing, with the Proj of a graded sheaf of algebras and by universal property. To conclude we will talk about blow-ups of regular varieties along regular subvarieties, computing their Picard groups and their canonical invertible sheaves.

To all my teachers and professors.

Contents

Chapter 1

The blow-up construction for algebraic varieties

In this chapter we will give a quick overview of the blow-up construction in the context of classical algebraic geometry. Some properties will already be treated here, but most of them will be stated and proven in the next chapter in the more general setting of schemes.

We will use the term affine (resp. projective) algebraic set for closed subsets of affine (resp. projective) space. We will use the term affine (resp. projective) variety for irreducible algebraic sets of affine (resp. projective) space. A quasi-affine (resp. quasi-projective) variety is a non-empty open subset of an affine (resp. projective) variety. If we don't specify, by (algebraic) variety we mean a quasi-projective variety. We will work over a fixed algebraically closed field k of characteristic zero. Since we are working on noetherian topological spaces, we can always decompose algebraic sets in their irreducible components. For this reason we will mainly work with algebraic varieties.

To define the blow-up we need to talk about the product of varieties. The usual cartesian product is a product in the category of affine varieties with the usual set theoretical projections. On the other hand, to define a product in the category of varieties we have to be more careful. We need to use the *Segre embedding* (product of coordinates in lexicographic order) to endow the cartesian product of projective spaces with a structure of projective variety (see [12] Chapter 5 Section 5.1. for a complete discussion on this regard). The conclusion is the following: if $X \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^m_k$ are both affine varieties, the closed subsets of $X \times Y$ are the common zero loci of sets of polynomials in the variables $x_1, ..., x_n, y_1, ..., y_m$; if $X \subset \mathbb{A}^n_k$ is affine and $Y \subset \mathbb{P}^m_k$ is projective, then the closed subsets of $X \times Y$ are the common zero loci of sets of polynomials in the variables $x_1, ..., x_n, y_0, ..., y_m$ which are homogeneous in the variables $y_0, ..., y_m$; and if $X \subset \mathbb{P}^n_k$ and $Y \subset \mathbb{P}^m_k$ are both projective varieties, then the closed subsets of $X \times Y$ are the common zero loci of sets of polynomials in the variables $x_0, ..., x_n, y_0, ..., y_m$ which are homogeneous separately in each set of variables $x_0, ..., x_n$ and $y_0, ..., y_m$.

Blowing up points

Consider the affine space $\mathbb{A}_k^n$ with coordinate ring $k[x_1, ..., x_n]$ and the projective space $\mathbb{P}_k^{n-1}$ with homogeneous coordinate ring $k[y_1, ..., y_n]$.

We define the blow-up of $\mathbb{A}_k^n$ at the origin O to be the closed subset $\widetilde{\mathbb{A}_k^n}$ of $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ defined by the equations

$$x_i y_j = x_j y_i \tag{1.1}$$

for $i, j = 1, ..., n$. The inclusion $\widetilde{\mathbb{A}_k^n} \hookrightarrow \mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ and the projection $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1} \twoheadrightarrow \mathbb{A}_k^n$ are algebraic morphisms, so their composition gives rise to a morphism $\pi \colon \widetilde{\mathbb{A}_k^n} \to \mathbb{A}_k^n$. The restriction of π to $\widetilde{\mathbb{A}_k^n} - \pi^{-1}(O)$ is an isomorphism onto its image $\mathbb{A}_k^n - O$, for we have an inverse morphism

$$\begin{aligned} \psi \colon \mathbb{A}_k^n - O &\to \widetilde{\mathbb{A}_k^n} - \pi^{-1}(O) \\ (a_1, ..., a_n) &\mapsto (a_1, ..., a_n) \times (a_1, ..., a_n) \end{aligned}$$

given by the universal property of the product and well defined (with image in $\widetilde{\mathbb{A}_k^n} - \pi^{-1}(O)$) because it satisfies the equations 1.1.

Since the equations 1.1 become trivial for the point $O \in A_k^n$, we also have $\pi^{-1}(O) \cong \mathbb{P}_k^{n-1}$. In particular, points of $\pi^{-1}(O)$ are in bijection with the set of lines passing through O in $\mathbb{A}^n$. We call $\pi^{-1}(O)$ the *exceptional divisor*[1], denoted by E.

Now, $\widetilde{\mathbb{A}_k^n} - \pi^{-1}(O) \cong \mathbb{A}_k^n - O$ is irreducible. Moreover, every point in $\pi^{-1}(O)$ is in the closure (in $\widetilde{\mathbb{A}_k^n}$) of the image by ψ of the corresponding line in $\mathbb{A}_k^n$ without the origin. Therefore, $\widetilde{\mathbb{A}_k^n} - \pi^{-1}(O)$ is dense in $\widetilde{\mathbb{A}_k^n}$. Since irreducibility passes to the closure, we deduce that $\widetilde{\mathbb{A}_k^n}$ is irreducible.

Remark. We may interpret geometrically this construction bearing in mind the original purpose of the blow-up: the extension of morphisms. We can think of what we did as replacing the origin by the set of directions through it, but with some care: we deform the affine space without the origin giving to every point the "height" in the new axis $\mathbb{P}_k^{n-1}$ corresponding to the line passing through that point and the origin in the affine space. In this way, we can extend the quotient map $\mathbb{A}_k^n - O \twoheadrightarrow \mathbb{P}_k^{n-1}$ to a morphism $\widetilde{\mathbb{A}_k^n} \to \mathbb{P}_k^{n-1}$ sending a point in the $\mathbb{P}_k^{n-1}$ axis to the same point in $\mathbb{P}_k^{n-1}$.

This extension property will become more clear later on when we generalize our first definition and regard the blow-up as the closure of the graph of the original map inside the product, allowing us to blow up more than just points.

[1] Because of the universal property of the blow-up (see last section of Chapter 2).

Definition. Let Y be an affine variety in $\mathbb{A}^n_k$ passing through the origin[2]. We define the *blow-up of* Y *at* O to be the closure in $\widetilde{\mathbb{A}^n_k}$ of $\pi^{-1}(Y - O)$. We denote it by $\widetilde{Y}$, and we also denote by $\pi\colon \widetilde{Y} \to Y$ the restriction of π to $\widetilde{Y}$.

As we saw before, $\pi\colon \widetilde{Y} \to Y$ induces an isomorphism of $\widetilde{Y} - \pi^{-1}(O) = \widetilde{Y} - \widetilde{Y} \cap E$ to $Y - O$. Since $\widetilde{Y} \cap E$ is a proper closed subset of $\widetilde{Y}$ (as long as $Y \neq O$), π is a birational morphism.

Example. Let Y be the *cuspidal cubic curve* given by the polynomial $f(x, y) = y^2 - x^3$ in $\mathbb{A}^2_k$, which has only one singular point (the origin).

Denote by $X = \widetilde{\mathbb{A}^2_k}$ and consider the blow-up of the affine plane at the origin

$$\begin{aligned} \pi\colon X &\to \mathbb{A}^2_k \\ (x, y, t, u) &\mapsto (x, y) \end{aligned}$$

Denote $\Omega_t = \{(t, u) \in \mathbb{P}^1_k \mid t \neq 0\}$ and $\Omega_u = \{(t, u) \in \mathbb{P}^1_k \mid u \neq 0\}$. Denote also $X_t = X \cap (\mathbb{A}^2_k \times \Omega_t)$, $X_u = X \cap (\mathbb{A}^2_k \times \Omega_u)$, $\widetilde{Y}_t = \widetilde{Y} \cap X_t$ and $\widetilde{Y}_u = \widetilde{Y} \cap X_u$. With this notations we have $X = X_t \cup X_u$ and therefore $\widetilde{Y} = \widetilde{Y}_t \cup \widetilde{Y}_u$.

First we observe that $X_t \cong \mathbb{A}^2_k$ through $(x, y, 1, u) \mapsto (x, u) \mapsto (x, xu, 1, u)$ and $X_u \cong \mathbb{A}^2_k$ through $(x, y, t, 1) \mapsto (y, t) \mapsto (ty, y, t, 1)$.

Now $\widetilde{Y}_t = \widetilde{Y} \cap X_t = \overline{\pi^{-1}(Y - O)}^X \cap X_t = \overline{\pi^{-1}(Y - O) \cap X_t}^{X_t}$. Inside X_t we can take $t = 1$ and we get the equations for $\pi^{-1}(Y - O) \cap X_t$:

$$y = xu \quad \text{and} \quad y^2 = x^3 = x^2u^2$$

and therefore, since $x \neq 0$,

$$x = u^2 \quad \text{and} \quad y = u^3$$

To take the closure inside X_t we use the isomorphism given before, which sends $\pi^{-1}(Y - O) \cap X_t$ to $\{(u^2, u) \in \mathbb{A}^2_k \mid u \neq 0\}$, whose closure inside $\mathbb{A}^2_k$ is $\{(u^2, u) \in \mathbb{A}^2_k \mid u \in k\}$, and therefore $\overline{\pi^{-1}(Y - O) \cap X_t}^{X_t} = \{(u^2, u^3, 1, u) \in X_t \mid u \in k\}$.

We proceed in a similar way with $\widetilde{Y}_u$. In this case, $\pi^{-1}(Y - O) \cap X_u = \{(\frac{1}{t^2}, \frac{1}{t^3}, t, 1) \in X_u \mid t \neq 0\}$, which corresponds to $\{(\frac{1}{t^3}, t) \in \mathbb{A}^2_k \mid t \neq 0\} = Z(y^3x - 1)$, which is already closed in $\mathbb{A}^2_k$. Hence $\pi^{-1}(Y - O) \cap X_u$ was already closed in X_u and $\widetilde{Y}_u = \{(\frac{1}{t^2}, \frac{1}{t^3}, t, 1) \in X_u \mid t \neq 0\}$.

In particular, $\widetilde{Y}_u \cap (X - X_t) = \varnothing$, so we finally obtain

$$\widetilde{Y} = \widetilde{Y}_t = \{(u^2, u^3, 1, u) \in X \mid u \in k\}$$

[2] We may assume that Y passes through the origin without loss of generality, because we can always perform a translation, which is an automorphism of $\mathbb{A}^n_k$.

This shows also that $\widetilde{Y}$ is non-singular, as $\widetilde{Y} \cong \mathbb{A}_k^1$ through $(x, y, t, u) \mapsto u \mapsto (u^2, u^3, 1, u)$. Since $\widetilde{Y}$ is isomorphic through $\pi\colon \widetilde{Y} \to Y$ to Y outside of the point $\pi^{-1}(O) = \widetilde{Y} \cap E = \{(0, 0, 1, 0)\}$, π is a birational morphism and we have resolved the singularity of Y.

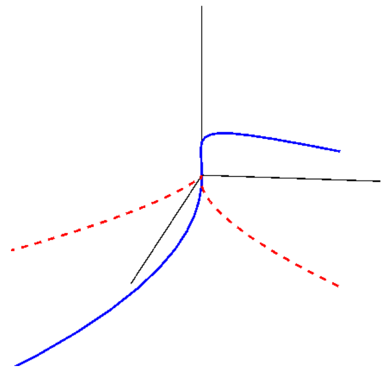

Figure 1.1: The cuspidal cubic curve, in red (dashed), and its blow-up at the origin, in blue.

As Harris suggests (see [5] Example 7.17.), we may regard $\widetilde{\mathbb{A}_k^n}$ as the closure in $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ of the graph of the quotient map $\mathbb{A}_k^n - O \twoheadrightarrow \mathbb{P}_k^{n-1}$. Indeed, we saw in order to prove irreducibility of $\widetilde{\mathbb{A}_k^n}$ that $\widetilde{\mathbb{A}_k^n} - \pi^{-1}(O)$ (which is precisely the graph of the quotient map) is dense in $\widetilde{\mathbb{A}_k^n}$.

This has a natural generalization. Consider the projection from the point $p = (1, 0, ..., 0) \in \mathbb{P}_k^n$ onto $\mathbb{P}_k^{n-1}$

$$\varphi\colon \mathbb{P}_k^n - p \to \mathbb{P}_k^{n-1}$$
$$(a_0, a_1, ..., a_n) \mapsto (a_1, ..., a_n)$$

We define the blow-up of $\mathbb{P}_k^n$ at p to be the closure in $\mathbb{P}_k^n \times \mathbb{P}_k^{n-1}$ of the graph of φ, denoted $\widetilde{\mathbb{P}_k^n}$. As before, we have a morphism $\pi\colon \widetilde{\mathbb{P}_k^n} \to \mathbb{P}_k^n$ obtained from the restriction of $\mathbb{P}_k^n \times \mathbb{P}_k^{n-1} \to \mathbb{P}_k^n$ to the first factor.

This construction coincides with the one given by Shafarevich (see [12] Chapter 2 Section 4.1). Moreover, if we take $\Omega_0 = \{(a_0, a_1, ..., a_n) \in \mathbb{P}_k^n \mid a_0 \neq 0\} \cong \mathbb{A}_k^n$ and we restrict φ to Ω_0, projecting onto $H_\infty = \{(a_0, a_1, ..., a_n) \in \mathbb{P}_k^n \mid a_0 = 0\} \cong \mathbb{P}_k^{n-1}$, we obtain the same quotient map $\mathbb{A}_k^n - O \twoheadrightarrow \mathbb{P}_k^{n-1}$ as before. Thus, the closure of the graph of the restriction of φ to Ω_0 inside $\Omega_0 \times H_\infty \cong \mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ coincides with our previous construction for $\widetilde{\mathbb{A}_k^n}$.

Hence, we can give the following generalization of our first construction:

Definition. Let $X \subset \mathbb{P}_k^n$ be a variety passing through p (recall footnote 2). The blow-up of X at $p \in X$, denoted $\widetilde{X}$, is the closure in $X \times \mathbb{P}_k^{n-1}$ of the graph of φ restricted to $X - p$, together with the morphism $\pi\colon X \to \mathbb{P}_k^n$ obtained as before.

The *exceptional divisor* of the blow-up is $E = \pi^{-1}(p) \subset \widetilde{X}$. Since E is a proper closed subset of $\widetilde{X}$ (as long as $X \neq p$) and π induces an isomorphism $\widetilde{X} - E \cong X - p$ (arguing as we did at the beginning of this section), π is a birational morphism.

We know that singular points in a variety are a proper closed subset (see [6] Theorem I.5.3 or [12] Chapter 2 Corollary to Theorem 2.9). Therefore curves have only finitely many singularities and we can deal with them just by blowing up points. But already for surfaces we may have whole lines of singular points: consider for example the surface given in space by the equation of any singular plane curve, which will have then singular vertical lines over the singular points of the curve.

Blowing up subvarieties

Our last definition is easy to generalize to blow up entire subvarieties:

Definition. Let $X \subset \mathbb{A}^n$ be an affine variety[3] and $Y \subset X$ a subvariety. Let $f_1, ..., f_r \in A(X)$ be a set of generators for the ideal of Y in X and set $U = X - Y = X - Z(f_1, ..., f_r)$. Consider the well-defined morphism

$$\begin{aligned} \varphi\colon U &\to \mathbb{P}_k^{r-1} \\ a &\mapsto \varphi(a) = (f_1(a), ..., f_n(a)) \end{aligned}$$

obtained by composition with the quotient morphism $\mathbb{A}_k^r - O \twoheadrightarrow \mathbb{P}_k^{r-1}$.

Let $\Gamma_\varphi = \{(a, \varphi(a)) \mid x \in U\} \subset X \times \mathbb{P}_k^{r-1}$ be its graph. We define *the blow-up of X along Y*, denoted $\widetilde{X}_Y$, to be the closure of Γ_φ inside $X \times \mathbb{P}_k^{r-1}$, together with the natural projection $\pi\colon \widetilde{X} \to X$. We call Y the *center* of the blow-up. The *exceptional divisor* of the blow-up is $E = \pi^{-1}(Y) \subset \widetilde{X}_Y$. If $Z \subset X$ is another subvariety, $\widetilde{Z}_{Z\cap Y} \subset \widetilde{X}_Y$ is called the *strict transform* of Z in the blow-up of X along Y.

As before, we have an induced isomorphism $U \cong \Gamma_\varphi$. Since X is irreducible and $Y \neq X$, U is a non-empty open subset of an irreducible space, U is also irreducible. Thus, Γ_φ is also irreducible, and so is $\widetilde{X}_Y$. So π is a birational morphism, because it induces an isomorphism on a non-empty (dense) oepn subset.

The trivial case $Y = X$ gives $\widetilde{X}_X = \varnothing$

[3] The irreducibility assumption is again not necessary in this definition. Indeed, if we drop it and we blow up arbitrary algebraic sets, we can consider the strict transforms of the irreducible components in the blow-up of our algebraic set. Since taking closures commutes with finite unions, the strict transforms of the irreducible components are precisely the irreducible components of the blow-up. So we can compute the blow-up component-wise and we get the same result (see [2] Chapter 9 Remark 9.12).

About the affine assumption, we may define in an analogous way (with homogeneous polynomials) the blow-up of projective varieties along subvarieties. But we will see that the blow-up is a local construction, so we can restrict our attention to the affine case.

We should check now that the previous construction is well-defined, meaning that it is independent of the choice of generators for the ideal of Y in X.

Proposition 1. *If we choose a different set of generators for the ideal $I_X(Y) \subset A(X)$, we obtain a variety $\widetilde{X}'_Y$ which is isomorphic to $\widetilde{X}_Y$ through an isomorphism ϕ making the following diagram commute:*

$$\begin{array}{ccc} \widetilde{X}_Y & \xrightarrow{\ \phi\ } & \widetilde{X}'_Y \\ & \searrow^{\pi} \quad \swarrow_{\pi'} & \\ & X & \end{array}$$

Proof. (from [2] Chapter 9 Lemma 9.16) Let $f'_1, ..., f'_s \in I_X(Y)$ be another set of generators. Since both sets generate the ideal, we can find $g_{11}, ..., g_{rs} \in A(X)$ and $h_{11}, ..., h_{sr} \in A(X)$ such that

$$f_i = \sum_{j=1}^{s} g_{ij} f'_j \quad \text{in } A(X) \text{ for each } i = 1, ..., r, \text{ and} \tag{1.2}$$

$$f'_j = \sum_{k=1}^{r} h_{jk} f_k \quad \text{in } A(X) \text{ for each } j = 1, ..., s. \tag{1.3}$$

Now define

$$\phi\colon \widetilde{X}_Y \longrightarrow \widetilde{X}'_Y$$
$$(a, b) = (a; b_1, ..., b_r) \longmapsto (a, b') = \left(a; \sum_{k=1}^{r} h_{1k}(a) b_k, ..., \sum_{k=1}^{r} h_{sk}(a) b_k\right)$$

We check that ϕ is well-defined. Let $(a, b) \in \Gamma_\varphi$. Since $(b_1, ..., b_r) = (f_1(a), ..., f_r(a)) \in \mathbb{P}_k^{r-1}$, we can find $\lambda \in k - 0$ such that $b_i = \lambda f_i(a)$ for all $i = 1, ..., r$, one of them at least being non-zero. Plugging equations 1.3 into equations 1.2, we obtain the new equations

$$f_i(a) = \sum_{j=1}^{s} g_{ij}(a) \left(\sum_{k=1}^{r} h_{jk}(a) f_k(a) \right) \quad \text{for each } i = 1, ..., r.$$

We can multiply the previous equations by λ and obtain the relations

$$b_i = \sum_{j=1}^{s} g_{ij}(a) \left(\sum_{k=1}^{r} h_{jk}(a) b_k \right) = \sum_{j=1}^{s} g_{ij}(a) b'_j$$

So if $b' = 0$, then $b = 0$, which is a contradiction with $b \in \mathbb{P}_k^{r-1}$. And since the equations remain valid in the closure, the same holds for any $(a, b) \in \widetilde{X}_Y$. Moreover, by construction we have that $\phi(a, b) \in \widetilde{X}'_Y$ for all points $(a, b) \in \Gamma_\varphi$, and therefore for all points (a, b) in the closure $\widetilde{X}_Y$. Hence, ϕ is well-defined. To check that it is an isomorphism, we construct ϕ^{-1} in the same way (changing the roles of the sets of generators). The commutativity of the diagram is straightforward. $\square$

From our definition, it follows immediately that for points $(a,b) = (a; b_1, ..., b_r) \in \Gamma_\varphi$ we have $b_i = f_i(a)$ for all $i = 1, ..., r$, and therefore $b_i f_j(a) = b_j f_i(a)$ for all $i, j = 1, ..., r$. Since this equations also hold on the closure $\widetilde{X}_Y$ of Γ_φ, we get the inclusion

$$\widetilde{X}_Y \subset \{(a,b) \in X \times \mathbb{P}_k^{r-1} \mid b_i f_j(a) = b_j f_i(a) \text{ for all } i, j = 1, ..., r\} \tag{1.4}$$

In particular, if we take Y to be the blow-up of the affine space at the origin according to our first definition 1.1, we can take the generators $x_1, ..., x_n$ in $k[x_1, ..., x_n]$. It follows that $\widetilde{\mathbb{A}_k^n}_O \subset Y$. It is in fact a closed subset, since they are both closed in the product $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$. But Y and $\widetilde{\mathbb{A}_k^n}_O$ are both irreducible and they share the non-empty open subset U, so they are birationally equivalent. By irreducibility of Y, this implies that $Y = \widetilde{\mathbb{A}_k^n}_O$, for otherwise we could write $Y = \widetilde{\mathbb{A}_k^n}_O \cup (Y - U)$ as a union of two proper closed subsets. Therefore, this construction generalizes the previous one in a very natural way (the constructions are equal, not only isomorphic).

Theorem 1. *The blow-up is a local construction, i.e. if X is an arbitrary variety, $Y \subset X$ is a subvariety, $U \subset X$ is a non-empty open subset and $\pi\colon \widetilde{X}_Y \to X$ is the blow-up of X along Y, then $\pi^{-1}(U) \subset \widetilde{X}_Y$ is the blow-up of U along $Y \cap U$.*

Proof. By irreducibility of Y, $U \cap Y$ is dense in Y. If a polynomial vanishes in a set of points, then it also vanishes in its closure. This gives the inclusion $I_X(U \cap Y) \subset I_X(Y)$. And since $U \cap Y \subset Y$, we also have the other inclusion $I_X(Y) \subset I_X(U \cap Y)$. So they are actually equal.

Let $f_1, ..., f_r \in A(X)$ be a set of generators of $I_X(Y) = I_X(U \cap Y)$. The corresponding map φ_U to construct $\widetilde{U}_{Y \cap U}$ is then just the restriction of the map φ to construct $\widetilde{X}_Y$ to the open subset $U - U \cap Y$.

Hence, $\Gamma_{\varphi_U} = \Gamma_\varphi \cap (U \times \mathbb{P}_k^{r-1})$. But $\pi^{-1}(U)$ is precisely the set of points $(a,b) \in \overline{\Gamma_\varphi}^{X \times \mathbb{P}_k^{r-1}}$ such that $a \in U$. So we get

$$\pi^{-1}(U) = \overline{\Gamma_\varphi}^{X \times \mathbb{P}_k^{r-1}} \cap (U \times \mathbb{P}_k^{r-1}) = \overline{\Gamma_\varphi \cap (U \times \mathbb{P}_k^{r-1})}^{U \times \mathbb{P}_k^{r-1}} = \overline{\Gamma_{\varphi_U}}^{U \times \mathbb{P}_k^{r-1}} = \widetilde{U}_{Y \cap U}$$

□

Example. Let X be the surface given by the polynomial $f(x,y,z) = x^3 - y^2 z$ in $\mathbb{A}_k^3$, whose singular locus is the z-axis (its intersection with the z-axis).

We compute its blow-up $\widetilde{X}$ along (its intersection with) the z-axis.

Denote $U = X - Z(x,y)$ and consider

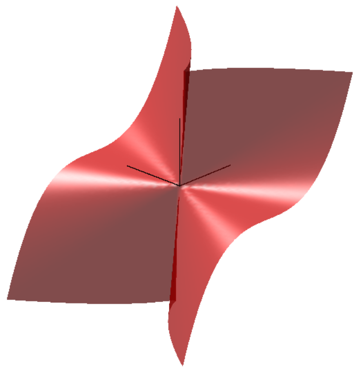

Figure 1.2: The surface given by $x^3 = y^2 z$ in $\mathbb{A}^3_k$.

$$\begin{aligned} \varphi \colon U &\to \mathbb{P}^1_k \\ (x,y,z) &\mapsto (x,y) \end{aligned}$$

Let $\Gamma = \{(x,y,z,t,u) \in X \times \mathbb{P}^1_k \mid t = x, u = y\} \subset X \times \mathbb{P}^1_k$ be its graph. Recall that $\widetilde{X} = \overline{\Gamma}^{X \times \mathbb{P}^1_k}$.

Denote again $\Omega_t = \{(t,u) \in \mathbb{P}^1_k \mid t \neq 0\}$ and $\Omega_u = \{(t,u) \in \mathbb{P}^1_k \mid u \neq 0\}$. Denote also $\widetilde{X}_t = \widetilde{X} \cap (X \times \Omega_t)$ and $\widetilde{X}_u = \widetilde{X} \cap (X \times \Omega_u)$.

Then $\widetilde{X}_t = \overline{\Gamma \cap (X \times \Omega_t)}^{X \times \Omega_t}$ and $\widetilde{X}_u = \overline{\Gamma \cap (X \times \Omega_u)}^{X \times \Omega_u}$, as in the previous example.

We start for example with $\widetilde{X}_t$:

$$\Gamma \cap (X \times \Omega_t) = \{(x,y,z,1,\frac{y}{x}) \in X \times \Omega_t \mid x^3 = y^2 z, x \neq 0\}$$

In particular, $y \neq 0$ and $z = \frac{x^3}{y^2}$. So $\Gamma \cap (X \times \Omega_t) = \{(x,y,\frac{x^3}{y^2},1,\frac{y}{x}) \in X \times \Omega_t \mid xy \neq 0\} \cong \{(z,u) \in \mathbb{A}^2_k \mid z \neq 0\}$ through $(x,y,z,1,u) \mapsto (z,u) \mapsto (zu^2, zu^3, z, 1, u)$. This shows that $\Gamma \cap (X \times \Omega_t)$ has dimension 2. But if we see $\Gamma \cap (X \times \Omega_t)$ inside $\mathbb{A}^4_k$ through $(x,y,z,1,u) \mapsto (x,y,z,u)$, it is contained in $Z(y - xu, x - zu^2)$, which is a 2-dimensional affine variety because its coordinate ring is $k[z,u]$. Hence, its closure in $\mathbb{A}^4_k$ is $Z(y - xu, x - zu^2)$ and $\widetilde{X}_t = \{(x,y,z,1,u) \in X \times \Omega_t \mid y = xu, x = zu^2\}$.

We do the same now for $\widetilde{X}_u$:

$$\begin{aligned}\Gamma \cap (X \times \Omega_u) &= \{(x, y, z, \frac{x}{y}, 1) \in X \times \Omega_u \mid x^3 = y^2 z, y \neq 0\} \\ &= \{(x, y, \frac{x^3}{y^2}, \frac{x}{y}, 1) \in X \times \Omega_u \mid y \neq 0\}\end{aligned}$$

Again, $\Gamma \cap (X \times \Omega_u) \cong \mathbb{A}^2_k$ through $(x, y, z, t, 1) \mapsto (y, t) \mapsto (ty, y, t^3 y, t, 1)$, so it has dimension 2. If we see it inside $\mathbb{A}^4_k$ through $(x, y, z, t, 1) \mapsto (x, y, z, t)$, it is contained in $Z(x - ty, z - t^3 y)$, which is a 2-dimensional affine variety because its coordinate ring is $k[y, t]$. Hence, its closure in $\mathbb{A}^4_k$ is $Z(x - ty, z - t^3 y)$ and $\widetilde{X}_u = \{(x, y, z, t, 1) \in X \times \Omega_u \mid x = ty, z = t^3 y\}$.

Notice finally that $\widetilde{X}_t \cong \widetilde{X}_u \cong \mathbb{A}^2_k$ through the previously mentioned mappings. Since $\widetilde{X} = \widetilde{X}_t \cup \widetilde{X}_u$ and both $\widetilde{X}_t$ and $\widetilde{X}_u$ are open in $\widetilde{X}$, we deduce that $\widetilde{X}$ is non-singular. And the blow-up morphism $\pi\colon \widetilde{X} \to X$ is a birational morphism, so we have successfully resolved the singularities of X.

Remark. In the previous example, we managed to resolve the singularities by only one blow-up with appropiate center. The choice of the center plays an important role in the resolution problem. For example, the *Whitney-umbrella*, given by the equation $x^2 = y^2 z$ in $\mathbb{A}^3_k$, has also its intersection with the z-axis for singular locus. As Hauser points out in [7] (Section 2 Exercise 9), if we blow-up along the z-axis, we resolve the singularity. But if we decide to blow-up the origin first, we will obtain still a singular variety and we will need to do more blow-ups to resolve it.

How to choose the center is not a trivial question. In fact, all existing resolution algorithms choose for the Whitney-umbrella the origin as the first center of blow-up. More about this discussion can be found in [7].

Chapter 2

The blow-up construction for schemes

In this chapter we will study the blow-up construction using the language of schemes. We will use the term scheme[1] for any ringed space which is locally isomorphic to a ringed space of the form $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, for some ring A (all rings are assumed to be commutative and unitary). If S is a scheme, we will use the term S-scheme for an object in the category of schemes over S (see [11] *slice category*). For affine schemes $S = \operatorname{Spec} A$ we will use the term A-scheme instead of $\operatorname{Spec} A$-scheme.

If k is an algebraically closed field, every variety over k can be seen as a k-scheme: there is a fully faithful functor from the category of varieties over k to the category of k-schemes (see [6] Proposition II.2.6.). A fully faithful functor induces bijections on the hom-sets. In particular, fully faithful functors are injective on objects up to isomorphism: the identity on $F(X) = F(Y)$ comes from unique morphisms $f \colon X \to Y$ and $g \colon Y \to X$ whose compositions map also to the identity on $F(X) = F(Y)$, and by injectivity on the hom-sets they must be respectively equal to the identity on X and the identity on Y. So we may safely regard every variety over k as a k-scheme. Moreover, we know which k-schemes represent varieties over k: the image of this functor is precisely the subcategory of integral quasi-projective k-schemes (see [6] Proposition II.4.10.). These are always integral separated finite type k-schemes, so from now on we will use the term variety to refer to an integral separated scheme of finite type over k. Hence we will achieve a greater generality with this setting and every result that we prove will also apply to the previous setting.

For a ring A and an ideal I of A we will denote by $V(I)$ the set of prime ideals of A which contain I. For an element $f \in A$ we will denote by $D(f)$ the set of prime ideals of A which do not contain f.

[1]This nomenclature differs from the original nomenclature used by Grothendieck in [3], where he used the term prescheme to refer to locally affine ringed spaces and reserved the term scheme for separated preschemes.

The Proj construction

Similarly to what happened on the previous chapter with the product of varieties, the Proj construction will be a key ingredient in the definition of the blow-up. But unlike in the case of varieties, we need a bit more than a smart embedding to define it, so we will devote this section to its study.

Let $S = \bigoplus_{n\geqslant 0} S_n$ be an $\mathbb{N}$-graded ring. Then S_0 is a subring of S and $S_+ = \bigoplus_{n>0} S_n$ is an ideal of S. Denote by S^h the set of homogeneous elements of S.

As a set, we define $\mathrm{Proj}(S)$ to be the set of prime ideals $\mathfrak{p}$ of S that are homogeneous ($\mathfrak{p} = \bigoplus_{n\geqslant 0}(\mathfrak{p} \cap S_n)$) and such that $S_+ \nsubseteq \mathfrak{p}$.

For every homogeneous ideal $\mathfrak{a}$ of S, denote $V_h(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$. The sets of this form are the closed subsets of a topology on $\mathrm{Proj}(S)$. This topology is also called the *Zariski topology*, because it is the topology induced in $\mathrm{Proj}(S)$ by the Zariski topology on $\mathrm{Spec}\, S$. For every $f \in S$ homogeneous, denote $D_h(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}$. These sets form a basis for the Zariski topology, because $\mathrm{Proj}(S) - V_h(\mathfrak{a}) = \mathrm{Proj}(S) - \cap_i V_h(f_i) = \cup_i D_h(f_i)$, where the f_i are a set of homogeneous generators of $\mathfrak{a}$. In particular, $\mathrm{Proj}(S)$ is covered sets of the form $D_h(f_i)$ for f_i a set of homogeneous generators of S_+.

For $\mathfrak{p} \in \mathrm{Proj}(S)$, denote $T_\mathfrak{p} = \{f \in S^h \mid f \notin \mathfrak{p}\}$. This is a multiplicative subset of S, because $1 \in T_\mathfrak{p}$ and if $f, g \in T_\mathfrak{p}$, then by primality of $\mathfrak{p}$ it follows that $fg \in T_\mathfrak{p}$. Therefore we may consider the ring of fractions $T_\mathfrak{p}^{-1}S$ of S with respect to $T_\mathfrak{p}$. This is a $\mathbb{Z}$-graded ring through the formula $\deg(\frac{g}{f}) = \deg(g) - \deg(f)$. Denote by $S_{(\mathfrak{p})}$ its subring on degree zero, explicitly given by $S_{(\mathfrak{p})} = (T_\mathfrak{p}^{-1}S)_0 = \{\frac{s}{t} \mid s \in S^h, t \in T_\mathfrak{p},\text{ and } \deg(s) = \deg(t)\}$. We will now define a sheaf of rings $\mathcal{O}$ on $\mathrm{Proj}(S)$ whose stalk at the prime ideal $\mathfrak{p}$ will be precisely $S_{(\mathfrak{p})}$.

Call $\pi\colon \sqcup S_{(\mathfrak{p})} \longrightarrow \mathrm{Proj}(S)$ the projection sending every element in $S_{(\mathfrak{p})}$ to $\mathfrak{p}$. For every open subset $U \subseteq \mathrm{Proj}(S)$, let $\mathcal{O}(U)$ be the set of sections of π defined on U (i.e. functions $s\colon U \longrightarrow \bigsqcup_{\mathfrak{p}\in\mathrm{Proj}(S)} S_{(\mathfrak{p})}$ whose composition with π is just the inclusion) such that for all $\mathfrak{p} \in U$ there exists an open neighbourhood $V \subseteq U$ of $\mathfrak{p}$ and homogeneous elements of S with the same degree f and g such that $V \subseteq D_h(f)$ and for all $\mathfrak{q} \in V$ we have $s(\mathfrak{q}) = \frac{g}{f} \in S_{(\mathfrak{q})}$.

The restriction maps are just the usual restrictions. With this, we have defined a presheaf of sets on $\mathrm{Proj}(S)$. But a couple of observations will show that it is indeed a sheaf of rings. First note that $\mathcal{O}(U)$ is the set of functions defined on U which locally look like a quotient of homogeneous elements of S of the same degree (analogous to the sheaf of regular functions on a projective variety). It makes sense therefore to define their addition and product pointwise, just adding and multiplying the corresponding fractions, and in this way we get a presheaf of rings and not just sets. And second note that the gluing axiom is satisfied because we are talking about actual functions defined on the open subsets of the topological space and their restrictions. So we have defined a sheaf of rings on $\mathrm{Proj}(S)$.

We should check now that the stalk at every $\mathfrak{p} \in \operatorname{Proj}(S)$ is indeed $S_{(\mathfrak{p})}$ as we wanted. So let $\mathfrak{p} \in \operatorname{Proj}(S)$. Define a ring homomorphism from $\mathcal{O}_{\mathfrak{p}}$ to $S_{(\mathfrak{p})}$ sending the germ of a section s defined in some neighborhood of $\mathfrak{p}$ to the fraction $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$. This is well defined, as two germs are equal if and only if they agree on a neighborhood of $\mathfrak{p}$, so in particular they must agree on $\mathfrak{p}$. It is also a ring homomorphism because of the way we defined the addition and multiplication in $\mathcal{O}_{\mathfrak{p}}$. To see that it is surjective, take an element $\frac{g}{f} \in S_{(\mathfrak{p})}$. Then $D_h(f)$ is an open neighborhood of $\mathfrak{p}$ and the germ of the section sending $\mathfrak{q} \in D_h(f)$ to $\frac{g}{f} \in S_{(\mathfrak{q})}$ maps to $\frac{g}{f} \in S_{(\mathfrak{p})}$. To see that it is injective, take two germs $s_{\mathfrak{p}}, t_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ mapping to the same element in $S_{(\mathfrak{p})}$. Then we can find sections s and t defined on a sufficiently small neighborhood V of $\mathfrak{p}$ such that $s(\mathfrak{q}) = \frac{g_1}{f_1} \in S_{(\mathfrak{q})}$ and $t(\mathfrak{q}) = \frac{g_2}{f_2} \in S_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$. It follows that $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ in $S_{(\mathfrak{p})}$. This means that there is some $h \in T_{\mathfrak{p}}$ such that $h(g_1 f_2 - g_2 f_1) = 0$ in S. But then again $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ in every $S_{(\mathfrak{q})}$ such that $\mathfrak{q} \in V \cap D_h(f_1) \cap D_h(f_2)$, so s and t agree on this open neighborhood of $\mathfrak{p}$ and they define the same germ in $\mathcal{O}_{\mathfrak{p}}$.

We will check now that for every homogeneous f of positive degree, there is a canonical isomorphism of ringed spaces from $(D_h(f), \mathcal{O}\restriction_{D_h(f)})$ to $\operatorname{Spec}(S_{(f)})$, where $S_{(f)}$ denotes again the subring of degree zero of the $\mathbb{Z}$-graded ring S_f obtained by inverting the powers of f in S.

First define the continuous map $\varphi\colon D_h(f) \longrightarrow \operatorname{Spec}(S_{(f)})$ by $\mathfrak{p} \longmapsto (\mathfrak{p}S_f) \cap S_{(f)}$. Note that $(\mathfrak{p}S_f) \cap S_{(f)}$ is the prime ideal of $S_{(f)}$ formed by fractions with numerators in $\mathfrak{p}$ and hence φ is well defined. By properties of the localization (see [1] Proposition 3.11.) the map φ is a bijection. Moreover, since $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $\varphi(\mathfrak{p}) \subseteq \varphi(\mathfrak{q})$ (just by set theoretical contemplation), the map φ is a homeomorphism for the Zariski topology.

Now we will see that the natural transformation $\varphi^{\#}$ of sheaves of rings on $\operatorname{Spec}(S_{(f)})$ induced by φ is an isomorphism. For each $V \subseteq \operatorname{Spec}(S_{(f)})$ open subset, the component at V of $\varphi^{\#}$ induced by φ is given by

$$\begin{aligned} \varphi^{\#}(V)\colon \mathcal{O}_{(\operatorname{Spec}(S_{(f)}))}(V) &\longrightarrow \varphi_*(\mathcal{O}\restriction_{D_h(f)})(V) = \mathcal{O}\restriction_{D_h(f)}(\varphi^{-1}(V)) \\ s &\longmapsto \varphi^{\#}(V)(s) \end{aligned}$$

To explicitly describe $\varphi^{\#}(V)(s)$, notice first that $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\varphi(\mathfrak{p})}$ are isomorphic for every $\mathfrak{p} \in D_h(f)$ (in fact they are practically equal), because in the end we are inverting the same elements and then taking the subring in degree zero of the result. And since φ is a homeomorphism, we may write every $\mathfrak{q} \in \operatorname{Spec}(S_{(f)})$ as the image of some $\mathfrak{p} \in D_h(f)$. Now, let $\mathfrak{q} = (\mathfrak{p}S_f) \cap S_{(f)} \in V$. Then $s(\mathfrak{q}) \in (S_{(f)})_{\mathfrak{q}} = (S_{(f)})_{\varphi(\mathfrak{p})}$ can be seen as an element of $S_{(\mathfrak{p})}$ through the previously mentioned isomorphism, and hence we may safely define $\varphi^{\#}(V)(s)(\mathfrak{q}) = s(\mathfrak{q}) \in S_{(\mathfrak{p})}$. Then $\varphi^{\#}$ is an isomorphism at every component, so it is an isomorphism of sheaves on $\operatorname{Spec}(S_{(f)})$ and $(\varphi, \varphi^{\#})$ is an isomorphism of ringed spaces.

Since $D_h(f_i)$ are an open cover of $\operatorname{Proj}(S)$ for a family $\{f_i\}_{i\in I}$ of homogeneous generators of S_+, we conclude that $\operatorname{Proj}(S)$ is a scheme. Furthermore, $\operatorname{Proj}(S)$ is an

S_0-scheme. Indeed, for each $i \in I$ there is a natural S_0-algebra structure on $S_{(f)}$ induced by $S_0 \subseteq S \to S_f$. Hence every $D_h(f_i)$ is an S_0-scheme, and this S_0-scheme structure agrees on the intersections $D_h(f_i) \cap D_h(f_j) = D_h(f_i f_j)$: the restriction of the respective scheme morphisms to the intersection is induced by the ring morphisms $S_0 \longrightarrow S_{(f_i)} \longrightarrow (S_{(f_i)})_{(f_j)}$ and $S_0 \longrightarrow S_{(f_j)} \longrightarrow (S_{(f_j)})_{(f_i)}$. But by the same argument that we gave before, the two rings $(S_{(f_i)})_{(f_j)}$ and $(S_{(f_j)})_{(f_i)}$ are the same as $D_h(f_i f_j)$. Thus the corresponding scheme morphisms are equal in the intersections. Since for any schemes X and Y the functor $\operatorname{Hom}_{\mathbf{Sch}}(-, Y)$ is a sheaf of sets on X, we obtain with the gluing axiom a unique morphism $\operatorname{Proj}(S) \longrightarrow \operatorname{Spec}(S_0)$.

Example. Let A be a ring and let $S = A[T_0, ..., T_n]$ be the polynomial ring over A in $n+1$ variables. Then S is an $\mathbb{N}$-graded ring with the usual degree of polynomials and $\mathbb{P}^n_A = \operatorname{Proj}(S)$ is called the projective n-space over A. The ideal S_+ is generated by the variables T_i, so if we denote $\Omega_i = D_h(T_i)$, we get an open cover of $\mathbb{P}^n_A$ by $n+1$ affine n-spaces over A: indeed, $\Omega_i \cong \operatorname{Spec}(A[T_0, ..., T_n]_{(T_i)}) = \operatorname{Spec}(A[X_0, ..., X_{i-1}, X_{i+1}, ..., X_n]) = \mathbb{A}^n_A$, with $X_j = \frac{T_j}{T_i}$.

Proposition 2. *Let S be an $\mathbb{N}$-graded ring and let $S_0 \longrightarrow B$ be an S_0-algebra. Then $S \otimes_{S_0} B$ is an $\mathbb{N}$-graded ring through the formula $\deg(s \otimes b) = \deg(s)$ and we have an isomorphism*

$$\operatorname{Proj}(S \otimes_{S_0} B) \cong \operatorname{Proj}(S) \times_{\operatorname{Spec} S_0} \operatorname{Spec} B$$

Proof. Since the sets of the form $D_h(f)$ for homogeneous elements of positive degree $f \in S$ cover $\operatorname{Proj}(S)$, the sets of the form $D_h(f) \times_{\operatorname{Spec} S_0} \operatorname{Spec} B$ cover the fiber product $\operatorname{Proj}(S) \times_{\operatorname{Spec} S_0} \operatorname{Spec} B$. But we have seen already that $D_h(f) \cong \operatorname{Spec}(S_{(f)})$, so the fiber product is covered by sets of the form $\operatorname{Spec}(S_{(f)}) \times_{\operatorname{Spec} S_0} \operatorname{Spec} B \cong \operatorname{Spec}(S_{(f)} \otimes_{S_0} B)$.

By the way we defined the degree on $S \otimes_{S_0} B$, the images $f \otimes 1 \in S \otimes_{S_0} B$ of the homogeneous $f \in S$ with positive degree generate the ideal $(S \otimes_{S_0} B)_+$. So for the same reason as before $\operatorname{Proj}(S \otimes_{S_0} B)$ is covered by sets of the form $\operatorname{Spec}((S \otimes_{S_0} B)_{(f \otimes 1)})$ for homogeneous elements $f \in S$ of positive degree. Hence it suffices to prove that for any such $f \in S$ the rings $S_{(f)} \otimes_{S_0} B$ and $(S \otimes_{S_0} B)_{(f \otimes 1)}$ are isomorphic. But we know that $S_f \otimes_{S_0} B \cong (S \otimes_{S_0} B)_{f \otimes 1}$ as rings, and this isomorphism preserves the grading. So the subrings on degree zero are also isomorphic. □

Since $\mathbb{Z}$ is an initial object in the category of rings, we obtain as a consequence of the previous proposition that for every ring A there is an isomorphism $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$. This motivates the definition of the projective n-space over an arbitrary scheme X as $\mathbb{P}^n_X = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X$.

In the affine case, we already checked that $\mathbb{P}^n_A$ is an A-scheme. It is still true in general that $\mathbb{P}^n_X$ is an X-scheme, because the fiber product projects to the second factor: $\mathbb{P}^n_X \xrightarrow{\pi_X} X$.

Definition. We will say that a scheme morphism $f\colon X \to Y$ is *projective*[2] if it factors as

$$\begin{array}{ccc} & \mathbb{P}^n_Y & \\ {}^{i}\nearrow & & \searrow^{\pi_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

for some $n \in \mathbb{N}$, where $i\colon X \hookrightarrow \mathbb{P}^n_Y$ is a closed immersion.

We will say that a scheme morphism $f\colon X \to Y$ is *quasi-projective* if it factors as

$$\begin{array}{ccc} & X' & \\ {}^{j}\nearrow & & \searrow^{g} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where $j\colon X \hookrightarrow X'$ is an open immersion and $g\colon X' \to Y$ is a projective morphism.

Theorem 2. *Projective morphisms are proper.*

Hartshorne proves the noetherian case in [6] (Theorem II.4.9.). A more general proof (using the same definition as Hartshorne of projective morphism) is given by Liu in [9] (Theorem 3.30.).

Blowing up affine schemes

As in the case of varieties (cf. Theorem 1), the blow-up construction for schemes will be a local construction. Therefore we will define it for affine schemes and then define it for general schemes by gluing the blow-ups of an open affine cover along the intersections in the separated case (and with a bit more of work in the general case).

Let A be a ring, I an ideal of A and $Z = \operatorname{Spec}(A/I)$ the corresponding closed subscheme of $\operatorname{Spec} A$ (there is a bijection between closed subschemes of an affine scheme and the ideals of the ring). The direct sum $\bigoplus_{n\geqslant 0} I^n$ is an $\mathbb{N}$-graded ring, where we adopt the convention $I^0 = A$.

Definition. The *blow-up* of $\operatorname{Spec} A$ along Z is defined as the scheme $\operatorname{Proj}(\bigoplus_{n\geqslant 0} I^n)$, denoted by $\widetilde{\operatorname{Spec} A}_Z$, together with the corresponding morphism

$$\widetilde{\operatorname{Spec} A}_Z \xrightarrow{\ \pi\ } \operatorname{Spec} A$$

[2]This definition of projective morphism is the one given by Hartshorne in [6]. Grothendieck gave a more general definition of projective morphism in [3]. A complete discussion about this can be found in the Stacks Project (see [14, Tag 01W7]).

We also say that $\widetilde{\operatorname{Spec} A}_Z$ is the blow-up of Z in $\operatorname{Spec} A$. We call Z the *center* of the blow-up and $\pi^{-1}(Z)$ the *expectional divisor*. Sometimes we may say that we are blowing up the ideal instead of the corresponding closed subscheme and use the notation $\widetilde{\operatorname{Spec} A}_I$.

There are again some trivial cases, namely $I = 0$ (which gives the empty scheme) and $I = A$ (which gives again $\operatorname{Spec} A$).

If A is reduced, then $A[T]$ is also reduced (a polynomial is nilpotent if and only if all its coefficients are nilpotent). Every localisation of a reduced ring is again reduced (because if $f^n = 0 \in S^{-1}A$ then $sf^n = 0 \in A$ for some $s \in S$, hence $sf = 0 \in A$ and $f = 0 \in S^{-1}A$). Since $\widetilde{A}_I$ can be covered by the affine opens $D_h(f) \cong \operatorname{Spec}(\oplus_{n\geqslant 0} I^n)_{(f)}$ for $f \in I$ (on degree 1) and $\operatorname{Spec}(\oplus_{n\geqslant 0} I^n)_{(f)}$ is a subring of $A[T]_f$, we get that $\widetilde{A}_I$ is a reduced scheme whenever A is reduced.

Since $\bigoplus_{n\geqslant 0} I^n$ is a subring of $\bigoplus_{n\geqslant 0} A = A[T]$, which is an integral domain whenever A is an integral domain, we get that $\widetilde{\operatorname{Spec} A}_I$ is an integral scheme whenever A is an integral domain and $I \neq 0$ (reduced because of our previous observation and irreducible because the Proj in this case is the closure of the zero ideal).

Proposition 3. *If I is a finitely generated ideal of A, then π is projective (and in particular proper).*

Proof. Let $I = (f_1, ..., f_r)$ and consider $\phi\colon A[T_1, ..., T_r] \longrightarrow \bigoplus_{n\geqslant 0} I^n$ sending T_i to the element f_i on degree 1. This is a surjective morphism of $\mathbb{N}$-graded A-algebras. Therefore ϕ induces a closed immersion (see [15] Exercise 8.2.B.)

$$\begin{array}{ccc} \widetilde{\operatorname{Spec} A}_I & \xhookrightarrow{\operatorname{Proj}(\phi)} & \mathbb{P}^r_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

which makes the triangle commute, because locally it is given by A-algebra morphisms and the category of affine schemes is equivalent to the the category of rings with reversed arrows (see [4] Theorem 2.35.). Therefore π is projective, and by Theorem 2, π is also proper. □

So in the affine case the blow-up of a variety is again a variety: an integral domain finite type k-algebra is noetherian, so π is a proper morphism and the blow-up is an integral separated finite type k-scheme.

Now we will check that the blow-up is indeed a local construction, as we wanted it to be. To prove this, we will prove a more general statement. Let A be a ring and I an ideal of A. Let $\varphi\colon A \to B$ be an A-algebra and J the ideal generated by $\varphi(I)$ in B. By Proposition 2 we have an isomorphism

$$\widetilde{\operatorname{Spec} A}_I \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \operatorname{Proj}((\oplus_{n\geqslant 0} I^n) \otimes_A B) = \operatorname{Proj}(\oplus_{n\geqslant 0}(I^n \otimes_A B)) \tag{2.1}$$

In general we have a surjective $\mathbb{N}$-graded A-algebra homomorphism $\bigoplus_{n\geqslant 0}(I^n \otimes_A B) \longrightarrow \bigoplus_{n\geqslant 0} J^n$, because on degree zero it is an isomorphism and on degree one it is surjective. This means that $\widetilde{\operatorname{Spec} B}_J$ is a closed subscheme of $\operatorname{Proj}(\bigoplus_{n\geqslant 0}(I^n \otimes_A B))$ (cf. proof of Proposition 3). But in general they are not isomorphic because the $\mathbb{N}$-graded A-algebra homomorphism fails to be injective, so in general we cannot say that blow-ups commute with pullbacks. But we can already say the following:

Proposition 4. *The blow-up is functorial, i.e. if* $\operatorname{Spec}\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *is an affine scheme morphism, then there is a unique scheme morphism* $\widetilde{\operatorname{Spec}\varphi}$ *such that making the following square commutative:*

$$\begin{array}{ccc} \widetilde{\operatorname{Spec} B}_J & \xrightarrow{\widetilde{\operatorname{Spec}\varphi}} & \widetilde{\operatorname{Spec} A}_I \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} B & \xrightarrow{\operatorname{Spec}\varphi} & \operatorname{Spec} A \end{array} \tag{2.2}$$

Proof. Indeed, the composition with the mentioned closed immersion with the projection from the fiber product above gives us a unique morphism $\widetilde{\operatorname{Spec}\varphi}$ making the diagram 2.2 commutative.

□

In the case where $\operatorname{Spec} B \longleftrightarrow \operatorname{Spec} A$ we have that $\widetilde{\operatorname{Spec} B}_J \longleftrightarrow \widetilde{\operatorname{Spec} A}_I$, because the pullback of a closed immersion is a closed immersion and the composition of closed immersions is a closed immersion. In this case we call $\widetilde{\operatorname{Spec} B}_J$ the *strict transform* of $\operatorname{Spec} B$ under the blow-up of $\operatorname{Spec} A$ at I.

Coming back to our main thread of argumentation, let us see now under what conditions blow-ups commute with pullbacks:

Proposition 5. *If B is a flat A-module, then the diagram 2.2 is cartesian.*

Proof. We check that in this case $\bigoplus_{n\geqslant 0}(I^n \otimes_A B) \longrightarrow \bigoplus_{n\geqslant 0} J^n$ is indeed an isomorphism. Consider the inclusion of A-algebras $\bigoplus_{n\geqslant 0} I^n \hookrightarrow \bigoplus_{n\geqslant 0} A = A[T]$. Tensor it over A with the flat A-module B to obtain an inclusion of A-algebras $(\bigoplus_{n\geqslant 0} I^n)\otimes_A B = \bigoplus_{n\geqslant 0}(I^n \otimes_A B) \hookrightarrow A[T] \otimes_A B = B[T] = \bigoplus_{n\geqslant 0} B$. The image of this inclusion is precisely $\bigoplus_{n\geqslant 0} J^n$ where J is the ideal generated in B by $\varphi(I)$. □

Lemma 1. *A ring morphism $\varphi\colon A \to B$ is flat if and only if the corresponding affine scheme morphism* $\operatorname{Spec}\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *is flat.*

Proof. This lemma follows from the fact that flatness is a local property (see [1] Proposition 3.10.) and from the following result: for any ring morphism $\varphi\colon A \to B$ and any B-module N (which in this case we take to be B itself), N is a flat A-module if and only if for every prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ the localization $N_{\mathfrak{q}}$ is a flat $A_{\varphi^{-1}(\mathfrak{q})}$-module (see [10] Theorem 7.1.). □

Proposition 6. *The blow-up is a local construction, i.e. if* $\operatorname{Spec} B \xhookrightarrow{\operatorname{Spec}\varphi} \operatorname{Spec} A$ *is an open immersion, then the diagram 2.2 is cartesian.*

Proof. By the Proposition 5 we only have to prove that the corresopnding ring morphism $\varphi\colon A \to B$ is flat. By the previous lemma, it suffices to prove that j is a flat morphism. But j is an open immersion, so it induces an isomorphism on every stalk and therefore j is flat. □

Now we can obtain one of the main features of the blow-up as a consequence of this proposition, namely that the blow-up induces an isomorphism outside of the center. Indeed, if $U = \operatorname{Spec} A - V(I)$, we have a cartesian square

$$\begin{array}{ccc} \pi^{-1}(U) & \hookrightarrow & \widetilde{\operatorname{Spec} A}_I \\ \downarrow & & \downarrow \pi \\ U & \hookrightarrow & \operatorname{Spec} A \end{array}$$

And then $\pi^{-1}(U) \cong \widetilde{U}_\varnothing$. But we already noted that this trivial case gives $\widetilde{U}_\varnothing = U$, hence $\pi^{-1}(U) \cong U$. In particular, if A is integral and $I \neq 0$, then π is a birational morphism.

The blow-up construction for general schemes

Most of the work was already done in the previous section. Now we have to generalize our construction by gluing the local pictures together and prove the analogous results for general schemes reducing to the affine case.

Let us see first that we can glue the local affine blow-ups together to define a global blow-up.

Lemma 2. *Let X be a scheme and let $Z \hookrightarrow X$ be a closed subscheme. There exists a unique X-scheme $\pi\colon \widetilde{X}_Z \to X$ such that its corestriction to each open affine subscheme is isomorphic to the corresponding affine blow-up.*

Proof. Uniqueness follows from the definition, so let us prove existence. Let X be a separated scheme. Let $\{X_i\}_{i\in I}$ be an affine open cover and for each $i \in I$ denote by Z_i the (scheme theoretical) intersection of Z with X_i given by $Z \times_X X_i$. Furthermore, for each $(i,j) \in I^2$ denote by X_{ij} the intersection of X_i and X_j and by Z_{ij} the intersection of Z and X_{ij}. Observe that $Z_{ij} = Z_i \cap X_j = Z_j \cap X_i$ (they are not equal, but canonically isomorphic, so we may safely identify them). In this situation, each X_{ij} is affine (because X is separated) and Z_{ij} is a closed subscheme of the affine scheme X_{ij}, hence also affine. We may then apply the affine case (Proposition 6) to obtain a cartesian square

$$\begin{array}{ccc} \widetilde{X_{ij}} & \hookrightarrow & \widetilde{X_i} \\ \downarrow & & \downarrow \\ X_{ij} & \hookrightarrow & X_i \end{array} \tag{2.3}$$

where we omit the centers of the blow-ups to simplify the notation (the center is always the intersection of Z with the corresponding subscheme of X). By existence and uniqueness of the affine case, we may glue the $\widetilde{X}_i$ along $\widetilde{X}_{ij}$ and we are done (for details on the gluing construction see [13] Chapter 5 Section 3.2 or [4] Proposition 3.10.).

If X is not separated, the gluing still works, but in this case one has to be a bit more careful: the interesction of two open affines is not affine in general. The way we solve this obstacle is by covering the intersections with basic affine open sets, which will then glue together by the same argument as above. Again, covering with the basic affine opens of one of the affine schemes or the other will yield the same result, so we can glue again along these glued pieces.

□

Definition. The scheme $\widetilde{X}_Z$ together with the morphism $\pi \colon \widetilde{X}_Z \longrightarrow X$ is called the *blow-up* of X along the closed subscheme Z (or with *center* Z). If the center is clear we will omit it to simplify the notation. We call $\pi^{-1}(Z)$ the *exceptional divisor* of the blow-up, where $\pi^{-1}(Z)$ denotes the scheme theoretical preimage $Z \times_X \widetilde{X}_Z$.

With this description, the blow-up construction is automatically local.

Theorem 3. *The blow-up is a local construction, i.e. it is compatible with open immersions meaning that for all* $U \hookrightarrow X$ *we have the following cartesian square:*

$$\begin{array}{ccc} \widetilde{U}_{Z\cap U} & \hookrightarrow & \widetilde{X}_Z \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_X} \\ U & \hookrightarrow & X \end{array} \tag{2.4}$$

where $Z \cap U$ *denotes the scheme theoretical intersection* $Z \times_X U$.

Proof. We may check that the diagram is cartesian on an open cover of the base. Hence we can reduce to the affine case, where the result was already proven. □

We have again the same trivial cases as before, namely $\widetilde{X}_X = \varnothing$ and $\widetilde{X}_\varnothing = X$. In what follows we will ignore such trivial cases. For example, we will say that any blow-up of a variety is a variety (which is false if we blow up the whole variety, for the empty scheme is not integral).

Corollary 1. *The blow-up induces an isomorphism outside of the center, i.e.* $X - Z \cong \widetilde{X}_Z - \pi^{-1}(Z)$.

Proof. The same as in the affine case. □

Theorem 4. *The blow-up is functorial, i.e. if* $f\colon Y \to X$ *is a scheme morphism, then there is a unique scheme morphism* $\widetilde{f}_Z$ *such that making the following square commutative:*

$$\begin{array}{ccc} \widetilde{Y}_{f^{-1}(X)} & \xrightarrow{\widetilde{f}_Z} & \widetilde{X}_Z \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow{f} & X \end{array} \tag{2.5}$$

Moreover, if f *is flat, then the square 2.5 is cartesian.*

Proof. Functoriality follows from the affine case, since we can always take an affine open neighborhood of the image of a point and an affine open neighborhood of the point intself mapping to the neighborhood of the image. We get a map between the affine local blow-ups making the square commute and then we glue all of them together to obtain $\widetilde{f}_Z$.

The second part of the theorem follows also from the affine case: if the morphis $f\colon Y \to X$ is flat, then for all open subschemes $V \subseteq Y$ and $U \subseteq X$ such that $f(V) \subseteq U$, the restriction $f\restriction_V\colon V \to U$ is flat (see [14, Tag 01U2] Lemma 28.24.3). □

If $Y \overset{f}{\hookrightarrow} X$ is a closed immersion, then $\tilde{f}$ is also a closed immersion. Indeed, if U_i is an affine open cover of X, then each corestriction $f^{-1}(U_i) \longrightarrow U_i$ is also a closed immersion. By the affine case, the corestriction $\tilde{f}^{-1}(\widetilde{U}_i) \longrightarrow \widetilde{U}_i$ is also a closed immersion. Since being a closed immersion is local on the target and the $\widetilde{U}_i$ cover $\widetilde{X}$, we get the result.

Definition. Let X be a scheme, $Z \hookrightarrow X$ a closed subscheme of X and $\widetilde{X}_Z$ the blow-up of X along Z. Let $Y \hookrightarrow X$ be another closed subscheme of X. Then the closed subscheme $\widetilde{Y}_{Z\cap Y} \hookrightarrow \widetilde{X}_Z$ of $\widetilde{X}_Z$ is called the *strict transform* of Y under the blow-up.

Our next goal is to show that the blow-up of a variety is again a variety.

It follows immediately from the affine case that the blow-up of a reducible scheme is reducible. Irreducibility and finite type need a bit more work.

Lemma 3. *If $X = \operatorname{Spec} A$ is an affine scheme and $Z = \operatorname{Spec}(A/I)$ is a closed subscheme of X, then the open subscheme $\widetilde{X}_Z - \pi^{-1}(Z)$ of the blow-up $\widetilde{X}_Z$ is dense.*

Proof. First note that if $f \in A$ is not a zero divisor, then $D(f)$ is dense in $\operatorname{Spec} A$. Indeed, if f is not a zero divisor then the localisation $A \to A_f$ is injective. Hence the corresponding affine scheme morphism $D(f) = \operatorname{Spec} A_f \to \operatorname{Spec} A$ is dominant and $D(f)$ is dense in $\operatorname{Spec} A$.

Let us call $S = \oplus_{n\geqslant 0} I^n$. We can cover $\widetilde{X}_Z$ by affine open subschemes of the form $D_h(fT)$ for f generators of the ideal I (the notation fT means that we refer to the element $f \in S$ on degree 1, we reserve f to denote the element $f \in S$ on degree 0). To show that $\widetilde{X}_Z - \pi^{-1}(Z)$ is dense in $\widetilde{X}_Z$ it suffices to show that each $D_h(fT) \cap (\widetilde{X}_Z - \pi^{-1}(Z))$ is dense in $D_h(fT) = \operatorname{Spec} S_{(fT)}$. We may compute the exceptional divisor using the formula 2.1, which gives $\pi^{-1}(Z) = \operatorname{Proj}(\oplus_{n\geqslant 0}(I^n \otimes_A A/I)) = \operatorname{Proj}(\oplus_{n\geqslant 0} I^n/I^{n+1})$. Hence $\pi^{-1}(Z)$ is covered by the open subschemes $D_h(fT + I^2)$ and inside $\operatorname{Spec} S_{(fT)}$ the closed subscheme $\pi^{-1}(Z) \cap D_h(fT)$ is then $D_h(fT + I^2) = \operatorname{Spec}((\oplus_{n\geqslant 0} I^n/I^{n+1})_{(ft+I^2)})$. The canonical surjective morphism $\phi\colon S_{(fT)} \to (\oplus_{n\geqslant 0} I^n/I^{n+1})_{(ft+I^2)}$ has $fS_{(fT)} \subseteq \operatorname{Ker}\phi$, because if $xT^n \in I^n$, then $fxT^n \in I^{n+1}$ (still on degree n) and therefore $f\frac{xT^n}{f^nT^n} \mapsto 0$ (recall that f denotes $f \in S$ on degree 0). Conversely, if $xT^n \in I^{n+1} \subseteq I^n$ then $\frac{xT^n}{f^nT^n} = \frac{fTxT^n}{f^{n+1}T^{n+1}} = f\frac{xT^{n+1}}{f^{n+1}T^{n+1}}$ in $S_{(fT)}$. Hence $x \in fS_{(fT)}$ and $\operatorname{Ker}\phi \subseteq fS_{(fT)}$. This shows that the closed subscheme $\pi^{-1}(Z) \cap \operatorname{Spec}(S_{(fT)})$ of $\operatorname{Spec}(S_{(fT)})$ is given by the ideal $fS_{(fT)}$. Hence its complement is $D(f)$ and by our first observation we only have to prove that f is not a zero divisor in $S_{(fT)}$. But $S_{(fT)}$ is a subring of S_{fT}, where fT is invertible (and therefore f is not a zero divisor in S_{fT}). So f is also not a zero divisor in $S_{(fT)}$ and we are done.

□

With this lemma we see that blow-ups preserve irreducibility: every non empty open subset of an irreducible space is irreducible. The blow-up induces an isomorphism outside of the center, so the complement of the exceptional divisor is also irreducible. By the lemma, it is dense in the blow-up. Since irreducibility is preserved by the closure, we conclude that the blow-up is irreducible.

Unlike in the affine case, we cannot guarantee that blow-ups are Hartshorne-projective in general: in the proof of Proposition 3, we had to make a choice of generators of the ideal I. The embedding to projective space is not canonical because of this choice, and therefore the local embeddings to projective space (we could take them all to the same projective space with the Segre embedding) will not necessarily glue to a global embedding of the whole blow-up[3].

But we can say that locally noetherian schemes have proper blow-ups, because properness is local on the target and we already know it for the affine case:

Theorem 5. *Blow-ups of locally noetherian schemes are proper.*

In fact there is a more general statement which says that blow-ups of locally noetherian schemes are Grothendieck-projective (see [4] Proposition 13.96.).

We can finally conclude that any blow-up of a variety is a variety. Indeed, we saw that irreducibility and reducedness are preserved (hence integrality). Since composition of finite type morphisms is again finite type, the blow-up of a variety is again a finite type k-scheme. And the same argument implies that it is separated over k. Therefore:

Theorem 6. *Any blow-up of a variety is again a variety.*

In particular, blow-ups are birational morphisms in the category of varieties. But not every birational morphism is a blow-up. For example normalizations, which are the best way to deal with curve singularities.

Proj description of the blow-up

Let X be a scheme. We will start with some quick overview of the basic definitions regarding sheaf of modules. We will give two equivalent descriptions of sheaves of modules. The first description, although slightly more abstract, is much more natural (due to the analogy with the category of modules over a ring). This description uses some category theory, which will not be explained here in detail. We refer to [8] in this regard.

The category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X is abelian (see [6] Section II.1 for the details) and monoidal with the tensor product (over $\mathbb{Z}$), which is defined as the sheaf associated to the naive tensor product presheaf

$$\begin{aligned} \otimes\colon \mathfrak{Ab}(X) \times \mathfrak{Ab}(X) &\to \mathfrak{Ab}(X) \\ (\mathscr{M}, \mathscr{N}) &\mapsto [U \mapsto \mathscr{M}(U) \otimes_{\mathbb{Z}} \mathscr{N}(U)] \end{aligned}$$

The unit object is the sheaf $\underline{\mathbb{Z}}$, which is the sheafification of the constant sheaf $\mathbb{Z}$. Moreover, sheaves of rings $\mathcal{O}$ on X are monoids in this category with multiplication $\mu\colon \mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$ induced by the presheaf morphism which for an open U of X is given by $\mu(U)\colon \mathcal{O}(U) \otimes_Z \mathcal{O}(U) \to \mathcal{O}(U)$, $a \otimes b \mapsto ab$, and with unit $\eta\colon \underline{\mathbb{Z}} \to \mathcal{O}$ induced by the only presheaf morphism between $\mathbb{Z}$ and $\mathcal{O}$.

[3] Not everything glues, just as not every diagram commutes! One has to be careful when dealing with non canonical morphisms.

Definition. An $\mathcal{O}_X$-module $\mathscr{M}$ is a pair $(\mathscr{M}, \lambda)$ where $\mathscr{M}$ is a sheaf of abelian groups on X and $\lambda\colon \mathcal{O}_X \otimes \mathscr{M} \to \mathscr{M}$ is a morphism of sheaves of abelian groups such that the following diagrams commute:

$$
\begin{CD}
(\mathcal{O}_X \otimes \mathcal{O}_X) \otimes \mathscr{M} @>{\cong}>> \mathcal{O}_X \otimes (\mathcal{O}_X \otimes \mathscr{M}) @>{\mathrm{id} \otimes \lambda}>> \mathcal{O}_X \otimes \mathscr{M} \\
@V{\mu \otimes \mathrm{id}}VV @. @VV{\lambda}V \\
\mathcal{O}_X \otimes \mathscr{M} @>{\lambda}>> @>>> \mathscr{M}
\end{CD}
\qquad
\begin{CD}
\underline{\mathbb{Z}} \otimes \mathscr{M} @>{\eta \otimes \mathrm{id}}>> \mathcal{O}_X \mathscr{M} \\
@V{\cong}VV @VV{\lambda}V \\
\mathscr{M} @= \mathscr{M}
\end{CD}
$$

Definition. Equivalently, an $\mathcal{O}_X$-module $\mathscr{M}$ is a sheaf of abelian groups such that for every open U in X, $\mathscr{M}(U)$ is an $\mathcal{O}_X(U)$-module and for every open immersion $V \subseteq U$ the restriction map $\mathscr{M}(U) \to \mathscr{M}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathscr{M}(V)$ is given by the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

In this case we get the morphism $\lambda\colon \mathcal{O}_X \otimes \mathscr{M} \to \mathscr{M}$ by taking the sheafification of the presheaf morphism $\lambda(U)\colon \mathcal{O}_X(U) \otimes_Z \mathscr{M}(U) \to \mathscr{M}(U)$ sending $a \otimes x \mapsto ax$.

We denote the tensor product in the category of $\mathcal{O}_X$-modules by $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$. We define sheaves of $\mathcal{O}_X$-algebras the same way (changing the word module for algebra in the previous definition).

If $X = \operatorname{Spec} A$ is an affine scheme and M is an A-module we define the sheaf of modules $\tilde{M}$ associated to M in an analogous way to the sheaf of rings on the Proj of an $\mathbb{N}$-graded ring or the sheaf of rings on the spectrum of a ring, namely associating to every open U of X the module of functions defined on U which locally look like fractions $\frac{x}{s}$ with $x \in M$ and $s \in A$.

Definition. A sheaf of $\mathcal{O}_X$-modules $\mathscr{M}$ is called *quasi-coherent* if X can be covered by open affines $U_i = \operatorname{Spec} A_i$, such that each $\mathscr{M}|_{U_i}$ is the sheaf of modules $\tilde{M}_i$ associated to an A_i-module M_i. If furthermore each M_i can be taken to be a finitely generated A_i-module, then we say that $\mathscr{M}$ is *coherent*.

Over an affine scheme $X = \operatorname{Spec} A$ the functor $M \mapsto \tilde{M}$ induces an equivalence of categories between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of A-modules with inverse the functor "taking global sections" $\mathscr{M} \mapsto \mathscr{M}(X)$ (see [6] Corollary II.5.5). In particular both functors are exact, because the categories are abelian, and being an abelian category is a property and not an extra structure[4].

[4]It is common to find in the literature definitions of abelian category which start from preadditive categories, because every abelian category is indeed preadditive. But preadditive categories are categories with some extra structure on the hom-sets, whereas being an abelian category is just a property: an abelina category is one in which finite coproducts (sums) and products exist and coincide (in particular, the empty product and coproduct gives us the zero object and the zero map between any two objects obtained by composition of the corresponding morphisms to and from the zero object), kernel and cokernel of any morphism exist (equalizer and coequalizer of the morphism and the zero morphism) and the factorization theorem holds ([8] Proposition VII.3.1). If a category is abelian, there is a canonical induced abelian group structure on the hom-sets (hint: given $M \rightrightarrows N$, use the universal properties of the product of M and N to define the sum of the morphisms).

Given a scheme morphism $f\colon X \to Y$ and a $\mathcal{O}_X$-module $\mathscr{M}$ we may define its *pushforward* or *direct image* to be just the usual pushforward of sheaves $f_*\mathscr{M}$, which is naturally a $\mathcal{O}_Y$-module through $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. On the other hand, if $\mathscr{N}$ is a $\mathcal{O}_Y$-module we cannot endow $f^{-1}\mathscr{N}$ with an $\mathcal{O}_X$-module structure in the same way, because the map that we obtain from $f^\#$ with the pullback of sheaves goes in the wrong direction ($f^{-1}\mathcal{O}_Y \to f^{-1}f_*\mathcal{O}_X \to \mathcal{O}_X$). For this reason we define its *pullback* or *inverse image* as $f^*\mathscr{N} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{N}$, which is naturally a $\mathcal{O}_X$-module. This two constructions are functorial. Moreover, we have an adjunction $f^* \dashv f_*$ (in particular f^* is right exact and f_* is left exact).

Definition. A graded $\mathcal{O}_X$-algebra $\mathscr{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras with a grading $\mathscr{S} = \oplus_{n\geqslant 0}\mathscr{S}_n$, where the $\mathscr{S}_n$ are quasi-coherent sub-$\mathcal{O}_X$-modules. We say that $\mathscr{S}$ is a homogeneous $\mathcal{O}_X$-algebra if $\mathscr{S}_1$ is coherent and generates $\mathscr{S}$ as a sheaf of algebras. In this case, for any affine open subscheme U of X, $\mathscr{S}(U)$ is then a homogeneous $\mathcal{O}_X(U)$-algebra.

The main example that we should keep in mind for our purposes is the case of $\mathscr{I}$ being a quasi-coherent sheaf of ideals on X. In this case $\mathscr{S} = \oplus_{n\geqslant 0}\mathscr{I}_n$ is naturally a homogeneous $\mathcal{O}_X$-algebra.

Let us now check that we can define the global Proj of a graded $\mathcal{O}_X$-algebra by gluing up the affine pieces.

Lemma 4. *For any scheme X and graded $\mathcal{O}_X$-algebra $\mathscr{S}$ there is a unique X-scheme $f\colon \operatorname{Proj}\mathscr{S} \to X$ such that for any affine open subscheme U of X, we have an isomorphism of U-schemes $f^{-1}(U) \cong \operatorname{Proj}\mathscr{S}(U)$ that is compatible with the restriction to any affine open subscheme V of U.*

Proof. The uniqueness follows from the definition itself, so let us show the existence. Assume first that X is affine. Then set $\operatorname{Proj}\mathscr{S} = \operatorname{Proj}\mathscr{S}(X)$. Let V be an affine open subscheme of X. Then $\operatorname{Proj}\mathscr{S}(V) \cong \operatorname{Proj}(\mathscr{S}(X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(V)) \cong (\operatorname{Proj}\mathscr{S}(X)) \times_X V$ by Proposition 2, so $\operatorname{Proj}\mathscr{S}(V) \cong f^{-1}(V)$.

Let us now consider an arbitrary separated [5] scheme X and cover it by affine open subschemes X_i. Then $\operatorname{Proj}\mathscr{S}|_{X_i}$ glue by the existence and uniqueness of the $\operatorname{Proj}\mathscr{S}|_{X_i\cap X_j}$. □

Definition. The resulting X-scheme is called $\operatorname{Proj}\mathscr{S}$ (we omit the morphism $\operatorname{Proj}\mathscr{S} \to X$ when clear or when not needed).

For any scheme X there is a bijection between closed subschemes and quasi-coherent sheaves of ideals on X. This bijection sends a closed subscheme $Y \overset{i}{\hookrightarrow} X$ to its *ideal sheaf* $\mathscr{I}_Y$, which is defined as the kernel of the morphism $i^\#\colon \mathcal{O}_X \to i_*\mathcal{O}_Y$ (which is quasi-coherent because this property is preserved by pullback of quasi-coherent sheaves of

[5] Again we assume without much loss of generality that we are working on a separated base scheme for simplicity.

modules and by kernels of maps between quasi-coherent sheaves of modules). Conversely, given a quasi-coherent sheaf of $\mathcal{O}_X$-ideals $\mathscr{I}$ denote by $V(\mathscr{I})$ the set of points $x \in X$ such that $\mathscr{I}_x \neq \mathcal{O}_{X,x}$ (which is a closed subspace of X) and by $j\colon V(\mathscr{I}) \to X$ the inclusion (which is a continuous map). The ringed space $(V(\mathscr{I}), j^{-1}(\mathcal{O}_X/\mathscr{I}))$ is a closed subscheme of X, and these two constructions are inverse of each other (see [6] Proposition II.5.9). This correspondence is the key to define the blow-up of a scheme along a closed subscheme.

Definition. Let X be a scheme, $Z \overset{i}{\hookrightarrow} X$ be a closed subscheme and let $\mathscr{I} \subseteq \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. The *blow-up* of X along Z is the X-scheme $\widetilde{X}_Z = \operatorname{Proj}(\oplus_{n\geqslant 0}\mathscr{I}^n)$.

By construction of the Proj (cf. Lemma 4) this last definition of blow-up verifies the defining property of our previous blow-up (cf. Lemma 2), so the two definitions are equivalent.

Remark. Although this second definition of blow-up is more natural and elegant in the sense that it is analogous to the construction of the affine case, it requires more theory to be studied[6]. For this reason we proved all the properties that we have seen so far with our first definition.

It follows from the affine case (cf. proof of Lemma 3) that the exceptional divisor is given by the formula

$$E = \operatorname{Proj} \oplus_{n\geqslant 0}\mathscr{I}^n/\mathscr{I}^{n+1} \tag{2.6}$$

Universal property of the blow-up

We will close this chapter with the universal property of blow-ups. This universal property can be taken also as a starting point to define the blow-up, but in that case existence has to be proven afterwards. What we will do is state this universal property and check that our definition of blow-up verifies it. This will constitute also a proof of its existence, so we will have given then 3 different equivalent definitions of the blow-up.

Definition. Let X be a scheme and E be a closed subscheme of X. Then E is an *effective Cartier divisor*[7] in X if for any $x \in X$ we can find an affine open neighbourhood $\operatorname{Spec} A$ of x in X such that $E \cap \operatorname{Spec} A$ is the closed subscheme $V(f)$ of $\operatorname{Spec} A$ defined by a non zero divisor $f \in A$.

Reading this definition should immediately evoque the proof of Lemma 3. Indeed, what we proved there is that the preimage of the center is an effective Cartier divisor. We proved only the affine case, but being an effective Cartier divisor is a local property, so the general case follows from the affine case.

[6]The theory of queasi-coherent sheaves, which we can only sketch here.

[7]This definition (from [7] Section 4) is not the best definition from the conceptual point of view, for we didn't define what a Cartier divisor is. But it keeps things more elementary (cf. [6] Section II.6).

Theorem 7. *Let X be a scheme and Z a closed subscheme. Let $\pi\colon \widetilde{X}_Z \to X$ be the blow-up of X along Z. Then $\pi^{-1}(Z)$ is an effective Cartier divisor and every X-scheme $\pi'\colon Y \to X$ with the same property factors uniquely through π:*

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & \widetilde{X}_Z \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

Proof. It remains to show that every other X-scheme π' with this property factors uniquely through π.

As usual, the general case follows from the affine case: being an effective Cartier divisor is a local property and the uniqueness of the factorization implies that we can glue the local factorizations to a globally defined morphism.

It suffices then to prove the affine case $X = \operatorname{Spec} A$. Recall the local description that we made of the blow-up in Lemma 3. If we blow up the ideal $I \subseteq A$, then the blow-up $\widetilde{X}_Z = \widetilde{\operatorname{Spec} A_I}$ is covered by the affine open subschemes $D_h(fT)$ when the f run along a generating system of I. In each of the $D_h(fT) = \operatorname{Spec} S_{(fT)}$ the center (image of I under $A \to S_{(fT)}$) is given by the ideal $fS_{(fT)}$, where f is a regular element (not a zero divisor) in the A-algebra $S_{(fT)}$. Moreover, if $\varphi\colon A \to C$ is another A-algebra such that $\varphi(f)$ is a regular element and generates $\varphi(I)C$, then there exists a unique A-algebra morphism $S_{(fT)} \to C$ sending each $\frac{xT}{fT}$ to the unique element $c \in C$ such that $\varphi(f)c = \varphi(x)$, which is the factorization that we were looking for. $\square$

In the case of noetherian and locally noetherian schemes we have many results involving effective Cartier divisors, so the universal property turns out to be especially useful (we refer to the Stacks Project in this regard, see [14, Tag 0B3Q]). The following is a good example:

Corollary 2. *Let X be a regular noetherian scheme and Z be a closed integral subscheme of codimension 1 in X. Then the blow-up is an isomorphism.*

Proof. If X is a noetherian scheme and Z is a closed integral subscheme of codimension 1 in X such that $\mathcal{O}_{X,x}$ is a UFD for all $x \in Z$, then Z is an effective Cartier divisor (see [14, Tag 0B3Q] Lemma 30.15.7.). But X is regular, so each $\mathcal{O}_{X,x}$ is a regular local ring, hence UFD (see [14, Tag 0AG0] Lemma 15.96.7.). Thus Z is an effective Cartier divisor and the identity on X satisfies the universal property. $\square$

Chapter 3

Blowing up regular varieties along regular subvarieties

In this last chapter we study the blow-up of regular varieties over an algebraically closed field k along regular subvarieties[1]. Recall that by a variety we mean an integral separated finite type k-scheme. And by subvariety we will always mean closed subvarieties.

Our goal will be to express the Picard group of the blow-up in terms of the Picard group of the variety and to compute the canonical invertible sheaf on the blow-up. To get there, we need some preliminary notions and results. Just as with sheaves of modules, these notions are not that central to our main discussion and they are fairly common notions, so we will go quickly over them giving enough references for further details. We will also use some facts without proving them, such as the fact that the blow-up of a regular variety along a regular subvariety is again regular (see [6] Theorem II.8.24.).

Invertible sheaves, Weil divisors and the Picard group

Let X be a scheme. An $\mathcal{O}_X$-module $\mathscr{E}$ is called *locally free of finite type* (or rank) if X can be covered by open subschemes U such that each restriction $\mathscr{E}|_U$ is isomorphic to $\oplus_{i=1}^r \mathcal{O}_U$ as an $\mathcal{O}_U$-module for some $r \in \mathbb{N}$. Note that locally the rank is well-defined and in particular if X is irreducible we may define $\mathrm{rk}(\mathscr{E}) \in \mathbb{N}$ (by looking at the stalks it boils down to the fact that isomorphic finite dimensional vector spaces have the same dimension). If r is the same in each element of the cover we say that $\mathscr{E}$ is a locally free sheaf of rank r.

A locally free $\mathcal{O}_X$-module of rank 1 is called an *invertible sheaf* on X. The name is justified as follows. The set of (isomorphism classes of) invertible sheaves on X is closed

[1] One could ask at this point how many subvarieties of a regular variety are again regular. In projective space, Bertini's Theorem tells us that there are plenty (see [6] Theorem II.8.18.).

with respect to the tensor product, because the tensor product commutes with restriction to open substets and $\mathcal{O}_X \otimes \mathcal{O}_X = \mathcal{O}_X$. For any $\mathcal{O}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, define the *hom-sheaf* as the sheafification of the presheaf sending every open U of X to the abelian group $\operatorname{Hom}_{\mathcal{O}_U}(\mathscr{M}|_U, \mathscr{N}|_U)$, and denote it by $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, \mathscr{N})$. With this definition we get the usual adjunction $\mathscr{M} \otimes (-) \dashv \mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, -)$. For an $\mathcal{O}_X$-module $\mathscr{M}$, define its dual as $\mathscr{M}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X)$. Since hom-sheaf commutes with restriction to open subsets, if a sheaf of modules is locally free of finite type then so is its dual. And since free modules of finite type are reflexive, the canonical morphism $\mathscr{E} \to (\mathscr{E}^\vee)^\vee$ (adjoint to the evaluation morphism $\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{E}^\vee \to \mathcal{O}_X$) is an isomorphism. Thus, if $\mathscr{L}$ is an invertible sheaf on X, its dual is again an invertible sheaf on X and their tensor product is isomorphic to $\mathcal{O}_X$. Hence the name invertible.

Since the tensor product is associative and commutative (when considered on isomorphism classes), the previous discussion shows that the set of (isomorphism classes of) invertible sheaves on X forms a group with the tensor product.

Definition. The group of invertible sheaves on a scheme X with the tensor product is called the *Picard group* of X, denoted by $\operatorname{Pic}(X)$.

The Picard group is functorial: every morphism of schemes $f\colon X \to Y$ induces a canonical group homomorphism $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$, sending the equivalence class of $\mathscr{L}$ to the equivalence class of $f^*\mathscr{L}$. The pullback of an inverse sheaf is an inverse sheaf, because pullbacks commute with restriction to open subsets and $f^*\mathcal{O}_Y = \mathcal{O}_X$ (unlike with the pushforward, sheaves of modules behave good with pullbacks, see [6] Proposition II.5.8.).

Remark. Invertible sheaves on a smooth variety X correspond to classical line bundles through the usual correspondence: invertible sheaves are sheaves of sections of line bundles and line bundles are global sections of invertible sheaves. The philosophy is as usual to understand the geometric object via the functions on it, which are algebraic objects easier to deal with. The tensor product of invertible sheaves corresponds then to multiplication of transition functions. For this reason it is also common to refer to invertible sheaves as line bundles and more generally to locally free of finite type sheaves as vector bundles.

Let now X be a regular variety and K its fraction field. A *prime divisor* or irreducible divisor on X is an integral closed subscheme Z of codimension 1.

Definition. A *Weil divisor* on X is a formal $\mathbb{Z}$-linear combination of prime divisors on X. The group of Weil divisors is denoted by $\operatorname{Div}(X)$. A Weil divisor is called *effective* if all its coefficients are non-negative.

Since X is regular, every prime divisor is an effective Cartier divisor (cf. proof of Corollary 2). If Y is a prime divisor on X, then the local ring at its generic point is a discrete valuation ring of K. The corresponding discrete valuation is called *valuation of* Y, and denoted by v_Y. Since X is separated, Y is uniquely determined by its valuation. For every nonzero rational function $f \in K^\times$, we say that f has a zero along Y (resp. a pole along Y) of order $v_Y(f)$ (resp. of order $-v_Y(f)$) if the integer $v_Y(f)$ is positive (resp. negative). This can only be the case for finitely many prime divisors (which are called the support of the divisor). Hence it makes sense to define the *divisor* of f, denoted by (f), as $(f) = \sum v_Y \cdot Y \in \mathrm{Div}(X)$. A divisor which is equal to the divisor of a function is called a *principal divisor*. Note that, by the properties of valuations, sending a function to its divisor gives a group homomorphism $K^\times \to \mathrm{Div}(X)$, and in particular principal divisors form a subgroup of $\mathrm{Div}(X)$. Hence we may consider the quotient group called the *divisor class group* of X and denoted by $\mathrm{Cl}(X)$. Two divisors in the same equivalence class are said to be *linearly equivalent*. See [6] Section II.6. for more details.

Proposition 7. *Let X be a regular variety, Z be a proper closed subet of X and $U = X - Z$. If Z has codimension at least 2, then $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$. If Z is irreducible and has codimension 1, we have an exact sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$ (where the maps are defined in the proof below).*

Proof. First we define the surjective homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ as follows. Let Y be a prime divisor on X. Then $Y \cap U$ is either empty or a prime divisor on U. Send $D = \sum n_i Y_i$ to $\sum n_i (Y_i \cap U)$, ignoring those indices for which $Y_i \cap U$ is empty. This gives a surjective map $\mathrm{Div}(X) \to \mathrm{Div}(U)$, because every divisor on U is the restriction of its closure in X. Now if $f \in K^\times$ has $(f) = \sum n_i Y_i$, then f as a rational function on U has $(f)_U = \sum n_i (Y_i \cap U)$, hence we have a well-defined surjective map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$.

By definition, the groups $\mathrm{Div}(X)$ and $\mathrm{Cl}(X)$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension at least 2 does not change them and the previous map is an isomorphism.

Finally, if Z has codimension 1, the kernel of the previous map consists precisely of those divisors whose support is contained in Z. So if Z is also irreducible, the kernel is just the subroup of $\mathrm{Cl}(X)$ generated by $1 \cdot Z$, and we get the map $\mathbb{Z} \to \mathrm{Cl}(X)$ defined as $1 \mapsto 1 \cdot Z$ making the sequence exact. □

For regular varieties X we have a natural isomorphism $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ (see [6] Corollary II.6.16.). This isomorphism sends (the class of) a Weil divisor D to (the class of) the sheaf $\mathcal{O}_X(D)$ which sends an open subset U of X to the ring of rational functions defined on U that are either zero or have poles and zeros “constrained by D”, meaning that the corresponding divisor plus the restriction of D to the open U is effective: a positive coefficient in D *allows* a pole of that order, a negative coefficient *demands* a zero of that order and without any restrictions away from the support of D (see [15] Definition 14.2.2.). In particular, if D is an effective Cartier divisor on X, it follows that the corresponding ideal sheaf $\mathscr{I}$ is precisely $\mathcal{O}_X(-D)$.

Let S be a graded ring which is generated by S_1 as an S_0-algebra and let $X = \mathrm{Proj}(S)$. Analogous to the functor $M \mapsto \tilde{M}$ defined for affine schemes, we define a functor from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules: given a graded S-module M there exists a unique quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ such that

$\tilde{M}(D_h(f)) = M_{(f)}$ for every homogeneous element $f \in S_+$ and the restriction maps are the $S_{(f)}$-linear maps $M_{(f)} \to M_{(g)}$ induced by the corresponding S_f-linear map $M_f \to M_g$. This generalizes the Proj construction, as $\tilde{S} = \mathcal{O}_X$.

For a graded S-module M and an integer $n \in \mathbb{Z}$, define a new graded S-module $M(n)$ by $M(n)_d = M_{n+d}$, which is just M "shifted". We define now *Serre's twisting sheaf* on X as $\widetilde{S(n)}$, and denote it by $\mathcal{O}_X(n)$. This sheaf is invertible, as for every $f \in S_1$ we have an isomorphism $\mathcal{O}_X(n)|_{D_h(f)} \cong \mathcal{O}_X|_{D_h(f)}$ given by multiplication with f^n. And a similar argument shows that more generally for any pair of integers $n, m \in \mathbb{Z}$ we have an isomorphism $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$. Using Serre's twisting sheaf we may define the twist of any $\mathcal{O}_X$-module $\mathscr{F}$ as $\mathscr{F} \otimes \mathcal{O}_X(n)$, and denote it by $\mathscr{F}(n)$.

Now consider $X = \mathbb{P}^n_k$. We study the twisting sheaf $\mathcal{O}_X(m)$ on X. This sheaf corresponds to the bundle which is trivial on the usual open cover of $\mathbb{P}^n_k$ by affine n-spaces U_i and has "multiplication by $(\frac{x_i}{x_j})^m$" as transition function from U_i to U_j. A global section consists then of a family of polynomials $f_i \in k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$ such that $f_i(\frac{x_0}{x_1}, ..., \frac{x_n}{x_i}) \cdot (\frac{x_i}{x_j})^m = f_j(\frac{x_0}{x_j}, ..., \frac{x_n}{x_j})$. Then $f = f_i x_i^m \in k(x_0, ..., x_n)$ is a rational function independent of i such that, divided by x_i^m, gives a polynomial in the variables $\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}$. So the denominator of f has to be a power of x_i. But there are at least two indices, hence the denominator of f is a unit and $f \in k[x_0, ..., x_n]$. Since f divided by x_i^m is in $k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$, if $m \geqslant 0$, every global section of $\mathcal{O}_X(m)$ corresponds to a homogeneous polynomials f of degree m, and if $m < 0$, there are no global sections. This bijection for $m \geqslant 0$ gives us that $\dim_k(\mathcal{O}_X(m)(X)) = \binom{n+m}{n}$, because the vector space of homogeneous polynomials of degree m in $n+1$ variables has dimension $\binom{n+m}{n}$.

We compute now the Picard group of X using the isomorphism with $\mathrm{Cl}(X)$. Define the degree of a Weil divisor D on $X = \mathbb{P}^n_k$ as $\deg(D) = \sum n_i \deg(f_i)$, where the f_i are the corresponding irreducible homogeneous polynomials of $k[x_0, ..., x_n]$. The degree then defines a group homomorphism $\mathrm{Div}(X) \to \mathbb{Z}$. Since a rational function on $\mathbb{P}^n_k$ is an element of $k(x_0, ..., x_n)$ on degree zero, the degree of any principal divisor is zero and we have an induced map $\mathrm{Cl}(X) \to \mathbb{Z}$. Conversely, a degree zero divisor has to be principal: given any divisor $D = \sum n_i(f_i)$, if $\deg(D) = 0$, then $\prod f_i^{n_i}$ is a rational function of degree zero inducing D. This shows that $\mathrm{Cl}(X) \cong \mathbb{Z}$, hence $\mathrm{Pic}(X) \cong \mathbb{Z}$.

By the previous discussion, the image of $\mathcal{O}_X(1)$ is 1 and therefore $\mathcal{O}_X(1)$ generates the Picard group of X. In particular, every invertible sheaf on X is of the form $\mathcal{O}_X(m)$ for some $m \in \mathbb{Z}$. Also note that we didn't use that k is algebraically closed, so everything remains true over arbitrary fields.

Definition. Let X be a variety and $\mathscr{E}$ be a locally free sheaf of finite type on X. Let $\mathscr{S} = S(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}$ (the sheafification of $U \mapsto S_{\mathcal{O}_X(U)}(\mathscr{E}(U))$, see [6] Exercise II.5.16.). Then $\mathscr{S} = \oplus_{d \geqslant 0} S^d(\mathscr{E})$ is a sheaf of graded $\mathcal{O}_X$-algebras and we can consider its Proj as defined on the previous chapter. We define the associated *projective space bundle* as the X-scheme $\mathbb{P}(\mathscr{E}) = \mathrm{Proj}(\mathscr{S})$, which comes together with the morphism

$\pi\colon \mathbb{P}(\mathscr{E}) \to X$ and an invertible sheaf $\mathcal{O}(1)$ as in projective space.

Lemma 5. *Let X be a regular variety and $\mathscr{E}$ be a locally free sheaf of finite type and rank r at least 2 on X. Then $\mathrm{Pic}(\mathbb{P}(\mathscr{E})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.*

Proof. Consider the morphism $\alpha\colon \mathrm{Pic}(X) \oplus \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}(\mathscr{E}))$ defined by $(\mathscr{L}, n) \mapsto (\pi^*\mathscr{L}) \otimes \mathcal{O}(n)$. Let us check that this is an isomorphism. Let $i\colon x \hookrightarrow X$ be a point with residue field $\kappa(x)$. Consider an open affine neighbourhood U of x on which $\mathscr{E}$ is free. Then on U the symmetric algebra is the polynomial algebra on r variables, hence $\pi^{-1}(U) = \mathbb{P}^{r-1}_U$ and we obtain an embedding $\mathbb{P}^{r-1}_{\kappa(x)} \to \mathbb{P}^{r-1}_U \to \mathbb{P}(\mathscr{E})$. Since $\mathcal{O}_{\mathbb{P}(\mathscr{E})}|_U \cong \mathcal{O}_U(n)$ and $\mathrm{Pic}(\mathbb{P}^{r-1}_{\kappa(x)}) = \mathbb{Z}$, we get a left inverse to $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}(\mathscr{E}))$. So it remains to show that α is surjective and that $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{P}(\mathscr{E}))$ is injective.

For the injectivity, suppose that $\pi^*\mathscr{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. Then $\pi_*(\pi^*\mathscr{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$ (using [6] Proposition II.7.11.) and by the *projection formula* (see [6] Exercise II.5.1.(d)) we get that $\mathscr{L} \otimes \pi^*\mathcal{O}(n) \cong \mathcal{O}_X$. Now $\pi_*\mathcal{O}(n)$ is the degree n part of the symmetric algebra on $\mathscr{E}$ (again [6] Proposition II.7.11.) and since $\mathrm{rk}(\mathscr{E}) \geqslant 2$ we obtain that $n = 0$ and $\mathscr{L} \cong \mathcal{O}_X$.

Let us check that α is also surjective, which is slightly more delicate. Let U_i be an affine open cover by integral affine (hence separated) schemes (X is integral) on which $\mathscr{E}$ is free (of rank r). Define $V_i = \mathbb{P}(\mathscr{E}|_{U_i}) \cong U_i \times \mathbb{P}^{r-1} = \mathbb{P}^{r-1}_{U_i} = \pi^{-1}(U_i)$, which form an open cover of $\mathbb{P}(\mathscr{E})$. It is easier to see for this V_i that $\mathrm{Pic}(V_i) \cong \mathrm{Pic}(U_i \times \mathbb{Z})$ (see [6] Exercise II.6.1. The two morphisms in the exact sequence of Proposition 7 split, so we get an isomorphism between the divisor class groups which are isomorphic to the Picard groups).

Now if $\mathscr{L} \in \mathrm{Pic}(\mathbb{P}(\mathscr{E}))$, we consider its restriction for each element of the cover to get an element $\mathcal{O}_i(n_i) \otimes \pi_i^*\mathscr{L}_i \in \mathrm{Pic}(V_i) \cong \mathrm{Pic}(U_i \times \mathbb{Z})$ together with transition isomorphisms $\alpha_{ij}\colon (\mathcal{O}_i(n_i) \otimes \pi_i^*\mathscr{L}_i)|_{V_i \cap V_j} \to (\mathcal{O}_j(n_j) \otimes \pi_j^*\mathscr{L}_j)|_{V_j \cap V_i}$ verifying the cocycle conditions. We consider the pushforward by π of these isomorphisms and use the projection formula again to get isomorphisms $\alpha_{ij}\colon \pi_*(\mathcal{O}_i(n_i)|_{V_i \cap V_j}) \otimes \mathscr{L}_i \to \pi_*(\mathcal{O}_j(n_j)|_{V_j \cap V_i}) \otimes \mathscr{L}_j$. Then we get that $n_i = n_j$ is an integer n independent of the index (by [6] Proposition II.7.11. looking at the ranks). But the sheaf $\mathcal{O}(n)$ is compatible with restriction to opens, and so we can rewrite the previous isomorphism as $\mathcal{O}_{ij}(n) \otimes \pi_i^*\mathscr{L}_i|_{V_i \cap V_j} \to \mathcal{O}_{ij}(n) \otimes \pi_j^*\mathscr{L}_j|_{V_i \cap V_j}$. Tensor with the inverse sheaf $\mathcal{O}_{ij}(-n)$ to get isomorphisms $\mathcal{O}_{ij} \otimes \pi_i^*\mathscr{L}_i|_{V_i \cap V_j} \to \mathcal{O}_{ij} \otimes \pi_j^*\mathscr{L}_j|_{V_i \cap V_j}$. Finally, use the projection formula together with Proposition II.7.11. to get isomorphisms $\mathscr{L}_i|_{U_i \cap U_j} \cong \mathscr{L}_j|_{U_i \cap U_j}$ verifying the cocycle conditions (because α_{ij} verifies them). So we can glue them together to obtain a sheaf on X mapping to $\mathscr{L}$. □

Using this lemma we can proof the first main result of this chapter.

Theorem 8. *Let X be a regular variety and let Y be a regular subvariety of codimension*[2] *r at least 2. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X along Y and let $E = \pi^{-1}(Y)$ be the exceptional divisor. Then the map $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ given by functorialtiy of the*

[2]If it has codimension 1, nothing happens: Corollary 2.

Picard group and the map $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X})$ *defined by* $n \mapsto nE$ *define an isomorphism* $\mathrm{Pic}(X) \oplus \mathbb{Z} \cong \mathrm{Pic}(\widetilde{X})$.

Proof. Let $U = X - Y$. By Proposition 7 we have an exact sequence $\mathbb{Z} \to \mathrm{Cl}(\widetilde{X}) \to \mathrm{Cl}(U) \to 0$ and an isomorphism $\mathrm{Cl}(U) \cong \mathrm{Cl}(X)$. The composition $\mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(U)$ is the same as $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ (which is an isomorphism) and therefore $\mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X)$ is the identity. Moreover, the composition $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X)$ is zero (because the sequence is exact). So it suffices to find a splitting for $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X})$.

Consider the closed immersion $E \hookrightarrow \widetilde{X}$ and the corresponding morphism $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(E)$. We know (by [6] Theorem II.8.24.(b)) that E is a projective bundle over Y. By the previous Lemma, we deduce that $\mathrm{Pic}(E) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}$. Consider the composition $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(E) \to \mathrm{Pic}(Y) \oplus \mathbb{Z} \to \mathbb{Z}$. Through the first map, 1 is sent to $\mathcal{O}_{\widetilde{X}}(E) \in \mathrm{Pic}(\widetilde{X})$. Since E is an effective Cartier divisor (by the universal property of the blow-up), we noted already that the corresponding sheaf of ideals $\mathscr{I}_E$ is the inverse of $\mathcal{O}_{\widetilde{X}}(E)$. But we also know that $\mathscr{I}_E \cong \mathcal{O}_{\widetilde{X}}(1)$ (cf. proof of [6] Proposition II.7.13.), so we get that $\mathcal{O}_{\widetilde{X}}(E) \cong \mathcal{O}_{\widetilde{X}}(-1)$. Through the next map, $\mathcal{O}_{\widetilde{X}}(-1)$ is sent to $\mathcal{O}_E(-1)$, which is then sent to -1. Although this is not the identity on $\mathbb{Z}$, it is an isomorphism, so composing with $1 \mapsto -1$ we obtain the desired splitting. □

Kähler differentials and the canonical invertible sheaf

Definition. Let A be a ring, B be an A-algebra and M be a B-module. An A-derivation of B into M is an A-linear map $d\colon B \to M$ such that the Leibniz rule is verified: for all $b_1, b_2 \in B$ we have $d(b_1 b_2) = b_1 db_2 + b_2 db_1$ (in particular $da = 0$ for all $a \in A$).

We denote the set of these derivations by $\mathrm{Der}_A(B, M)$. The *module of relative differential forms of* B *over* A or *module of Kähler differentials* is a B-module $\Omega_{B/A}$ together with an A-derivation $d\colon B \to \Omega_{B/A}$ with the following universal property: for any B-module M and for any A-derivation $d'\colon B \to M$ there exists a unique homomorphism of B-modules $\phi\colon \Omega_{B/A} \to M$ such that $d' = \phi \circ d$.

If it exists, it is unique up to unique isomorphism (because it solves a universal problem). To see that it exists, consider the free B-module given by linear combinations of the symbols $db, b \in B$ and quotient out by the submodule generated by appropiate elements.

Given a morphism of schemes $f\colon X \to S$, we can as usual glue the differential forms on affine open subsets to define the unique quasi-coherent sheaf of relative differentials $\Omega_{X/S}$ on X such that for each open affine V of S and each open affine U contained in $f^{-1}(V)$, the restriction $\Omega_{X/S}|_U$ is (isomorphic to) the sheaf of modules associated to the module of relative differential forms of $\mathcal{O}_X(U)$ over $\mathcal{O}_S(V)$, and for each $x \in U$ the stalk of $\Omega^1_{X/S}$ at

x is (isomorphic to) the module of differential forms of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{S,f(x)}$ (see [9] Chapter 6 Proposition 1.26.).

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. The *first fundamental exact sequence* of sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

follows from the first fundamental exact sequence for rings (see [10] Theorem 25.1). If $f\colon X \to Y$ is a scheme morphism and Z is a closed subscheme of X with ideal sheaf $\mathscr{I}$, then the *second fundamental exact sequence* of sheaves on Z

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0$$

follows from the second fundamental exact sequence for rings (see [10] Theorem 25.2).

In the case which occupies us (the base being an algebraically closed field k) the previous description simplifies (for V is either empty or $\operatorname{Spec} k$) and we have the following result (see [6] Theorem II.8.15. for the proof):

Theorem 9. *Let X be an irreducible separated scheme of finite type over k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $\dim X$ if and only if X is a regular variety over k.*

Let X be a regular variety of dimension n. The previous theorem allows us to define the *canonical invertible sheaf* on X as $\det \Omega_{X/k} = \wedge^n \Omega_{X/k}$ (the sheafification of $U \mapsto \wedge^n_{\mathcal{O}_X(U)} \Omega_{X/k}(U)$, see [6] Exercise II.5.16.), and we denote it by $\mathscr{K}_X$. Moreover, if Y is a nonsingular irreducible closed subscheme of X of codimension r given by the ideal sheaf $\mathscr{I}$, then the second fundamental exact sequence is also exact on the left (plugging 0 on the left) and $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank r on Y (see [6] Theorem II.8.17.). Thus for a regular subvariety Y of X we may define the *conormal sheaf* of Y in X as the locally free sheaf $\mathscr{I}/\mathscr{I}^2$, and the *normal sheaf* of Y in X as its dual $\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_Y)$.

If Y is a codimension r regular subvariety of X, then we have the *adjunction formula* for $\mathscr{K}_X$ (see [6] Proposition II.8.20. or [15] Exercise 21.5.B.):

$$\mathscr{K}_Y \cong \mathscr{K}_X \otimes (\wedge^r \mathscr{N}_{Y/X}) \otimes \mathcal{O}_Y$$

which follows by taking highest exterior powers (determinants) in the second fundamental (short) exact sequence and using that forming highest exterior powers commutes with taking the dual sheaf. In the particular case of a divisor ($r = 1$) we get the nicer formula

$$\mathscr{K}_Y \cong \mathscr{K}_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y \tag{3.1}$$

We compute now the canonical invertible sheaf of the projective space[3] $X = \mathbb{P}^n_k$. We know that it has to be of the form $\mathcal{O}_X(m_n)$ for some $m_n \in \mathbb{Z}$. The right m_n, as we will see

[3] There is a cleaner way to do this (see [6] Example II.8.20.1.) using the Euler sequence (the exact sequence in [6] Theorem II.8.13.).

now, is $-n-1$. For notational convenience, we do the case $n=2$, so let $X=\mathbb{P}_k^2$, and U_0, U_1 and U_2 the usual affine open cover. The sections of $\mathscr{K}_X$ over U_0 are $p(u_1,u_2)du_1\wedge du_2$. We look at the section $du_1\wedge du_2$ over U_0 and we look for its zeros and poles outside U_0 (since $x_0=0$ is the only divisor outside U_0, we only have to check this divisor). We look over U_1 with coordinates $v_0=\frac{1}{u_1}$ and $v_2=u_2v_0$, so that $(1:u_1:u_2)=(v_0:1:v_2)$. Then our section is

$$du_1\wedge du_2=\left(-\frac{1}{v_0^2}dv_0\right)\wedge\left(\frac{v_0dv_2-v_2dv_0}{v_0^2}\right)=-\frac{1}{v_0^3}dv_0\wedge dv_2$$

So along $x_0=0$ we have a pole of order 3 and $m_2=-3$. Now for every $n\in\mathbb{N}$ the divisor $\mathbb{P}_k^{n-1}=V(x_n)$ in $\mathbb{P}_k^n$ is given by a homogeneous polynomial of degree 1. On each element of the usual affine open cover U_i, we have an isomorphism $\mathcal{O}_{\mathbb{P}_k^n}(1)(U_i)\to\mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}_k^{n-1})(U_i)$ given by $x_i^1h\mapsto\frac{h}{x_0}$. Thus $\mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}_k^{n-1})=\mathcal{O}_{\mathbb{P}_k^n}(1)$ and we may inductively apply the adjunction formula 3.1 to obtain for $n=3$ the integer $-3=m_2+1$, and in general for any $n>2$ the integer $-m_{n-1}=m_n+1$. And for $n=1$ we get $m_1=m_2+1$. So we have that the canonical sheaf of $X=\mathbb{P}_k^n$ is $\mathcal{O}_X(-n-1)$.

Observe also that the previous argument works the same way if we replace x_0 by any homogeneous polynomial of degree d and we replace the occurences of 1 by d. Hence, if Y is a hypersurface of $\mathbb{P}_k^n$ given by a homogeneous polynomial of degree d, then

$$\mathcal{O}_{\mathbb{P}_k^n}(Y)=\mathcal{O}_{\mathbb{P}_k^n}(d) \tag{3.2}$$

Moreover, we didn't use that k is algebraically closed, so the result still holds over any field.

Lemma 6. *Let X and Y be regular varieties of dimensions n and m respectively. Then the canonical sheaf of their product is $p_1^*\mathscr{K}_X\otimes p_2^*\mathscr{K}_Y$ (where p_1 and p_2 are the projections from the fiber product).*

Proof. We start by proving that if X and Y are S-schemes, then $\Omega_{X\times Y/S}\cong p_1^*\Omega_{X/S}\oplus p_2^*\Omega_{Y/S}$.

We have $\Omega_{X\times Y/X}\cong p_2^*\Omega_{Y/S}$ and $\Omega_{X\times Y/Y}\cong p_1^*\Omega_{X/S}$ (by [6] Proposition II.8.10.). Apply this to the first fundamental sequence we get two exact sequences $\Omega_{X\times Y/X}\to\Omega_{X\times Y/S}\to\Omega_{X\times Y/Y}\to 0$ and $\Omega_{X\times Y/Y}\to\Omega_{X\times Y/S}\to\Omega_{X\times Y/X}\to 0$. So now we need to see that $\Omega_{X\times Y/S}$ decomposes into $p_1^*\Omega_{X/S}\oplus p_2^*\Omega_{Y/S}$. It suffices to consider the affine case, so let A and B be C-algebras. We check that the composition $\Omega_{A\otimes_C B/A}\cong\Omega_{B/C}\otimes_B(B\otimes_C A)\to\Omega_{A\otimes_C B/C}\to\Omega_{A\otimes_C B/A}$ is the identity. The first module is generated by elements of the form dx for $x\in A\otimes_C B$. Since d is a morphism of abelian groups and $d(a\otimes b)=d(a\otimes 1)+d(1\otimes b)=d(1\otimes b)$, it is enough to consider elements of the form $d(1\otimes b)$. Such an element is sent through the first map to $(1\otimes 1)\otimes db$, which then is sent to $d(1\otimes b)$, and finally back to $d(1\otimes b)$. So the composition is the identity and we get what we wanted (see [10] Chapter 25 for the definitions of these maps).

Now back to our varieties X and Y. By definition, $\mathscr{K}_{X\times Y} = \wedge^{nm}\Omega_{X\times Y}$. We apply what we just proved to this case and get $\wedge^{nm}((p_1^*\Omega_X)\oplus(p_2^*\Omega_Y))$. But by properties of the exterior product this is isomorphic to $(\wedge^n p_1^*\Omega_X)\otimes(\wedge^m p_2^*\Omega_Y)$, which is also isomorphic to $(p_1^*(\wedge^n\Omega_X))\otimes(p_2^*(\wedge^m\Omega_Y))$ (see [6] Exercise II.5.16. (d) and (e) respectively). And this last expression is again by definition $p_1^*(\mathscr{K}_X)\otimes p_2^*(\mathscr{K}_Y)$, which is what we wanted. □

We are redy now to prove the second main result of this chapter.

Theorem 10. *Let X be a regular variety and let Y be a regular subvariety of codimension*[4] *r at least 2. Let $\pi\colon \widetilde{X}\to X$ be the blow-up of X along Y and let $E=\pi^{-1}(Y)$ be the exceptional divisor. Then $\mathscr{K}_{\widetilde{X}}\cong\pi^*\mathscr{K}_X\otimes\mathcal{O}_{\tilde{X}}((r-1)E)$.*

Proof. Let $U=X-Y$. According to Theorem 8, we may write $\mathscr{K}_{\tilde{X}}$ as $\pi^*\mathscr{M}\otimes\mathcal{O}_{\tilde{X}}(qE)$ for some $\mathscr{M}\in\operatorname{Pic}(X)$ and some $q\in\mathbb{Z}$. Outside the center we have an isomorphism $U\cong\widetilde{X}-E$ and therefore $\mathscr{K}_{\tilde{X}}|_{\tilde{X}-E}\cong\mathscr{K}_U\cong\mathscr{K}_X|_U$. And by Proposition 7 we have an isomorphism $\operatorname{Pic}(X)\cong\operatorname{Pic}(U)$, so from $\mathscr{M}|_U\cong\mathscr{K}_X|_U$ we deduce $\mathscr{M}\cong\mathscr{K}_X$. By the adjunction formula 3.1 we obtain $\mathscr{K}_E\cong\mathscr{K}_{\tilde{X}}\otimes\mathcal{O}_{\tilde{X}}(E)\otimes\mathcal{O}_E\cong\pi^*\mathscr{K}_X\otimes\mathcal{O}_{\tilde{X}}((q+1)E)\otimes\mathcal{O}_E$. Since E is an effective Cartier divisor (by the universal property of the blow-up), if $\mathscr{I}_E$ is the corresponding ideal sheaf, we have that $\mathcal{O}_{\tilde{X}}(-E)\cong\mathscr{I}_E$, and so $\mathcal{O}_{\tilde{X}}((q+1)E)\cong\mathscr{I}_E^{-q-1}$. But we also know that $\mathscr{I}_E\cong\mathcal{O}_{\tilde{X}}(1)$ (cf. proof of [6] Proposition II.7.13.), hence $\mathscr{K}_E\cong\pi^*\mathscr{K}_X\otimes\mathcal{O}_E(-q-1)$.

Now let $y\in Y$ be a closed point and let $Z=y\times_Y E$ be the fiber of E over y. We apply our previous Lemma to get $\mathscr{K}_Z\cong p_1^*\mathscr{K}_y\otimes p_2^*\mathscr{K}_E$. We substitute $\mathscr{K}_E$ to get $p_1^*\mathscr{K}_y\otimes p_2^*(\pi^*\mathscr{K}_X\otimes\mathcal{O}_E(-q-1))$. Note that $\mathscr{K}_y$ is just $\mathcal{O}_y$, so we get $\mathcal{O}_Z\otimes p_2^*(\pi^*\mathscr{K}_X\otimes\mathcal{O}_E(-q-1))=p_2^*(\pi^*\mathscr{K}_X\otimes\mathcal{O}_E(-q-1))$. And now we can pull back $\mathscr{K}_X$ to Z through y where it becomes trivial:

$$\begin{array}{ccccc} Z & \xhookrightarrow{p_2} & E & \hookrightarrow & \tilde{X} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ y & \hookrightarrow & Y & \hookrightarrow & X \end{array}$$

Therefore we obtain $\mathcal{O}_Z(-q-1)$. But Z is a projective space of dimension $r-1$ (cf. [6] Theorem II.8.24.), so by the computations we did previously in projective space $q=r-1$ and $\mathscr{K}_{\tilde{X}}\cong\pi^*\mathscr{K}_X\otimes\mathcal{O}_{\tilde{X}}((r-1)E)$. □

[4] Again, if it has codimension 1 nothing happens.

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