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Applied Mathematics III

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Feynman: *To those who do not know mathematics, it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.*

References of the course include:

- *Mathews and Walker: Mathematical Methods of Physics.*

- *Trodden: Lecture notes: Methods of Mathematical Physics I*
  http://www.phy.syr.edu/ trodden/courses/mathmethods/

The materials to be covered in this course are the following:

- Complex analysis and evaluation of integrals (saddle-point approx.)

- Fourier transform (+ Laplace transform)

- Sturm-Liouville theory, linear space of functions and special functions

- Special functions

This note is provided as a supplement, not a substitute, to the references. We will use *Einstein's summation convention* in this note.

Chapter 1

Complex Analysis

In this chapter we will learn how to use complex analysis, in particular the residue theorem, to evaluate integrals. We will also learn how to compute the saddle-point approximation of an integral.

- holomorphic/analytic functions.

- singularity, branch cut.

- contour integral, residue theorem.

- using residual theorem to evaluate integrals.

- saddle-point approximation.

1.1   Introduction

Complex numbers and complex functions appear in physical problems as well as formulations of fundamental physics. Why is complex analysis useful? Are not all physical quantities real? Since $z = x + iy$, a complex function of z is just two real functions of (x, y)

$$f(z) = u(x, y) + iv(x, y). \tag{1.1}$$

Why don't we just reduce every problem in complex analysis to real analysis?

Holomorphic functions are rare, in the sense that there are a lot more functions on $\mathbb{R}^2$ than holomorphic functions on $\mathbb{C}$.

This is why *analytic continuation* $(f(x) \to F(z))$ is possible. Given a real function $f(x)$ of $x \in \mathbb{R}$, we can imagine that it is the restriction of a holomorphic function $F(z)$ to the real axis of the complex plane $\mathbb{C}$. Therefore, properties of $F(z)$ can be viewed as properties of $f(x)$. A lot of properties of $f(x)$ are hidden until we view it in the form of $F(z)$. These properties allow us to deal with problems involving $f(x)$ (such as the integration of $f(x)$) more easily.

An example of the use of complex fx's appeared in your E&M course when dealing with waves: $\cos(k \cdot x) \to e^{ik \cdot x}$. In QM, complex fx's are needed at a more fundamental level.

If we are not restricted to holomorphic functions, there are infinitely many functions $F(z, \bar{z})$ which reduce to a given $f(x)$ when $z = \bar{z} = x$. For example, for $f(x) = x^2$, we can have $F = z^2$, $|z|^2$, $z^{1/2}\bar{z}^{3/2}$ etc. Only the first choice is holomorphic.

A theorem that demonstrates this notion is Liouville's theorem (for complex analysis, not phase space).

Another example is that $z \to z' = z'(z)$ always defines a conformal map.

1.2 Preliminaries and Definitions

Definitions are not just about names, which help us to communicate more efficiently. More important is the concept behind a definition.

1. Change of variables on $\mathbb{R}^2$:

$$z = x + iy, \qquad z^* = x - iy, \tag{1.2}$$

$$z = re^{i\theta}, \qquad z^* = re^{-i\theta}. \tag{1.3}$$

2. We consider *functions* that are single-valued maps from *domains* in $\mathbb{C}$ to *ranges* in $\mathbb{C}$. A function which appears to be multiple-valued should be restricted to a smaller domain such that it is single-valued.

3. What is $\partial \equiv \partial_z = \partial/\partial z$? (What is $\bar{\partial} \equiv \partial_{\bar{z}} = \partial/\partial\bar{z}$?)

$$\partial = \frac{1}{2}(\partial_x - i\partial_y), \qquad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y). \tag{1.4}$$

4. The rules of differentiation (chain rule, Leibniz rule) are the same as for real functions.

Recall that the electric potential $V(r)$ of a point charge is $\propto \frac{1}{r}$. Gauss's law: $\nabla^2\frac{1}{r} = -4\pi\delta^{(3)}(\vec{r})$. The analogue in 2D is $V(r) \propto \log r$, and we need $\nabla^2 \log r \propto \delta^{(2)}(\vec{r})$. Now $\log r = \frac{1}{2}(\log z + \log \bar{z})$, and $\nabla^2 \propto \partial\bar{\partial}$, so we need $\partial\frac{1}{\bar{z}} \propto \bar{\partial}\frac{1}{z} \propto \delta^{(2)}(\vec{r})$.

General rule:
To compute the derivative of a function ill-defined at a point, we use Gauss' law or Stoke's theorem.

5. Caution: $\partial_z(1/\bar{z}) \neq 0$.
Stoke's theorem

$$\oint_C (A_x dx + A_y dy) = \int_R dxdy(\partial_x A_y - \partial_y A_x) \tag{1.5}$$

implies that

$$\oint_C f(z)dz = \oint_C (f(z)dx + if(z)dy)$$
$$= \int_R dxdy(i\partial_x - \partial_y)f(z) = \int_R dxdy2i\bar{\partial}f(z). \tag{1.6}$$

If $f(z)$ is well-defined in R, $\bar{\partial}f(z) = 0$, implying $\oint_C f(z)dz = 0$ (Cauchy's theorem). If $f(z) = \frac{1}{z}$, for C being an infinitesimal circle around the origin, $\oint_C f(z)dz = 2\pi i$, implying that

$$\partial\frac{1}{\bar{z}} = \bar{\partial}\frac{1}{z} = \pi\delta^{(2)}(x, y). \tag{1.7}$$

6. A function is *analytic* at a point if it has a derivative (complex-differentiable) there. That is,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad (h \in \mathbb{C}) \tag{1.8}$$

exists and is independent of the path by which the complex number approaches to 0.

7. $f(z) = u(x, y) + iv(x, y)$ satisfies *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.9}$$

So u and v are harmonic functions

$$\nabla^2 u = 0, \qquad \nabla^2 v = 0. \tag{1.10}$$

8. One can use CR-eqs to derive v if u is given (assuming that v exists).

> Let $u = xy$. One can solve v from CR-eqs and get $v = (y^2 - x^2)/2 + c$. Thus $f = -iz^2/2 + ic$. Alternative derivation: Try z^2 since u is quadratic.

9. Taylor expansion, analytic continuation, radius of convergence, singularity.

> Ex: $f(x) = \sum_n (-x)^n$ $\to f(z) = \sum_n (-z)^n$ with rad. of conv. $= 1$. One can compute $f^{(n)}(1/2)$ for all n and find $f(z) = \sum_n \frac{2}{3}(-\frac{2}{3}(z - \frac{1}{2}))^n$ with rad. of conv. $= 3/2$. This goes beyond the convergence region of the 1st expression.

10. [*Holomorphic/Analytic* function in R] = [function that is analytic everywhere in R].

> Theorem: The Taylor expansion of an analytic fx. around a point inside R has a finite radius of convergence.

11. The rad. of conv. R of $f(z)$ about z_0 is equal to the distance between z_0 and the nearest singularity of $f(z)$.

12. Composition of analytic functions:

$$af_1 + bf_2, \tag{1.11}$$
$$f_1 f_2, \tag{1.12}$$
$$f_1/f_2 \quad \text{when } f_2 \text{ is not } 0, \tag{1.13}$$
$$f_1(f_2(z)). \tag{1.14}$$

13. [*Entire* function] = [function that is holomorphic on the whole complex plane].

> Examples of entire fx's: $f(z) = z$, $f(z) = e^z$, etc.

14. [*Laurent series* of $f(x)$] = [$\sum_{n=-m}^{\infty} f_n z^n$].

- f has a *pole* (isolated *singularity*) of order m if $0 < m < \infty$.
- [*Simple poles*] = [poles of order 1].
- f has an *essential singularity* if $m = \infty$.

> Example: $f(z) = z^{-n}$ has a pole of order n at $z = 0$ for $n \in \mathbb{Z}_+$.

> An essential singularity is a singularity more severe than poles of any order: $\lim_{z \to z_0} (z - z_0)^n f(z) = \infty$ for any finite n. Example of essential sin-

A fx. $f(z)$ is meromorphic iff $f(z) = g(z)/h(z)$ for some holomorphic fx's $g(z)$ and $h(z)$.

A fx. is discontinuous across a branch cut. Examples of branch cut: $f(z) = z^{1/2}$, $f(z) = \log(z)$.

15. *Meromorphic* function in R = analytic except at isolated points in R with singularities of finite order.

16. *Branch cut* = curve in $\mathbb{C}$ taken out of the domain to avoid multiple-valuedness.

1.3   Residue Theorem

The *residue* of a function $f(z)$ at z_0 is defined by

$$\text{Res}_{z \to z_0} f(z) \equiv f_{-1} \tag{1.15}$$

if the Laurent expansion of f around z_0 is

$$f(z) = \cdots + \frac{f_{-2}}{z^2} + \frac{f_{-1}}{z} + f_0 + f_1 z + f_2 z^2 + \cdots . \tag{1.16}$$

The residue is nonzero only at singular points.

This number f_{-1} is special because it is singled out from all coefficients f_n if we integrate f over a small circle around z_0:

$$\oint_{S^1(z_0)} dz f(z) = 2\pi i f_{-1}. \tag{1.17}$$

The essence of the residue theorem is simply

$$\int_0^{2\pi} d\theta e^{im\theta} = 2\pi \delta_m^0, \tag{1.18}$$

which we used to derive (1.17).

Residue Theorem:

This theorem is easy to understand by replacing the contour by infinitely many infinitesimal circles.

The contour integral of an analytic fx. $f(z)$ is given by the sum of its residues at all singular points encircled in the contour:

$$\oint_C dz f(z) = 2\pi i \sum_i \text{Res}_{z \to z_i} f(z). \tag{1.19}$$

This is for C going around the poles once in the *counterclockwise* direction. In general we get an integer factor on the right hand side if C is going in the opposite direction or if C encircles the poles more than once.

To prove this theorem, we simply quote Cauchy's theorem proved in Sec. 1.2 and (1.17).

Immediate consequences of the residue theorem:

We used Cauchy's theorem to prove residue theorem, but once you accept and memorize residue theorem, it is convenient to view Cauchy's theorem as its consequence because

- *Cauchy's theorem*

$$\oint_C f(z) dz = 0 \tag{1.20}$$

if $f(z)$ is analytic in R and $C \in R$.

- *Cauchy's integral formula*
  For an analytic function $f(z)$ in R,

$$f(z) = \frac{1}{2\pi i} \oint_z d\zeta \frac{f(\zeta)}{\zeta - z}, \qquad (1.21)$$

  where C is a closed contour in R encircling the point z once in the counterclockwise direction.

- A contour integral of $f(z)$ in $\mathbb{C}$ with endpoints z_0, z_1 is invariant under continuous deformations of the contour with fixed endpoints if the deformation never crosses over any singularity.

- *Liouville's theorem*
  A bounded entire function must be constant.

Liouville's theorem implies that: If $f(z) \to 0$ as $z \to \infty$, $f(z)$ must have singularities (unless $f(z)$ is identically zero).

The residue theorem not only helps us to understand analytic functions better, it is also very powerful for evaluating integrals. However it takes experience (a lot of practice) and ingenuity (which usually only comes after hard working) in inventing the relevant analytic function and contour.

Hint of proof: Consider $|f(z_1) - f(z_2)|$ using Cauchy's integral formula.

A holomorphic function $f(z)$ with only simple poles is uniquely determined by the locations and residues of the simple poles, and its value at infinity.

1.3.1   Examples

In the following, let $a > 0$.

1.
$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = ? \qquad (1.22)$$

2.
$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = ? \qquad (1.23)$$

3.
$$\int_0^{\infty} \frac{dx}{1 + x^3} = ? \qquad (1.24)$$

4. For $a > b > 0$,
$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = ? \qquad (1.25)$$

5.
$$\int_0^{\infty} dx \frac{\sqrt{x}}{1 + x^2} = ? \qquad (1.26)$$

6. For $0 < a < 1$,
$$\int_{-\infty}^{\infty} dx \frac{e^{ax}}{e^x + 1} = ? \qquad (1.27)$$

7.
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(1+x^2)} = ? \tag{1.28}$$

8. For $f(z)$ with several isolated singularities, consider the contour integral

$$\oint \frac{f(z)dz}{\sin \pi z}. \tag{1.29}$$

Show that

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_k \frac{R_k}{\sin \pi z_k}, \tag{1.30}$$

where R_k is the residue of $f(z)$ at z_k.

9.
$$\int_{-\infty}^{\infty} dx \frac{x - \sin x}{x^3} = ? \tag{1.31}$$

10. A holomorphic function $f(z)$ has only 2 simple poles at $z = \pm 1$ with both residues equal to 1. $f(z)$ also approaches to 1 at infinities. Find $f(z)$ and prove that it is unique.

1.3.2   Comments

Holomorphicity

Given a complex number $z = x+iy \in \mathbb{C}$, its complex conjugate $\bar{z} = x-iy$ is known. But algebraically z and $\bar{z}$ are independent. Consider a general change of coordinates of $\mathbb{R}^2$

$$x' = ax + by, \qquad y' = cx + dy \tag{1.32}$$

for arbitrary coefficients a, b, c, d (as long as the inverse exists). A function $f(x, y)$ can be rewritten in terms of (x', y') as $f'(x', y')$ such that $f'(x'(x, y), y'(x, y)) = f(x, y)$. The new coordinates (x', y') are independent variables just as (x, y) are independent variables. One can consider derivatives and use the chain rule to derive

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}, \qquad \frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'}. \tag{1.33}$$

All of these algebraic relations remain the same for our special case of $z = x', \bar{z} = y'$ ($a = 1, b = i, c = 1, d = -i$). In this sense, $(z, \bar{z})$ are independent variables.

Analytic continuation

Analytic continuation means to extend the domain of a function from its original definition to other regions on the complex plane as much as possible,

with the function's analyticity preserved. For example, if we define $f(x)$ by the series

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{1.34}$$

its domain is $(-1, 1)$ because its radius of convergence is 1.

By analytic continuation, we extend the original definition of $f(x)$ to everywhere on the complex plane except the point $x = 1$, so that

$$f(x) = \frac{1}{1 - x}. \tag{1.35}$$

Suppose we find the expansion (1.34) in a physical problem, and we are interested in the case when x is a phase. Strictly speaking, based on the definition (1.34), the value of $f(x)$ for $x = e^{i\theta}$ is ill-defined. One might wonder whether we can define $f(e^{i\theta})$ as the limit

$$\lim_{\epsilon \to 0+} \sum_{n=0}^{\infty} (e^{i\theta - \epsilon})^n = \frac{1}{1 - e^{i\theta}}, \tag{1.36}$$

but how do we justify this choice, rather than the other choice ($\epsilon \to 0-$)?

In a physical problem, there should be physical reasons to justify one choice over the other. A simple criterion is that, if the quantity you are computing is physical/observable, then it must be finite and well-defined. Roughly speaking, this justifies the use of analytic continuation whenever possible.

As an example of the use of analytic continuation, imagine that in a physical problem, we need to solve the differential equation

$$(1 - x)f'(x) - f(x) = 0. \tag{1.37}$$

One might try to solve $f(x)$ as an expansion

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots . \tag{1.38}$$

Then we will get some recursion relations and find that (1.34), up to an overall constant factor, is the solution. For this problem, one will not hesitate to replace the series (1.34) by (1.35), because one can directly check that (1.35) satisfies the differential equation (1.37). The appearance of the series (1.34) is not a consequence of anything of physical significance, but only a result of the mathematical technique we apply to the problem.

Residue theorem

In the above, we see that the use of residue theorem is very powerful. We can use it to evaluate many integrals which we do not know how to compute otherwise. Even for some of the integrals which we can directly integrate, it is much easier to use the residue theorem. It is a really elegant technique.

The use of the residue theorem always involves complex numbers in the calculation, even when the integral is real. It is important to have a rough

idea about the integral, e.g. whether it is real, positive or negative, etc. before applying the residue theorem, so that you can easily detect an error in calculation. If, after using the residue theorem, you find a complex number for a real integral, or a negative number for an integral that is obviously positive definite, you must have made a mistake.

1.4 Saddle-Point Approximation

Most integrals can not be carried out to give an exact answer. Saddle-point method is one of the most important methods to estimate an integral. The integral must have a parameter whose value is "large". The larger the parameter, the better the approximation.

The basic idea is this. Consider a function $f(x)$ with a maximum at x_0. The contrast between values of $f(x)$ at different points x is magnified in the expression

$$e^{\alpha f(x)} \tag{1.39}$$

for large α. The value of the integral

$$\int dx e^{\alpha f(x)} \tag{1.40}$$

is expected to be dominated by the contribution from a small region around the point $x = x_0$. As a lowest order approximation, in the very near neighborhood of x_0 (i.e., $x \sim x_0$), we can approximate $f(x)$ by

We have $f'(x_0) = 0$ and $f''(x_0) < 0$ since x_0 is a maximum.

$$f(x) \simeq f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2. \tag{1.41}$$

Then the integral (1.40) is approximated by a Gaussian integral

$$\int dx e^{\alpha f(x)} \simeq e^{\alpha f(x_0)} \int dx e^{\frac{1}{2}\alpha f''(x_0)(x - x_0)^2}. \tag{1.42}$$

This can be carried out easily assuming that the range of integration covers the point x_0 and its neighborhood of several standard deviations. In case the range of integration misses a significant portion of the Gaussian function, we can still estimate the Gaussian integral by looking it up in a table or using a computer.

$$\int_{-\infty}^{\infty} dx e^{-x^2/2} = \sqrt{2\pi}. \tag{1.43}$$

The standard deviation of the Gaussian is small $\sim \frac{1}{\sqrt{\alpha}}$.

Whether this is a good approximation depends on whether the integral is dominated by a small region around x_0 within which the first few terms of the Taylor expansion (1.41) is a good approximation of $f(x)$.

Higher order approximations can be obtained by keeping more terms in the Taylor expansion

$$f(x) \simeq f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \cdots, \tag{1.44}$$

and Taylor-expand the integrand apart from the Gaussian factor

$$e^{\alpha f(x)} \simeq e^{\alpha f(x_0)} e^{\frac{1}{2}\alpha f''(x_0)(x-x_0)^2} \left(1 + A(x-x_0)^3 + B(x-x_0)^4 + \cdots \right). \quad (1.45)$$

Then we can carry out the integration term by term. To do so, you should know how to derive the following (using integration by parts)

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2} x^n = \begin{cases} 2^{(n+1)/2}\Gamma(\frac{n+1}{2}) = \sqrt{2\pi}[1\cdot 3\cdot 5 \cdots (n-1)], & n = \text{even}, \\ 0, & n = \text{odd}. \end{cases} \quad (1.46)$$

With the help of complex analysis, we can extend the discussions above from $\mathbb{R}$ to $\mathbb{C}$. The function $f(x)$ can be analytically continuated and the integration on the real axis can be deformed into the complex plane. The condition $f'(x) = 0$ which determines the maximum x_0 is replaced by

$$f'(z) = 0 \quad \rightarrow \quad z = z_0. \quad (1.47)$$

This point z_0 is not a local maximum nor minimum but a saddle point. In the two dimensional plane $\mathbb{C}$, one can choose to pick the contour with steepest descent or the one with stationary phase. Formally, the two approaches look almost the same

$$\int dz\, e^{f(z)} \simeq \int dz\, e^{f(z_0)+\frac{1}{2}f''(z_0)(z-z_0)^2} = e^{f(z_0)} \sqrt{\frac{2\pi}{-f''(z_0)}}. \quad (1.48)$$

The subtle difference lies in how to define the square root, which depends on how we choose the contour to pass through the saddle point z_0 (at what angle in the complex plane).

More precisely, if we choose a real parameter s on the contour so that

$$z \simeq z_0 + se^{i\phi} \quad (1.49)$$

around the point z_0 for a constant ϕ, we have

$$\int dz\, e^{\frac{1}{2}f''(z_0)(z-z_0)^2} = e^{i\phi} \int ds\, e^{\frac{1}{2}\rho\cos(\theta+2\phi)s^2 + i\frac{1}{2}\rho\sin(\theta+2\phi)s^2} \quad (1.50)$$

where we denoted $f''(z_0) = \rho e^{i\theta}$.

In the steepest descent approach, in order for the real part of $f(z)$ to be local maximum, we want to choose ϕ so that

$$\cos(\theta + 2\phi) = -1 \quad \Rightarrow \quad \sin(\theta + 2\phi) = 0, \quad (1.51)$$

so that the imaginary part is automatically roughly constant. Then the expression above is

$$e^{i\phi} \int ds\, e^{-\frac{1}{2}\rho s^2} = e^{i\phi}\sqrt{\frac{2\pi}{\rho}} \qquad \left(= \pm\sqrt{\frac{2\pi}{-f''(z_0)}} \right). \quad (1.52)$$

The sign depends on which of the two solutions of ϕ is chosen to satisfy (2.20).

For the stationary phase approach, we choose

$$\sin(\theta + 2\phi) = \pm 1 \quad \Rightarrow \quad \cos(\theta + 2\phi) = 0, \tag{1.53}$$

so that the real part is automatically roughly constant. Then (1.50) becomes

$$e^{i\phi} \int ds \, e^{\pm i \frac{1}{2}\rho s^2} = e^{i\phi}\sqrt{\frac{2\pi}{\mp i\rho}} = e^{i(\phi \pm \pi/4)}\sqrt{\frac{2\pi}{\rho}} \qquad \left(= \pm\sqrt{\frac{2\pi}{-f''(z_0)}} \right). \tag{1.54}$$

The sign depends on which solution of ϕ is chosen to satisfy (2.20).

Note that the final results (1.52) and (1.54) are exactly the same up to the ambiguity of a sign, which depends on the choice of the contour.

1.4.1 Examples

1. For large x,

$$\Gamma(x+1) = \int_0^\infty dt \, t^x e^{-t}. \tag{1.55}$$

2. For large α,

$$\int dx g(x) e^{\alpha f(x)} \tag{1.56}$$

3. For large $n \in \mathbb{Z}$,

$$\int_0^{2\pi} \sin^{2n}\theta d\theta. \tag{1.57}$$

4. For large α,

$$\int_C dz e^{\alpha f(z)}. \tag{1.58}$$

1.4.2 Comments

Before applying the saddle point approximation, it is necessary to have a rough idea about the behavior of the integrand. We might not want to go all the way to figure out the shape of the contour in the complex plane, but at least we should know, for example, whether the integral is dominated by the region close to a boundary of integration. This may happen even when there is a local maximum within the range of integration, and then the saddle point approximation can not be used without modification.

The ambiguity in choosing the direction of integration (due to different choices of the contour) when we pass over the saddle point results in the ambiguity of an overall sign of the integral. This ambiguity in the sign can also be associated with the ambiguity of the sign of the square root $\sqrt{\frac{2\pi}{-f''(z_0)}}$. Usually it is not too hard to guess which choice is correct for a given integral, and so it is usually not necessary to figure out the details of the contour. It is

sufficient to know that the integral is dominated by a small region somewhere along a certain contour.

In both case (steepest descent and stationary phase), the key point is to make sure that the dominating contribution of the integral comes from a "small" region of the contour, so that we can approximate the integrand $e^{f(z)}$ by Taylor-expanding $f(z)$ and keeping only the first few terms.

The difference in the two approaches lies in the reason why the integral is dominated by a small region. For the steepest descent approach, the reason is that the real part of the exponent $f(z)$ is large in a small region (while the imaginary part of $f(z)$ does not change quickly within that small region, so that there is no significant cancellation due to changes of phase).

For example, the integral of $f(x) = e^{-\frac{\alpha^2}{2}x^2 + \beta x^3}$ over the real line is dominated by the region close to $x = 0$, with a width of $\sim \frac{1}{\alpha}$. For larger α, the region is smaller. But how small is small enough? The small region is small enough if the term βx^3 can be ignored compared with the first term $-\frac{\alpha^2}{2}x^2$ within the region $(-\frac{1}{\alpha}, \frac{1}{\alpha})$ so that our saddle point approximation is good. That is, we want $|\beta \frac{1}{\alpha^3}| \ll |\frac{\alpha^2}{2} \frac{1}{\alpha^2}|$, i.e., $|\beta| \ll \alpha^3$. This necessary condition for the steepest descent approach can be directly written down by counting the "dimension" of each quantity. Imagine that this integral arises in a physical problem and the dimension of x is L. Then the dimension of α is $1/L$, and that of β is $1/L^3$. Since it makes no sense to talk about the precise meaning of "large" or "small" for a quantity with dimension, we can only talk about the dimensionless quantity β/α^3 being large or small. Since it is obvious that we want $|\beta|$ to be small in order to ignore it, what we really want must be that $|\beta/\alpha^3| \ll 1$.

For the stationary phase approach, the reason why the integral is dominated by a small region is that, except this small region, the imaginary part of $f(z)$ changes quickly (while the real part changes slowly) when we move along the contour, corresponding to a high frequency oscillation that leads to a large cancellation in the integration.

As an example, consider the integral of $f(x) = e^{i\frac{\alpha^2}{2}x^2 + \beta x^3}$ over the real line. If $|\beta/\alpha^3| \ll 1$ so that we can ignore the 2nd term, the first term of the exponent tells us that the wavelength of the oscillation around a point x in a small region (much smaller than the wavelength) is roughly $\frac{1}{\alpha^2 x}$. Integration over the fast oscillating region largely cancals, and so the integration is dominated by the region $(-1/\alpha, 1/\alpha)$, within which the wavelength is of order $1/\alpha$ or larger.

1.5   Exercises:

1. For $k, a \in \mathbb{R}$,

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dx \frac{e^{ikx}}{x^2 - a^2 + i\epsilon} = ? \tag{1.59}$$

2. For $a, b \in \mathbb{R}$, $m + n \geq 2$,

$$\int_{-\infty}^{\infty} dx \frac{1}{(x^2 + a^2)^m (x^2 - b^2)^n} = ? \tag{1.60}$$

3. For $a > b > 0$,

$$\int_{-1}^{1} dx \frac{1}{\sqrt{1 - x^2}(a + bx)} = ? \tag{1.61}$$

4. For $a > b > 0$,

$$\int_{0}^{2\pi} d\theta \frac{\sin^2 \theta}{a + b \cos \theta} = ? \tag{1.62}$$

5.

$$\int_{0}^{\infty} dx \frac{\log(x)}{x^3 + a^3} = ? \tag{1.63}$$

6. (a) Consider the contour integral

$$\oint dz f(z) \cot \pi z \tag{1.64}$$

around a suitable large contour, and obtain thereby a formula for the sum

$$\sum_{n=-\infty}^{\infty} f(n). \tag{1.65}$$

(b)

$$g(a) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = ? \tag{1.66}$$

7. Evaluate

$$I(x) = \int_{0}^{\infty} dt e^{xt - e^t} \tag{1.67}$$

approximately for large positive x.

8. (a) Check that the saddle-point approximation of the gamma function is

$$\Gamma(z) = \int_{0}^{\infty} dt e^{-t} t^{z-1} \simeq \sqrt{2\pi} z^{z-1/2} e^{-z}. \tag{1.68}$$

(b) Find the saddle-point approximation of the beta function

$$B(a, b) = \int_{0}^{1} dx x^{a-1} (1 - x)^{b-1}. \tag{1.69}$$

(c) Check that the results in (a) and (b) agrees with the relationship between the gamma function and the beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}. \tag{1.70}$$

9. The Bessel function can be expressed as the Bessel's integral

$$J_n(z) = \frac{1}{\pi} \int_0^\pi d\theta \cos(n\theta - z\sin\theta). \qquad (1.71)$$

Use saddle-point approximation to find the asymptotic behavior of the Bessel function $J_n(z)$ for large z.

10. (a) For large α, find the saddle-point approximation of

$$\int_0^1 dx \, x e^{-\alpha x^2/2}. \qquad (1.72)$$

(b) Find the exact result of the integral and compare.

Chapter 2

Fourier Transform

2.1 Motivation

Fourier transform is a rearrangement of the information contained in a function $f(x)$. It is based on the fact that any (smooth) function $f(x)$ can be obtained as a superposition of sinusoidal waves with suitable amplitudes and phases

$$f(x) = \int_{-\infty}^{\infty} dk \, \tilde{f}(k) e^{ikx}, \tag{2.1}$$

where

$$\tilde{f}(k) = |\tilde{f}(k)| e^{i\theta(k)} \quad \in \mathbb{C} \tag{2.2}$$

specifies the amplitude and phase of the sinusoidal wave e^{ikx} with wave number k. Fourier transform is the map from $f(x)$ to $\tilde{f}(k)$. There is no loss of information in this transformation. We can reconstruct $f(x)$ from $\tilde{f}(k)$ via the inverse Fourier transform.

Fourier transform can be generalized to higher dimensions. It implies, for example, that we only need to consider plane waves when we learn about electromagnetic waves, because all electromagnetic waves can be viewed as superposition of plane waves.

As we will see, sometimes it is very helpful to present the information encoded in a function by its Fourier transform. In particular, it is useful to solve differential equations that are translationally invariant. In later chapters, we will also generalize the notion of Fourier transform for more general differential operators.

2.2 Dirac Delta Function and Distribution

Dirac δ-function is defined by

$$\int_{x_1}^{x_2} dx \delta(x - x_0) f(x) = \begin{cases} f(x_0), & x_0 \in (x_1, x_2), \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

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where $f(x)$ is a good (smooth) function.

From its definition and integration by parts, one deduce that the derivative of the δ-function can be defined

$$\int dx \delta'(x - x_0) f(x) = -f'(x_0),\qquad (2.4)$$

assuming that the range of integral includes x_0.

Dirac δ-functions and its derivatives are not (good) functions. They are defined only through integrals with good functions. Such objects are called "distributions". Other operations, e.g. $\delta^2(x - x_0)$, or even $\delta(x - x_1)\delta(x - x_2)$, are ill-defined.

Since δ-functions are defined only via integrals, we can derive all properties that can be defined for δ-functions by considering integrals with functions.

We will allow integration by parts and use the fact

$$\delta(x) = 0, \quad x \neq 0.$$

2.2.1 Examples

Prove the following identities.

$$\int_{x_1}^{x_2} dx f(x) \frac{d}{dx}\delta(x - x_0) = -f'(x_0), \quad x_1 < x_0 < x_2, \qquad (2.5)$$

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0), \qquad (2.6)$$

$$\delta(a(x - x_0)) = \frac{1}{|a|}\delta(x - x_0), \qquad (2.7)$$

$$\delta(f(x)) = \sum_k \frac{1}{|f'(x_k)|}\delta(x - x_k), \quad \forall x_k \ni f(x_k) = 0. \qquad (2.8)$$

Note that the coefficients on the RHS in (2.7) and (2.8) are absolute values since the δ-functions are positive-definite.

2.2.2 An Identity for Delta Function

Consider the distribution

$$A(x) \equiv \lim_{k \to \infty} e^{ikx}. \qquad (2.9)$$

Firstly it is a distribution rather than a well-defined function of x. For $x \neq 0$, $A(x) = 0$, because

$$\int dx\, A(x)f(x) = \int dx \lim_{k \to \infty} e^{ikx} f(x) = 0 \qquad (2.10)$$

for any smooth function $f(x)$ that vanishes in the neighborhood of $x = 0$. (We used this property in the stationary phase approximation.) For $x = 0$, it is just 1. Hence this distribution $A(x)$ is nonzero on only one point (of measure 0). It is thus equivalent to 0.

Next we consider the distribution

$$B(x) = \int_{-\infty}^{\infty} dk e^{ikx} = \lim_{k \to \infty} \frac{1}{ix}(e^{ikx} - e^{-ikx}). \qquad (2.11)$$

Again, it vanishes for any $x \neq 0$, but it is ill-defined for $x = 0$. We may guess that it is proportional to $\delta(x)$, which is also 0 for $x \neq 0$ and ill-defined for $x = 0$. We can check this by computing

$$\int_{-a}^{a} dx\, B(x) = \int_{-a}^{a} dx \int_{-\infty}^{\infty} dk\, e^{ikx} = \int_{-\infty}^{\infty} dk \frac{e^{ika} - e^{-ika}}{ik} = 2\pi \qquad (2.12)$$

for any $a > 0$. (You can use the residue theorem to evaluate the last step.) Thus, if $B(x)$ is proportional to $\delta(x)$, it must be

$$B(x) = \int_{-\infty}^{\infty} dk\, e^{ikx} = 2\pi\delta(x). \qquad (2.13)$$

Let us try to be more rigorous in deriving this result (2.13). Since $B(x)$ vanishes for all $x \neq 0$, $\int dx\, B(x) f(x)$ should only depend on the the behavior of $f(x)$ at $x = 0$. Because $\int dx\, B(x) f(x)$ is a linear map of $f(x)$ (to $\mathbb{C}$), the most general situation is

$$\int_{-\infty}^{\infty} dx\, B(x) f(x) = a_0 f(0) + a_1 f'(1) + a_2 f''(2) + \cdots \qquad (2.14)$$

for constant coefficients $a_0, a_1, a_2, \cdots$. This means

$$\int dx\, \delta^{(n)}(x) f(x) = (-1)^n f^{(n)}(0). \qquad B(x) = \int_{-\infty}^{\infty} dk\, e^{ikx} = a_0\delta(x) - a_1\delta'(x) + a_2\delta''(x) - \cdots . \qquad (2.15)$$

What we can learn from (2.12) is only that $a_0 = 2\pi$.

From the definition of $B(x)$ one can check that it is an even function of x. This implies that all $a_n = 0$ for all odd n's. We can further fix all the rest coefficients a_n by considering a simple example

$$\delta^{(n)}(-x) = (-1)^n \delta^{(n)}(x).$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, B(x) e^{-\frac{\alpha}{2}x^2} &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha}{2}(x - ik/\alpha)} e^{-\frac{1}{2\alpha}k^2} \\
&= \sqrt{\frac{2\pi}{\alpha}} \int_{-\infty}^{\infty} dk\, e^{-\frac{1}{2\alpha}k^2} \\
&= 2\pi. \qquad (2.16)
\end{aligned}$$

This should be identified with

$$\int_{-\infty}^{\infty} dx(2\pi\delta(x) + a_2\delta''(x) + a_4\delta^{(4)}(x) + \cdots)e^{-\frac{\alpha}{2}x^2} = 2\pi + a_2\alpha + 3a_4\alpha^2 + \cdots . \qquad (2.17)$$

Hence we conclude that $a_n = 0$ for all n.

2.3   Review of Fourier Series

Periodic functions $f(x)$ of period 2π can be approximated by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) . \qquad (2.18)$$

Another equivalent expression is

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}.$$

(2.19)

If $f(x)$ is real, $a_n, b_n \in \mathbb{R}$ and $f_n = f_{-n}^* \in \mathbb{C}$.

The equal signs in (2.18) and (2.19) are not exact. There can be points of measure 0 where the equality breaks down. In particular there is the Gibbs' phenomenon: if there is a sudden change of the slope of $f(x)$, the value of the Fourier series typically overshoots.

The Fourier series for periodic functions is just a special case of the following general idea: For a given class of functions (e.g. periodic functions on $[-\pi, \pi)$, one can choose a complete basis of functions $\{\phi_n(x)\}$ so that it is always possible to express any function $f(x)$ in this class as a linear superposition of them: $f(x) = \sum_n f_n \phi_n(x)$, which holds almost everywhere on the domain of $f(x)$. The Fourier expansion to be discussed below is another example. Here we give two more examples.

We will talk about the general theory of complete basis of functions in Chap. 3.

If periodic functions on $[-\pi, \pi)$ can be expressed as (2.18), then even periodic functions can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx),$$

(2.20)

and the odd ones as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

(2.21)

These are functions defined on $[0, \pi]$, and their values in $[-\pi, 0]$ are determined by either $f(-x) = f(x)$ or $f(-x) = -f(x)$. When we consider a function $f(x)$ defined on $[0, \pi]$, we can use either $\cos(nx)$ or $\sin(nx)$ to expand it, and its values in the region $[-\pi, 0]$ will be determined accordingly. Note that the RHS of (2.20) has the property that its first derivative vanishes at the boundaries $x = 0, \pi$, while the RHS of (2.21) vanishes at $x = 0, \pi$.

2.4 Fourier Transform as a Limit of Fourier Series

Any periodic function $f(x)$ of period 2π can be approximated to arbitrary accuracy by $\sum_n f_n e^{inx}$. By a change of variable (scaling of x) we have

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi nx/L}, \qquad x \in [-L/2, L/2).$$

(2.22)

Using

$$\int_{-L/2}^{L/2} dx\, e^{i2\pi mx/L} e^{i2\pi nx/L} = L\delta_{m+n}^0,$$

(2.23)

we find

$$f_n = \frac{1}{L} \int_{-L/2}^{L/2} dx\, e^{-i2\pi nx/L} f(x). \tag{2.24}$$

Now we take the limit $L \to \infty$ and expect that any fx. $f(x)$ defined on $\mathbb{R}$ can be approximated to arbitrary accuracy by the expansion. To do so, we imagine that the fx $f_n : \mathbb{Z} \to \mathbb{C}$ defined on $\mathbb{Z}$ is promoted to a fx. $\hat{f}(n) : \mathbb{R} \to \mathbb{C}$ defined on $\mathbb{R}$ by interpolation, so that $\hat{f}(n) = f_n$ for $n \in \mathbb{Z}$. In the limit of large L, we focus on fx's f_n which are very smooth, so that they do not change much when $n \to n+1$. That is, $f_{n+1} - f_n \to 0$ as $L \to \infty$.

Consider the change of variables

$$n \to k = 2\pi n/L, \tag{2.25}$$

$$f_n \to \tilde{f}(k) = \mathcal{N}\hat{f}(n). \tag{2.26}$$

The first equation implies that

$$\sum_n \simeq \int dn = \int dk \frac{L}{2\pi}, \tag{2.27}$$

$$\delta_{m+n}^0 \simeq \delta(m+n) = \delta(\tfrac{L}{2\pi}(k-k')) = \tfrac{2\pi}{L}\delta(k-k'). \tag{2.28}$$

To make the connection between the Kronecker delta and Dirac delta fx., we examine a sum/integral of them

$$\sum_n f_n \delta_{m+n}^0 = f_{-m} \simeq \hat{f}(-m) = \int dn\, \hat{f}(n)\delta(m+n). \tag{2.29}$$

Thus eqs. (2.22), (2.23) and (2.24) become

$$f(x) = \int_{-\infty}^{\infty} dk \frac{L}{2\pi\mathcal{N}} \tilde{f}(k) e^{ikx}, \tag{2.30}$$

$$\int_{-\infty}^{\infty} dx\, e^{ikx} e^{ik'x} = 2\pi\delta(k+k'), \tag{2.31}$$

$$\tilde{f}(k) = \frac{\mathcal{N}}{L} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). \tag{2.32}$$

To get a well-defined limit for $L \to \infty$, the ratio $L/\mathcal{N}$ should be a finite number. On the other hand the choice of the finite number is merely a convention.

An often used convention is $L/\mathcal{N} = \sqrt{2\pi}$:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{f}(k) e^{ikx}, \tag{2.33}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x). \tag{2.34}$$

Repeating this decomposition for n-variables, we get

$$f(x) = \int_{-\infty}^{\infty} \frac{d^n k}{(2\pi)^{n/2}} \tilde{f}(k) e^{ikx}, \tag{2.35}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{d^n x}{(2\pi)^{n/2}} e^{-ikx} f(x). \tag{2.36}$$

2.5   Basics of Fourier Transform

Try to be familiar with the following identities:

$$\tilde{\tilde{f}}(x) = f(x), \tag{2.37}$$

$$\widetilde{(f+g)}(k) = \tilde{f}(k) + \tilde{g}(k), \tag{2.38}$$

$$\widetilde{(af)}(k) = a\tilde{f}(k), \tag{2.39}$$

$$\widetilde{(\partial_i f)}(k) = ik_i\tilde{f}(k), \tag{2.40}$$

$$(\widetilde{\partial_{i_1}\cdots\partial_{i_m}f})(k) = (ik_{i_1})\cdots(ik_{i_m})\tilde{f}(k), \tag{2.41}$$

$$\text{Parseval:}\quad \int d^n k |\tilde{f}(k)|^2 = \int d^n x |f(x)|^2, \tag{2.42}$$

$$\tilde{f}(0) = \int \frac{d^n x}{(2\pi)^{n/2}} f(x), \tag{2.43}$$

$$f(x)^* = f(x) \Leftrightarrow \tilde{f}(-k) = \tilde{f}^*(k), \tag{2.44}$$

$$\text{convolution:}\quad \widetilde{fg}(k) = \int \frac{d^n p}{(2\pi)^{n/2}} \tilde{f}(k-p)\tilde{g}(p). \tag{2.45}$$

2.5.1   Uncertainty Relation

What Fourier transform does to a function $f(x)$ is to decompose it into a superposition of sinusoidal waves. (This is the same as Fourier series, except that the period $\to \infty$.)

Spiky details of $f(x)$ correspond to the large k dependence of $\tilde{f}(k)$, while the large scale behavior of $f(x)$ is encoded in the small k dependence of $\tilde{f}(k)$.

An important property of Fourier transform is that it demonstrates the uncertainty principle $\Delta x \Delta p \geq 1$ of QM. If a wave function $f(x)$ has a characteristic length scale L so that it is natural to write $f(x) = F(x/L)$, then its Fourier transform will have a characteristic scale of L^{-1}. That is, if the wave function $f(x)$ is mostly concentrated within a range of length L, its Fourier transform $\tilde{f}(k)$ is mostly concentrated within a range of length $1/L$.

This can be seen by a scaling argument as follows:

$$\begin{aligned}
\tilde{f}(k) &= \int \frac{dx}{\sqrt{2\pi}} f(x) e^{ikx} \\
&= \int \frac{dx}{\sqrt{2\pi}} F(x/L) e^{iLkx/L} \\
&= \int \frac{L dx'}{\sqrt{2\pi}} F(x') e^{iLkx'} \\
&= L\tilde{F}(Lk). \tag{2.46}
\end{aligned}$$

In the 3rd step we let $x = Lx'$. (Note that the overall scaling of a wave function does not change the range of concentration.)

If we superpose two sinusoidal waves with almost the same wavelength $e^{i(k+\kappa)x} + e^{i(k-\kappa)x} = 2\cos(\kappa x)e^{ikx}$, the result is a seemingly sinusoidal wave with a slowly-varying amplitude. It takes a distance of scale $\Delta x \sim 1/\kappa$, which is the inverse of the uncertainty in wave number, to see the change in amplitude, and to tell its deviation from an exact sinusoidal wave.

2.5.2 Example

The most important example is the Gaussian

$$f(x) = Ae^{-\frac{x^2}{2a^2}}. \tag{2.47}$$

Its Fourier transform is

$$
\begin{aligned}
\tilde{f}(k) &= A \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2a^2} + ikx} \\
&= A \int \frac{adx'}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} e^{-\frac{a^2 k^2}{2}} \\
&= aAe^{-\frac{a^2 k^2}{2}}.
\end{aligned} \tag{2.48}
$$

In the 2nd step we used the change of variable $ax' = x - ia^2 k$ to complete square. The Gaussian we start with ($f(x)$) has an uncertainty of $\Delta x = a$, and its Fourier transform is again an Gaussian with the uncertainty of $\Delta k = 1/a$.

Find the Fourier transform of the following functions:

1. Identity $f(x) = 1$.

2. Dirac δ-fx. $f(x) = \delta(x - x_0)$.

3. $f(x) = \frac{1}{x^2 + a^2}$.

2.6 Applications to Differential Equations

Fourier transform can be used to reduce the action of differential operators to multiplications by variables. For example, the ODE with constant parameters a, b,

$$\left(\frac{d^2}{dx^2} + a\frac{d}{dx} + b \right) \phi(x) = \rho(x) \tag{2.49}$$

is reduced to

$$(-k^2 + iak + b)\tilde{f}(k) = \tilde{\rho}(k). \tag{2.50}$$

$\tilde{f}$ can be easily solved from this algebraic eq., and then $f(x)$ is know as the Fourier transform of $\tilde{f}$. This works also for ODE's of higher orders.

Fourier transform for several variables can also be used to solve PDE's.

Fourier transform is not so useful when the differential op. involves non-trivial fx's as coefficients. But if the coefficients are all independent of a certain variable, it is still helpful to perform Fourier transform on that variable. For example, for op's of the form

$$\mathcal{D} = \mathcal{D}_0 - \partial_t^2, \tag{2.51}$$

where $\mathcal{D}_0$ is independent of t, we can do Fourier transform on t so that it is reduced to

$$\mathcal{D} = \mathcal{D}_0 + k^2. \tag{2.52}$$

Fourier transform applies best to problems with translational symmetry.

2.6.1   Example

Find the general solution to the differential equation

$$\frac{d^2}{dx^2}y(x) + a\frac{d}{dx}y(x) + by(x) = A\cos(kx) \tag{2.53}$$

for given constants a, b, A. Note that we have a quite different situation if k happens to be a solution to the algebraic equation

$$-k^2 + iak + b = 0.$$

(This happens only when $a = 0$ if $k \in \mathbb{R}$.)

2.6.2   Electric Potential of Time-Dependent Source

The electric potential $V(x)$ in Lorentz gauge satisfies

$$\Box^2\phi(x) = -\rho(x), \tag{2.54}$$

where $\rho(x)$ is the charge density. Due to superposition principle (the fact that this equation is linear), the solution of V can be obtained by superposing the electric potential generated by a point charge at an instant of time

$$\Box^2 G(x, x') \equiv (\nabla^2 - \partial_t^2)G(x, x') = \delta^{(4)}(x - x'), \tag{2.55}$$

where $G(x, x')$ represents $V(x)$ generated by a point charge at the space-time point x'. If $G(x, x')$ is known, $\phi(x)$ is just

$$\phi(x) = -\int d^4x' G(x, x')\rho(x'). \tag{2.56}$$

Therefore, solving $G(x, x')$ reduces the problem of solving $V(x)$ for all kinds of charge distributions $\rho(x)$ to a problem of integration.

Due to translation symmetry, $G(x, x') = G(x - x')$. Its Fourier transform (in a different convention)

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{G}(k)e^{ik\cdot x}, \tag{2.57}$$

where $k \cdot x \equiv k_0 t - \vec{k} \cdot \vec{x}$, satisfies

$$(k_0^2 - \vec{k}^2)\tilde{G}(\omega, \vec{k}) \equiv k^2\tilde{G}(k) = 1. \tag{2.58}$$

So we find

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot x}. \tag{2.59}$$

However, this expression is ill-defined, because the integrand diverges at $k^2 \equiv k_0^2 - \vec{k}^2 = 0$.

The ambiguity involved in giving this ill-defined quantity a precise meaning corresponds to the ambiguity in choosing the boundary/initial condition for the differential eq.

Let us single out the integral of k_0

$$G(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} g(\vec{k}), \qquad g(\vec{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{k_0^2 - \vec{k}^2} e^{ik_0 t}. \qquad (2.60)$$

The ambiguity resides in $g(\vec{k})$.

It turns out that the problem is better understood in terms of complex analysis. The divergence of the integrand can be circumvented in the complex plane of k_0. That is, we imagine that k_0 is originally a complex number, although the integral of interest is an integral over the real line in the complex plane of k_0. The divergences of the integrand include a simple pole at $k_0 = -|\vec{k}|$ and another simple pole at $k_0 = |\vec{k}|$. We can circumvent the poles by passing around them either from above or from below. There are 4 possible combinations of choices for the 2 poles, each giving a different but well-defined (finite) integral.

For a given choice of definition of the integral, we can evaluate the integral using the residue theorem.

The contour in the complex plane of k_0 we need here must include the real axis of k_0. To make a complete contour we need to add half a circle at infinity either in the upper half plane or lower half plane. At infinity in the UHP, $k_0 = \pm\infty + i\infty$, while in the LHP, $k_0 = \pm\infty - i\infty$. The difference between the choices of UHP vs. LHP is thus $e^{ik_0 t} = e^{\pm i\infty t - \infty t}$ (for UHP) vs. $e^{ik_0 t} = e^{\pm i\infty t + \infty t}$ (for LHP). Therefore, for $t > 0$, this exponential factor is 0 at infinity in the UHP, and for $t < 0$, it is 0 at infinity in the LHP.

Since what we want to compute is the integral over the real axis, the residue theorem helps us only if the half circle at infinity has no contribution to the integral. This critirium determines whether we should choose the half circle in the UHP or LHP for the contour.

For example, what will we get if we decide to circumvent both poles of k_0 by passing them from below (deforming the contour from the real axis to the LHP around the poles)? Let us consider what the Green's fx. is for $t > 0$ and $t < 0$ separately. For $t > 0$, we should choose the contour to be the real axis plus a half circle at infinity in the UHP, so that the contour integral is the same as the integral over the real axis. Then the residue theorem tells us that

$$g(\vec{k}) = \frac{2\pi i}{2\pi} \left(\frac{1}{2|\vec{k}|} e^{i|\vec{k}|t} - \frac{1}{2|\vec{k}|} e^{-i|\vec{k}|t} \right). \qquad (2.61)$$

Similarly, for $t < 0$, we choose the contour to be the real axis plus a half circle at infinity in the LHP, so that the contour integral is the same as the integral

over the real axis. The residue theorem now gives

$$g(\vec{k}) = 0, \tag{2.62}$$

because neither of the poles is inside the contour. The fact that $G(x - x') = 0$ for $t < t'$ tells us that this is the *retarded* Green's fx.

If we had chosen to circumvent both poles from above, we would have gotten the *advanced* Green's fx. Other choices of circumventing the poles give other types of solutions, e.g. retarded potential for positive freq. modes and advanced potential for neg. freq. modes, which is what Feynman chose for QED.

Having explained the conceptual problems, let us now evaluate the retarded Green's fx. more explicitly. Plugging (2.61) into (2.60), we get

$$G(x) = \int \frac{dk\, d\cos\theta\, d\phi\, k^2}{(2\pi)^3} e^{-ikr\cos\theta} \frac{i}{2k}(e^{ikt} - e^{-ikt}), \tag{2.63}$$

where we denoted $|\vec{k}|$ by k, and defined θ w.r.t. $\vec{x}$. One can easily integrate over ϕ and θ, and find

$$G(x) = \int \frac{dk}{(2\pi)^2} \frac{1}{2r}(e^{ikr} - e^{-ikr})(e^{ikt} - e^{-ikt}). \tag{2.64}$$

Finally, integrating over t and recall that this is for $t > 0$, the retarded Green's fx. is

$$G(\vec{x}, t) = \begin{cases} -\frac{1}{4\pi}\frac{1}{r}\delta(t - r) & (t > 0), \\ 0 & (t < 0). \end{cases} \tag{2.65}$$

The sol. to the PDE

$$(\nabla^2 - \partial_t^2)\phi = \rho \tag{2.66}$$

is therefore

$$\phi(\vec{x}, t) = \int d^4x'\, G(x - x')\rho(x') = \int d^3\vec{x}' \frac{1}{8\pi^2} \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}', t - |\vec{x} - \vec{x}'|). \tag{2.67}$$

2.6.3   Diffusion Equation

The diffusion eq.

$$\partial_t u = \partial_x^2 u \tag{2.68}$$

describes the change of temperature over space and time. For the class of problems in which $u(t = 0, x) = u_0(x)$ is given at the initial time $t = 0$, we can first find the solution for $u(t = 0, x) = \delta(x - x')$, and then superpose the solution for a generic initial condition $u_0(x)$.

By Fourier transform,

$$u(t, x) = \int \frac{dk}{2\pi} \tilde{u}(t, k) e^{ikx}. \tag{2.69}$$

The diffusion eq. implies that

$$\partial_t \tilde{u}(t,k) = -k^2 \tilde{u}(t,k), \tag{2.70}$$

which is solved by

$$\tilde{u}(t,k) = \tilde{v}(k)e^{-tk^2}. \tag{2.71}$$

At $t = 0$, we want

$$u(t,x) = \int \frac{dk}{2\pi} \tilde{v}(k)e^{ikx} = \delta(x) = \int \frac{dk}{2\pi} e^{ikx}. \tag{2.72}$$

Therefore $\tilde{v}(k) = 1$, and

$$u(t,x) = \int \frac{dk}{2\pi} e^{tk^2 + ikx} = \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)}. \tag{2.73}$$

If the source is located at $x = x'$, the solution is obtained by $x \to (x - x')$.

Hence, for a generic initial condition, the solution is

$u(t,x)$ here plays a role resembling Green's fx. It is sometimes called a "boundary Green's fx."

$$u(t,x) = \int dx' u_0(x') \frac{1}{2\sqrt{\pi t}} e^{-(x-x')^2/(4t)} = \frac{1}{\sqrt{\pi}} \int dy u_0(x + 2\sqrt{t}y)e^{-y^2}. \tag{2.74}$$

2.7   Laplace Transform

Laplace transform can be viewed as an application of Fourier transform to functions $f(t)$ defined for $t > 0$ (or $t > t_0$), which is not necessarily decaying sufficiently fast as $t \to \infty$ for $\tilde{f}(k)$ to be well-defined.

For $f(t)$ defined for $t > 0$ ($f(t)$ is not defined for $t < 0$) and $f(t)e^{-ct}$ for some $c > 0$ decaying to 0 sufficiently fast, the combination

$\Theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$

$$f(t)e^{-ct}\Theta(t) \tag{2.75}$$

is a function whose Fourier transform can be defined while the Fourier transform of $f(t)$ may be ill-defined. But the information contained in this combination is as much as $f(t)$ itself. There is (almost) no loss of information to use the Fourier transform of this combination to describe $f(t)$ (except the value $\lim_{t \to \infty} f(t)$).

Applying the formulas of Fourier transform,

$$f(t)e^{-ct}\Theta(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k)e^{ikt}, \tag{2.76}$$

$$\tilde{f}(k) = \int_0^{\infty} dt f(t)e^{(-c-ik)t}. \tag{2.77}$$

Combining exponential factors in these equations, we find

$$f(t)\Theta(t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} F(s)e^{st}, \tag{2.78}$$

$$F(s) = \int_0^{\infty} dt f(t)e^{-st}, \tag{2.79}$$

where we renamed $\tilde{f}(k)$ by $F(s)$, and

$$s = c + ik. \tag{2.80}$$

These formulas (2.78) and (2.79) define Laplace transform.

2.7.1   Example

Find the Laplace transform for $f(t) = 1$. (When we talk about Laplace transform, we always implicitly assume that $f(t)$ is not defined for $t < 0$.) Applying (2.79), we find

$$F(s) = \int_0^\infty dt e^{-st} = \frac{1}{s}. \tag{2.81}$$

Let us also transform $F(k)$ back to $f(t)$ to check the validity of our formulas

$$f(t) = \int \frac{ds}{2\pi i} \frac{1}{s} e^{st} = 1, \qquad \text{for} \quad t > 0. \tag{2.82}$$

Here we used the residue theorem, adding a large semicircle to the left of complex plane of s to complete the contour. ($e^{st} \to 0$ for the semicircle on the left.)

For $t < 0$, we must choose the semicircle on the right. Then the residue theorem gives $f(t) = 0$.

2.8   Other Integral Transforms

In addition to Fourier and Laplace transforms, there are other types of integral transforms.

In general, an integral transform is a change of basis of the space of functions. A complete basis for a space of functions $\mathcal{V}$ is a set of functions $\{\phi_n(x)\}$, which will be denoted $\{|\phi_n\rangle\}$, that can be used to express any function $f(x) \in \mathcal{V}$ as a linear superposition

$$f(x) = \sum_n f_n \phi_n(x). \tag{2.83}$$

Typically we can find a *dual basis* $\{\psi_n(x)\}$ such that

$$\int dx \mu(x) \psi_m^*(x) \phi_n(x) = \delta_{mn}. \tag{2.84}$$

We call $\{\phi_n\}$ an *orthonormal basis* if $\psi_n = \phi_n$.

Introducing the notation of inner product on $\mathcal{V}$ as

$$\langle f|g \rangle \equiv \int dx \mu(x) f^*(x) g(x), \tag{2.85}$$

eq. (2.83) can be expressed as

$$|f\rangle = |\phi_n\rangle \langle \psi_n|f\rangle, \tag{2.86}$$

where n is summed over the basis and

$$f_n = \langle \psi_n | f \rangle \equiv \int dx \mu(x) \psi_n(x) f(x). \tag{2.87}$$

2.9 Exercises

1. Prove the following identities.

$$\int_{x_1}^{x_2} dx f(x) \frac{d^n}{dx^n} \delta(x - x_0) = (-1)^n \frac{d^n}{dx^n} f(x_0), \quad x_1 < x_0 < x_2, \tag{2.88}$$

$$\delta^{(n)}(\vec{F}(\vec{x})) = \sum_k \frac{1}{|\det(\frac{\partial F_i}{\partial x_j})|} \delta^{(n)}(\vec{x} - \vec{x}_k). \tag{2.89}$$

2. Find the Fourier transform of the following fx's:

   (a) $f(x) = \frac{d}{dx} \delta(x - x_0)$.

   (b) $f(x) = \frac{x}{x^2 + a^2}$.

   (c) $f(t) = e^{-t/T} \sin(\omega t)$ for $t > 0$.

   (d) $f(x) = x^n e^{-\alpha x}$ for $n = 0, 1, 2, \cdots$ for $x > 0$.

   (e) $f(x) = g(ax) e^{ibx}$ if $G(k)$ is the Fourier transform of $g(x)$.

3. Prove Parseval's equation (2.42) and convolution theorem (2.45).

4. Find the retarded Green's function G which satisfies

$$(-\partial_t^2 + \nabla^2 - m^2) G(x, x') = \delta^{(4)}(x - x'), \tag{2.90}$$

   where $m \in \mathbb{R}$ can be interpreted as the mass of the particle in propagation.

5. Find the most general solution of the differential equation

$$(\partial_t^2 - \partial_x^2) \phi(t, x) = 0 \tag{2.91}$$

   with the boundary condition

$$\phi(t, 0) = \phi(t, L) = 0 \tag{2.92}$$

   for $\phi(t, x) \in \mathbb{R}$.

6. Find the most general solution of the differential equation for given parameter a

$$(i\partial_t + \partial_x^2 + \partial_y^2 - a) \phi(t, x, y) = 0 \tag{2.93}$$

   with the boundary conditions

$$\phi(t, 0, y) = 0, \quad \partial_x \phi(t, L, y) = 0, \quad \phi(t, x, y + 2\pi R) = \phi(t, x, y). \tag{2.94}$$

7. If the Laplace transform of $f(t)$ is $F(k)$, what is the Laplace transform of $\frac{d}{dt}f(t)$?

8. Use Laplace transform to find the solution of the differential eq.

$$\frac{d^2}{dt^2}X(t) = -\omega^2 X(t). \tag{2.95}$$

9. Find the Laplace transform of the following functions:

   (a) $\delta(t - t_0)$.

   (b) $\sin \omega t$.

   (c) $\cos \omega t$.

   (d) t^n.

   (e) $e^{-\lambda t}$.

10. Find the inverse Laplace transform of:

    (a) $\frac{1}{(s-a)^n}$.

    (b) $\frac{1}{(s-a)^2 - b^2}$.

    (c) $\frac{1}{(s^2+1)(s-1)}$.

$+++$

Chapter 3

Sturm-Liouville Theory

3.1 Motivation

The purpose of this chapter is mainly to establish the concept of the following table of analogy:

Linear Algebra	Differential Equations	Quantum Mechanics
vector (colume) $$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$	function $f(x)$	state (wave function) $\lvert\psi\rangle \quad (\psi(x))$
linear space $\{v \in \mathbb{C}^n\}$	space of functions $\{f(x) : \mathrm{BC}\}$	Hilbert space $\{\lvert\psi\rangle\}$
inner product $v^T w \in \mathbb{C}$	integration $\int dx\, \mu(x) f_1^*(x) f_2(x)$	inner product $\langle\psi_1\lvert\psi_2\rangle$
matrix M_{ij}	differential operator $\mathcal{D} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$	operator $\mathcal{O}(\hat{x}, \hat{p})$
Hermitian matrix $M_{ij}^* = M_{ji}$	self-adjoint operator $\mathcal{D}^\dagger = \mathcal{D}$	observable $\mathcal{O}^\dagger = \mathcal{O}$
eigenvector $Mv = \lambda v$	eigenfunction $\mathcal{D}f = \lambda f$	eigenstate $\mathcal{O}\lvert\psi\rangle = \lambda\lvert\psi\rangle$

Except subtleties involved in the limit $n \to \infty$ (n is the dimension of the linear space), the analogy between linear and (linear) differential equations is almost exact. This analogy will be very useful for our understanding of differential equations.

3.2 Linear Algebra: Review

Review of linear algebra:

- Linear space $\mathcal{V}$:
  If $|\alpha\rangle, |\beta\rangle \in \mathcal{V}$, then $a|\alpha\rangle + b|\beta\rangle \in \mathcal{V}$ for $a, b \in \mathbb{C}$.

- $\{|\alpha\rangle\}$ is a *basis* of $\mathcal{V}$ if any state $|\psi\rangle \in \mathcal{V}$ is a superposition

$$|\psi\rangle = \sum_\alpha \psi_\alpha |\alpha\rangle. \tag{3.1}$$

If $\alpha = 1, \cdots, n$ and all ψ_α's are linearly independent, the dimension of $\mathcal{V}$ is n.

- Inner product of $\mathcal{V}$:
  The inner product $I(|\alpha\rangle, |\beta\rangle)$, which is often denoted $\langle\alpha|\beta\rangle$, is a map from two elements of $\mathcal{V}$ to $\mathbb{C}$. It should be linear and anti-linear in the two arguments

$$I(|\alpha\rangle, (a|\beta_1\rangle + b|\beta_2\rangle)) = a\langle\alpha|\beta_1\rangle + b\langle\alpha|\beta_2\rangle, \tag{3.2}$$
$$I((a|\alpha_1\rangle + b|\alpha_2\rangle), |\beta\rangle) = a^*\langle\alpha_1|\beta\rangle + b^*\langle\alpha_2|\beta\rangle \tag{3.3}$$

and Hermitian

$$\langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle. \tag{3.4}$$

The notation $\langle\alpha|\beta\rangle$ for inner product is suggestive of its properties. Define a linear space $\mathcal{V}^* = \{\langle\psi| : |\psi\rangle \in \mathcal{V}\}$ and call $\langle\psi|$ the Hermitian conjugate of $|\psi\rangle$. Let the linear relations in $\mathcal{V}^*$ be defined from a complex conjugation of those in $\mathcal{V}$. That is, a linear relation in $\mathcal{V}$

$$|\psi\rangle = a|\alpha\rangle + b|\beta\rangle$$

is mapped to

$$\langle\psi| = a^*\langle\alpha| + b^*\langle\beta|$$

in $\mathcal{V}^*$. Then the properties (3.2), (3.3) are naturally inferred from the notation $\langle\alpha|\beta\rangle$. We can also generalize the notion of Hermitian conjugate to $\mathbb{C}$ as complex conjugate. Then (3.4) also follows from the usual rule for Hermitian conjugation of a product

$$(AB)^\dagger = B^\dagger A^\dagger.$$

- $\mathcal{D}$ is a linear operator on $\mathcal{V}$ if it is a map that maps any state $|\psi\rangle \in \mathcal{V}$ to a state in $\mathcal{V}$ and

$$\mathcal{D}(a|\alpha\rangle + b|\beta\rangle) = a\mathcal{D}|\alpha\rangle + b\mathcal{D}|\beta\rangle. \tag{3.5}$$

- Eigenstates and eigenvalues:
  Solutions ($\lambda \in \mathbb{C}, |\psi\rangle \in \mathcal{V}$) of the equation

$$\mathcal{D}|\psi\rangle = \lambda|\psi\rangle \tag{3.6}$$

are called eigenvalues (λ) and eigenstates ($|\psi\rangle$) of $\mathcal{D}$.

- Hermitian/Self-Adjoint operators:
  The adjoint op. (denoted by $\mathcal{D}^\dagger$) of $\mathcal{D}$ is defined by the requirement

$$\langle\alpha|\mathcal{D}|\beta\rangle = (\mathcal{D}^\dagger|\alpha\rangle)^\dagger|\beta\rangle \qquad \forall |\alpha\rangle, |\beta\rangle \in \mathcal{V}. \tag{3.7}$$

$\mathcal{D}$ is self-adjoint if $\mathcal{D}^\dagger = \mathcal{D}$.

- For a finite dimensional linear space $\mathcal{V}$, the eigenvalues λ_n are real, and the eigenstates $\{|n\rangle\}$ form a complete basis of $\mathcal{V}$.

3.3   ODE: Review

An ordinary differential equation (ODE) of order n

$$f(y(x), y'(x), y''(x), \cdots, y^{(n)}) = 0 \qquad (3.8)$$

typically needs n initial conditions

$$y(0), y'(0), \cdots, y^{(n-1)}(0) \qquad (3.9)$$

to uniquely determine a solution.

Here we will only consider *linear* ODEs of order 2

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0. \qquad (3.10)$$

The linearity of the equation implies that if $y_1(x)$ and $y_2(x)$ are two (linearly independent) solutions, then

$$a_1 y_1(x) + a_2 y_2(x) \qquad (3.11)$$

is also a solution. The fact that this ODE is of order 2 implies that there are only two independent parameters in the general solution, and so this is already the most general solution.

There are several elementary ODE's that every physicist is expected to be able to solve.

ODE of order 1 are of the form $f(y, y') = 0$. One can solve this algebraic equation to the form $y' = g(y)$ and then it is solved by integrating both sides of

$$g^{-1}(y)dy = dx.$$

Later we will also consider inhomogeneous equations where 0 on the RHS is replaced by a given function $d(x)$.

-
$$a\frac{d^2}{dx^2}y + b\frac{d}{dx}y + cy = 0. \qquad (3.12)$$

  This equation has translation symmetry and one should try the ansatz

$$y(x) = e^{\alpha x}, \qquad \alpha \in \mathbb{C}. \qquad (3.13)$$

  This includes all the following possibilities: $\sin(kx), \cos(kx), \sin(kx)e^{\alpha x}$, etc.

Recall Fourier transform and Fourier series.

-
$$ax^2\frac{d^2}{dx^2}y + bx\frac{d}{dx}y + cy = 0. \qquad (3.14)$$

  This equation has scaling symmetry and one should try the ansatz

$$y(x) = x^{\alpha}, \qquad \alpha \in \mathbb{C}. \qquad (3.15)$$

- You are also expected to be able to solve those that can be put into the forms above via simple change of variables.

3.4 Space of Functions and Differential Operators

- A space of functions as a linear space.

- Differential operators as linear operators.

The first thing to note about a diff. eq. of the form

$$\mathcal{D}\phi = \rho \tag{3.16}$$

is that this equation is formally the same as an equation in linear algebra, with $\mathcal{D} = $ matrix (linear map), and $|\phi\rangle$ and $|\rho\rangle$ being vectors.

We will use $|\cdot\rangle$ to represent elements in a vector space $\mathcal{V}$, so we rewrite (3.16) as

$$\mathcal{D}|\phi\rangle = |\rho\rangle. \tag{3.17}$$

The space of functions is a linear space.

The differential operator $\mathcal{D}$ acts on a function to give another function, and its action is linear:

$$\mathcal{D}(a|f_1\rangle + b|f_2\rangle) = a\mathcal{D}|f_1\rangle + b\mathcal{D}|f_2\rangle. \tag{3.18}$$

Thus $\mathcal{D}$ is a linear map acting on the linear space of functions. This is the most salient feature of linear diff. eqs, and we will see that it is useful to view it as a problem in linear algebra.

Apart from the linear-space structure of the space of functions, the space of functions is also equipped with the structure of an algebra, i.e., that of multiplication: $f_1(x)f_2(x) = f_3(x)$. However, the latter is not relevant in the problem of solving diff. eq's.

Feynman: *"The same equations have the same solutions.*

3.4.1 Functions on a Lattice

In a numerical analysis of an ODE using computer softwares, the continuous domain is often approximated by a lattice. The diff. op. becomes a difference operator. $|\phi\rangle$ and $|\rho\rangle$ become columns with finite number of elements. One can imagine that the original problem is the continuum limit of this problem of linear algebra when the number of lattice sites goes to infinity.

The linear space of functions on a lattice has the natural basis in which each basis vector $|e_n\rangle$ is the function which is 1 at the n-th point and 0 everywhere else. A function can be expanded in this basis as $|f\rangle = |e_n\rangle f_n$, where f_n is the value of the function at the n-th point. (The continuum limit of f_n is $f(x)$.) We have the following correspondence:

$$n \leftrightarrow x, \quad f_n \leftrightarrow f(x), \quad \sum_n \leftrightarrow \int dx, \quad \delta_{mn} \leftrightarrow \delta(x - x'), \tag{3.19}$$

$$|f\rangle = f_n|n\rangle \leftrightarrow f(x) = \int dx' f(x')\delta(x - x'), \tag{3.20}$$

$$\langle f|g\rangle = \sum_n f_n^* g_n \leftrightarrow \int dx f(x)^* g(x), \tag{3.21}$$

$$|f\rangle\langle g| = f_m g_n^*|m\rangle\langle n|, \quad f_m g_n^* \leftrightarrow f(x)g(x')^*. \tag{3.22}$$

3.4.2   Change of Basis

One can also choose a different basis for the linear space $\mathcal{V}$ related to the previous basis by a linear map M: $|e_n\rangle = |E_a\rangle M_{an}$. In the new basis, a function is $|f\rangle = |E_a\rangle F_a$, with $F_a = M_{an} f_n$ In the continuum limit, it is $(n \to x, a \to k)$

$$F(k) = \int dx\; u(k, x) f(x). \tag{3.23}$$

This is the Fourier transform if $u(k, x) \propto e^{ikx}$.

Thus, functions do not have to be represented as $f(x)$ (in terms of the basis $\delta(x - x')$). They are vectors in a linear space and how they look depends on which basis you choose. (The linear space of fxs is infinite dimensional; we will worry about convergence later.)

3.4.3   Eigenfunctions

Recall that the complete set of eigenvectors of a Hermitian matrix M constitutes a basis of the linear space on which M acts. Recall also that one can always choose this basis to be orthonormal.

Understanding that a diff. eq. is a problem in linear algebra, we can apply techniques in linear algebra. If $\mathcal{D}$ is *Hermitian*, it is associated with a convenient basis of the linear space $\mathcal{V}$, i.e., its eigenvectors.

$$\mathcal{D}|e_n\rangle = \lambda_n |e_n\rangle. \tag{3.24}$$

The number of eigenvectors equals the dimension of the linear space $\mathcal{V}$. Any two eigenvectors with different eigenvalues must be orthorgonal, because

$$0 = \langle e_m | \mathcal{D} e_n \rangle - (|\mathcal{D}^\dagger e_m\rangle)^\dagger |e_n\rangle = (\lambda_n - \lambda_m)\langle e_m | e_n \rangle. \tag{3.25}$$

For the subspace of $\mathcal{V}$ spanned by eigenvectors with the same eigenvalue, we can always choose the eigenvectors to be *orthonormal*

$$\langle e_m | e_n \rangle = \delta_{mn}. \tag{3.26}$$

via Graham-Schmidt, Then we have the identity

$$|e_n\rangle\langle e_n| = I, \tag{3.27}$$

where I is the identity operator.

$M = |e_m\rangle M_{mn}\langle e_n|$ is a linear operator whose action on a state is obvious from its notation. As a matrix its elements are M_{mn}. One can think of $|e_n\rangle$ as the basis of columns, and $\langle e_n|$ as the basis of rows.

3.5   Graham-Schmidt Orthonormalization

The *Graham-Schmidt Orthonormalization* is a way to construct a complete basis for a linear space. The space of functions is a linear space so we can use this method to construct a complete basis of functions.

As a linear space, the space of functions $\mathcal{V}$ is equipped with an inner product

$$\langle \phi_1 | \phi_2 \rangle = \int dx \mu(x) \phi_1^*(x) \phi_2(x). \tag{3.28}$$

Starting with an element in $\mathcal{V}$, we can normalize it and call it $\phi_0(x)$. It can be chosen as the first element of the complete basis. Then we pick arbitrarily another function $\phi_1(x)$ which is linearly independent of $\phi_0(x)$. We can project out the part of $\phi_1(x)$ that is proportional to $\phi_0(x)$

$$\phi_1(x) \to \phi_1' \equiv \phi_1 - \phi_0 \langle \phi_0 | \phi_1 \rangle \tag{3.29}$$

so that it is now perpendicular to ϕ_0: $\langle \phi_0 | \phi_1' \rangle = 0$. We can normalize it and rename it as $\phi_1(x)$, and include it in the orthonormal basis. Then we pick a third function $\phi_2(x)$ which is linearly independent of $\{\phi_0, \phi_1\}$, that is,

$$a\phi_0(x) + b\phi_1(x) + c\phi_2(x) = 0 \qquad \forall x \tag{3.30}$$

only if $a = b = c = 0$. Again we project out the components of $\phi_2(x)$ along ϕ_0 and ϕ_1

$$\phi_2(x) \to \phi_2' \equiv \phi_2 - \phi_0 \langle \phi_0 | \phi_2 \rangle - \phi_1 \langle \phi_1 | \phi_2 \rangle. \tag{3.31}$$

You can check that the ϕ_2' is perpendicular to both ϕ_0 and ϕ_1:

$$\langle \phi_0 | \phi_2' \rangle = \langle \phi_1 | \phi_2' \rangle = 0. \tag{3.32}$$

Now we normalize ϕ_2' and rename it as ϕ_2. The first 3 elements of the complete basis are found.

The process of finding the $(n+1)$-th basis element goes on like this. Find a function ϕ_n that is linearly independent of $\{\phi_0, \cdots, \phi_{n-1}\}$. Define

$$\phi_n' \equiv \phi_n - \sum_{k=0}^{n-1} \phi_k \langle \phi_k | \phi_n \rangle. \tag{3.33}$$

It is orthogonal to every element already in the basis $\{\phi_0, \cdots, \phi_{n-1}\}$. We can normalize it, rename it as $\phi_n(x)$ and include it in the basis.

The construction guarantees that the set $\{\phi_0, \phi_1, \cdots\}$ is orthonormal. Usually we also adopt a systematic way to pick the n-th function so that the "completeness" of the basis is more manifest. For instance, often we choose ϕ_0 to be constant and ϕ_n to be a polynomial of order n.

3.6   Sturm-Liouville Differential Operator

For the *inner product* defined by

$$\langle f | g \rangle = \int dx \mu(x) f^*(x) g(x), \tag{3.34}$$

where $\mu(x)$ is called the *weight function*, a generic 2nd order ordinary differential operator is Hermitian if it can be written in the form

$$\mathcal{D} = \frac{1}{\mu(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + \frac{i}{2}\left(r(x)\frac{d}{dx} + \frac{d}{dx}r(x)\right) + q(x)\right], \qquad (3.35)$$

with real functions $p(x), r(x), q(x)$, assuming suitable BC's. This expression can be better understood by recalling the identity $(MN)^\dagger = N^\dagger M^\dagger$. If $\mathcal{V}$ is a space of real fx's, or fx's with Neumann BC, we need $r(x) = 0$, and $\mathcal{D}$ is called the *Sturm-Liouville* differential operator.

Eigenvectors ϕ_n of a *Sturm-Liouville operator*

$$\mathcal{D} = \frac{1}{\mu(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right], \qquad (3.36)$$

constitute a *complete* basis of the linear space of fxs $\mathcal{V}$ (on which $\mathcal{D}$ is self-adjoint), assuming suitable choice of $p(x), q(x), \mu(x)$ as well as BC's. (We don't need to consider eigenfx's which do not belong to $\mathcal{V}$.) It is complete in the sense that any well behaved (piecewise continuous function with a finite number of finite discontinuities) F can be approximated to arbitrary accuracy by a series $\sum_n a_n \phi_n$. That is,

$$\lim_{m\to\infty}\int_{x_0}^{x_1} dx\mu(x)\left(F(x) - \sum_{n=0}^{m} a_n\phi_n(x)\right)^2 = 0. \qquad (3.37)$$

For example the periodic function $f(x) = \Theta(x)$ on $(-\pi, \pi)$, equals the series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{2n+1} \qquad (3.38)$$

up to points at the discontinuities.

For an *orthonormal* basis ϕ_n, i.e., $\langle\phi_m|\phi_n\rangle = \delta_{mn}$, the coefficients a_n are

$$a_n = \langle\phi_n|F\rangle \equiv \int dx\mu(x)\phi_n^*(x)F(x). \qquad (3.39)$$

To be more rigorous, one should make some assumptions about the Sturm-Liouville operator. (Discussions on the properties of eigenfx's/eigenvalues of Sturm-Liouville op's is called the *Sturm-Liouville theory*.) A simple example is to assume that $p(x) > 0$ and $\mu(x) > 0$, and $p(x)$ is differentiable, while $\mu(x)$ and $q(x)$ are continuous, and that the space of fx's $\mathcal{V}$ should be defined as the set of differentiable, square-integrable fx's living on a finite interval $[a, b]$ either with homogeneous BC's of the form $\alpha\phi(a)+\alpha'\phi'(a) = 0$ and $\beta\phi(b)+\beta'\phi'(b) = 0$, or with the periodic BC. In this case it can be proven that the eigenvalues λ_i are real and discrete, and the ordered set $\lambda_1 < \lambda_2 < \cdots \to \infty$ is unbounded on one side. Furthermore, the number of zeros of the eigenfx's ϕ_i is larger for those with larger eigenvalues.

On the other hand, the nice properties (e.g. completeness of eigenfx's) are not limited to those satisfying all the conditions above. If the space on which fx's in $\mathcal{V}$ are defined is not compact, the index n is a continuous parameter and the sum $\sum_n$ should be replaced by an integral. The Kronecker delta above shall be replaced by Dirac delta fx.

3.7 Examples

Here we construct a complete basis for the op. $-\frac{d^2}{dx^2}$ for various boundary conditions on the interval $[0, 1]$.

3.7.1 Periodic BC

For the periodic BC

$$f(1) = f(0), \qquad f'(1) = f'(0), \tag{3.40}$$

the answer is the Fourier series

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(2n\pi x) + \sum_{n=1}^{\infty} B_n \sin(2n\pi x). \tag{3.41}$$

The eigenfx's of ∂_x^2 have eigenvalues $-(2n\pi)^2$.

3.7.2 Dirichlet BC

For the Dirichlet BC

$$f(0) = f(1) = 0, \tag{3.42}$$

the answer is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x). \tag{3.43}$$

The eigenvalues are $-(n\pi)^2$.

3.7.3 Neumann BC

For the Neumann BC

$$f'(0) = f'(1) = 0, \tag{3.44}$$

the answer is

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(n\pi x). \tag{3.45}$$

The eigenvalues are $-(n\pi)^2$.

3.8   Inequalities

3.8.1   Bessel's Inequality

For a positive definite inner product $\langle f|f \rangle \geq 0$, if $|f\rangle = \sum_n a_n |\phi_n\rangle$ for an orthonormal basis $|\phi_n\rangle$, then $\langle f|f \rangle \geq \sum'_n |a_n|^2$ for a sum $\sum'$ over a subset of the basis. The meaning of the equality is simply that the inner product can be represented in different ways. The equality of inner product in different representations is called *Parseval relation*.

3.8.2   Schwarz Inequality

The Schwarz inequality is simply

$$\|f\|^{\,2}\,\|g\|^{\,2} \geq |\langle f|g\rangle|^2, \tag{3.46}$$

where $\|f\|^{\,2} \equiv \langle f|f\rangle$. This is as obvious as $|\cos\theta| \leq 1$.

3.9   Theorems

For the Sturm-Liouville operator

$$\mathcal{D} = \frac{1}{\mu(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right], \tag{3.47}$$

where $\rho(x)$ and $q(x)$ are continuous, $p(x)$ is continuously differentiable, $\rho(x)$ and $p(x)$ are positive definite, and $q(x)$ is non-negative for the range $\alpha < x < \beta$, we have:

1. Let the eigenvalues be ordered

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \tag{3.48}$$

   then $0 \leq \lambda_0$ and $\lim_{n\to\infty}\lambda_n = \infty$.

2. $\phi_n(x)$ (the eigenfunction with eigenvalue λ_n) has n zeros in (α,β).

3. Between any two consecutive zeros of $\phi_n(x)$, there must be at least one zero of $\phi_m(x)$ if $\lambda_m > \lambda_n$.

4. If we increase p, q, or decrease ρ, or reduce the range (α,β), all eigenvalues increase.

3.10 Exercises:

1. The ODE

$$\left(P(x)\frac{d^2}{dx^2} + Q(x)\frac{d}{dx} + R(x) - \lambda S(x)\right)\phi(x) = 0$$

can be viewed as an eigenvalue problem for a Sturm-Liouville op. $\mathcal{D}$. What are the fx's $\mu(x), p(x), q(x)$ defining $\mathcal{D}$?

2. For the eigenvalue problem

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = y(L) = 0, \tag{3.49}$$

verify the 4 properties in sec. 3.9.

3. Find the complete set of eigenfx's and their eigenvalues for the operator $\frac{d^2}{dx^2}$ with the following BC's:

   (a) $f(0) = 0, \quad f'(1) = 0.$

   (b) $f(0) + 2f'(0) = 0, \quad f(1) = 0.$

   Find the coefficients f_n to expand a generic function $f(x)$ in terms of the eigenfunctions.

4. Find the complete set of eigenfx's for the eigenvalue problem

$$(x + 1)^2 y''(x) + \lambda y(x) = 0 \tag{3.50}$$

   with the boundary condition

$$y(0) = y(1) = 0. \tag{3.51}$$

   Find the coefficients f_n to expand a generic function $f(x)$ in terms of the eigenfunctions.

5. Use Schwarz inequality to prove the *uncertainty relation*

$$\Delta A \Delta B \geq \frac{1}{2}. \tag{3.52}$$

   It holds for any state $|\psi\rangle$ if A and B are Hermitian op's satisfying the commutation relation

$$[A, B] \equiv AB - BA = i. \tag{3.53}$$

   Here ΔA^2 is defined by $\langle\psi|(A - \bar{A})^2|\psi\rangle$ with $\bar{A} \equiv \langle\psi|A|\psi\rangle$.

adsfds

Chapter 4

Special Functions

4.1 Introduction

Some functions are special and arise naturally in elementary problems. Here are a few possible reasons how some functions are "special".

- It arises as part of the eigenfxs of the Laplace op.

$$\nabla^2 \phi + \lambda \phi = 0. \tag{4.1}$$

The Laplace op. in flat space $\nabla^2 = \sum_i \partial_i^2$ appears in almost every elementary problem in physics (wave eq, diffusion eq, Schrödinger eq., etc.)

In Cartesian coordinates, $e^{i\vec{k}\cdot\vec{x}}$ is special. (And hence sin, cos are special.)

In spherical coordinates, Legendre polynormials are special.

- It has a geometrical meaning.

- It has some interesting algebraic properties.

- They form a complete basis for a certain space of functions.

4.2 Legendre Polynomials

Orthogonality:

$$\int_{-1}^{1} dx\, P_m(x) P_n(x) = \frac{2}{2n+1} \delta_{mn}. \tag{4.2}$$

Examples:

$$
\begin{aligned}
P_0 &= 1 & (4.3)\\
P_1 &= x & (4.4)\\
P_2 &= \frac{1}{2}(3x^2 - 1) & (4.5)\\
P_3 &= \frac{1}{2}(5x^3 - 3x) & (4.6)
\end{aligned}
$$

General formula:
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{4.7}$$

Generating function:
$$g(t, x) = \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}. \tag{4.8}$$

Recurrence relations:
$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0 \tag{4.9}$$
$$(1 - x^2)P_n'(x) = -nxP_n(x) + nP_{n-1}(x). \tag{4.10}$$

Differential equation:
$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0. \tag{4.11}$$

4.3   Hermite Polynomials

Orthogonality:
$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x) H_n(x) = 2^n n! \sqrt{\pi}\, \delta_{mn}. \tag{4.12}$$

Examples:

The coefficient of the x^n term in H_n is 2^n.

$$H_0 \;=\; 1 \tag{4.13}$$
$$H_1 \;=\; 2x \tag{4.14}$$
$$H_2 \;=\; 4x^2 - 2 \tag{4.15}$$
$$H_3 \;=\; 8x^3 - 12x \tag{4.16}$$

Symmetry:
$$H_n(-x) = (-1)^n H_n(x). \tag{4.17}$$

General formula:
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{4.18}$$

Generating function:
$$e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{4.19}$$

Recurrence relations:
$$H_{n+1} = 2xH_n - 2nH_{n-1} \tag{4.20}$$
$$H_n'(x) = 2nH_{n-1}(x) \tag{4.21}$$

Differential equation:
$$y'' - 2xy' + 2ny = 0. \tag{4.22}$$

4.4 Laguerre Polynomial

Orthogonality:

$$\int_0^\infty dx\, e^{-x} L_m(x) L_n(x) = \delta_{mn}. \tag{4.23}$$

Example:

$$L_0 = 1 \tag{4.24}$$
$$L_1 = 1 - x \tag{4.25}$$
$$L_2 = 1 - 2x + \frac{1}{2}x^2 \tag{4.26}$$
$$L_3 = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3 \tag{4.27}$$

General formula:

$$L_n = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right). \tag{4.28}$$

Generating function:

$$g(x, z) = \sum_{n=0}^\infty z^n L_n = \frac{e^{-\frac{xz}{1-z}}}{1-z}. \tag{4.29}$$

Recurrence relations:

$$(n+1)L_{n+1} = (2n+1-x)L_n - nL_{n-1}, \tag{4.30}$$
$$xL_n'(x) = nL_n(x) - nL_{n-1}(x), \tag{4.31}$$

Differential equation:

$$xy'' + (1-x)y' + ny = 0. \tag{4.32}$$

4.5 Bessel Functions

General formula:

$$J_m(x) = \sum_{\ell=0}^\infty \frac{(-1)^\ell x^{2\ell+m}}{2^{2\ell+m}\ell!(m+\ell)!}. \tag{4.33}$$

Generating function:

$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^\infty t^n J_n(x). \tag{4.34}$$

Recurrance relation:

$$\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x). \tag{4.35}$$

Differential equation:

$$x^2 y'' + xy' + (x^2 - m^2)y = 0. \tag{4.36}$$

Other identities:

$$
\begin{aligned}
J_{-m}(x) &= (-1)^m J_m(x), & &\text{(4.37)}\\
J_m(x) &\to \frac{(x/2)^m}{\Gamma(m+1)}, & x \to 0, &\text{(4.38)}\\
J_n(x+y) &= \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y), & &\text{(4.39)}\\
J_n(x) &= \frac{1}{\pi} \int_0^\pi d\theta \, \cos(x \sin\theta - n\theta), & &\text{(4.40)}\\
J_m(x) &\to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right), & x \to \infty. &\text{(4.41)}
\end{aligned}
$$

More identities:

$$
\sum_{m=-\infty}^{\infty} J_m(x) = 1, \tag{4.42}
$$

$$
\int_0^1 dx\, x\, J_k(z_{km}x) J_k(z_{kn}x) = \tfrac{1}{2} J_{k+1}^2(z_{km}) \delta_{mn}, \tag{4.43}
$$

$$
\int_0^\infty dr\, r\, J_m(kr) J_m(k'r) = \tfrac{1}{k} \delta(k - k'), \tag{4.44}
$$

where $z_{km} = m$-th zero of $J_k(x)$.

The defintion of Bessel function J_n can be extended to the case when the index is real J_ν, $\nu \in \mathbb{R}$.

These functions $J_\nu(x)$ are sometimes called Bessel functions of the first kind. There are also Bessel functions of the second kind $Y_\nu(x)$, which are also called Neumann functions $N_\nu(x)$. They can be defined by

$$
N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}.
$$

This is ill-defined for $\nu = $ integer. In that case we take the limit $\nu \to n$. $N_\nu(x)$ is the other independent solution of the same differential equation (4.36) with $m \to \nu$. Hankel functions are just a change of basis

$$
H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x), \qquad H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x). \tag{4.45}
$$

The description above allows the argument x of the Bessel function $J_\nu(\mathrm{x})$ to be complex. When it is purely imaginary, we get the modified Bessel functions

$$
I_\nu(x) = i^{-\nu} J_\nu(ix), \qquad K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H^{(1)}(ix). \tag{4.46}
$$

They satisfy the differential equation

$$
x^2 y'' + xy' - (x^2 + \nu^2)y = 0. \tag{4.47}
$$

4.6 Other Special Functions

In this section we briefly introduce gamma function $\Gamma(x)$, beta function $B(x, y)$, and hypergeometric functions.

4.6.1   Gamma Function and Beta Function

The gamma function can be defined as

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}. \tag{4.48}$$

Using integration by parts, one can show from this that

$$\Gamma(x) = (x-1)\Gamma(x-1). \tag{4.49}$$

For an integer n, $\Gamma(n) = (n-1)!$.

Another useful property is

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}. \tag{4.50}$$

Beta function is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4.51}$$

4.6.2   Hypergeometric Function

Differential equation:

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{4.52}$$

A regular solution is

$${}_2F_1(a,b;c;x) = 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \ldots. \tag{4.53}$$

Another independent solution is

$$x^{1-c}\,{}_2F_1(a+1-c, b+1-c; 2-c; x). \tag{4.54}$$

Properties:

$$\frac{d}{dx}{}_2F_1(a,b;c;x) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;x), \tag{4.55}$$

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\,\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}. \tag{4.56}$$

The generalized hypergeometric functions are

$${}_pF_q\left[\begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array}; x\right] = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k}\frac{x^k}{k!}, \tag{4.57}$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+k-1). \tag{4.58}$$

4.7   Exercises:

1. Expand the function

$$f(x) = \begin{cases} +1, & 0 < x < 1 \\ -1, & -1 < x < 0. \end{cases} \qquad (4.59)$$

   as an infinite series of Legendre polynomials $P_n(x)$.

2. Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} P_n(x). \qquad (4.60)$$

   Hint: Use the generating function.

3. Use the Grahamm-Schmidt orthogonalization to work out the first few Hermite polynomials $H_n(x)$ for $n = 0, 1, 2$, assuming that $H_n(x)$ is a polynomial of order n of the form $H_n(x) = 2^n x^n + \cdots$. (The measure of integral is e^{-x^2}.)

4. (Fourier-Bessel transform)
   Using (4.44), we define the Fourier-Bessel transform (or Hankel transform)

$$f(r) = \int_0^\infty dk\, k\, J_n(kr) F(k), \qquad F(k) = \int_0^\infty dr\, r\, J_n(kr) f(r). \quad (4.61)$$

   Find $F(k)$ for $f(r) = e^{-ar}/r$.

   Hint: Use the generating function.

5. (Spherical Bessel function)
   Try to solve the following differential equation

$$x^2 y'' + 2xy' + (x^2 - n(n+1))y = 0 \qquad (4.62)$$

   by using the ansatz $y = x^\alpha J_\nu(x)$ and $y = x^\alpha Y_\nu(x)$. Show that the result is

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \qquad y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x). \qquad (4.63)$$

6. What linear homogeneous second-order differential equation has

$$x^\alpha J_{\pm n}(\beta x^\gamma) \qquad (4.64)$$

   as solutions? Give the general solution of

$$y'' + x^2 y = 0. \qquad (4.65)$$

7. Find an approximate expression for the Gamma function $\Gamma(-x)$ for large positive x.

   Use Stirling's formula and (4.50).