Lattice implementation of generalized symmetry and 't Hooft anomaly

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- M. Abe, OM and H. Suzuki, PTEP **2023**, no.2, 023B03 (2023) [2210.12967].
- M. Abe, OM, S. Onoda, H. Suzuki and Y. Tanizaki, JHEP **08**, 118 (2023) [2303.10977].
- OM, Gaugefields.jl, `https://github.com/o-morikawa/Gaugefields.jl`; OM and H. Suzuki, in progress

Symmetry

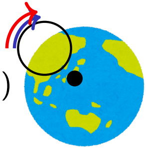
- Symmetry: fundamental tool in physics
  - Universal applications to high energy physics, condensed matter physics and mathematics
- Noether theorem and conservation law

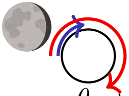
$$\text{Sym}: \phi \mapsto \phi'; S(\phi) = S(\phi'), \qquad \partial_\mu j^\mu = 0$$

$$\text{Charge } Q \equiv \int j^0 d^{D-1}x$$

- Structure in the nature: Lorentz symmetry, CPT theorem, Gauge invariance, Internal symmetries
  - Fundamental forces in the nature (e.g., photon: $U(1)$ sym)

$\theta \to \theta(x_{\mathrm{E}})$





$\theta \to \theta(x_{\mathrm{M}})$

't Hooft anomaly matching

- More aspects of Symmetry
  - Landau theory: phase transition (or vacuum structure) from viewpoint of symmetry
  - Symmetry breaking: quantum anomaly, spontaneous breaking
- 't Hooft anomaly matching for strongly coupled theories ['79]
  - Assume global symmetry G in system
  - Introduce background gauge field A assoc. G (by gauging G)

$$\mathcal{Z}[A] = \int \mathcal{D}\{\phi\}\, e^{-S(\{\phi\},A)}$$
$$\overset{?}{=} \mathcal{Z}[A^g] = \cdots = e^{\mathcal{A}[A,g]}\mathcal{Z}[A]$$

  $e^{\mathcal{A}} \neq 1 \xrightarrow{\text{Anomalous}}$ 't Hooft anomaly which is invariant at any energy scale (renormalization group inv.)
  - Restriction on low-energy dynamics: SSB, phase structure, SPT

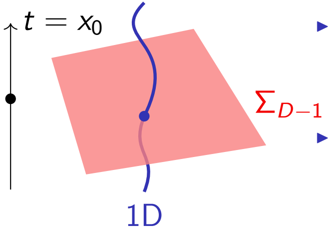
Summary: Recent generalization of symmetry

- Generalized global symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
  - Coupled with topological field theory
    - ★ Changing topological structure without changing local dynamics [Kapustin–Seiberg '14]:
    - ★ Nontrivial information by using generalized 't Hooft anomaly matching
- Basic property: fractionality of topological charge
  - Usually, topo. charge $Q \sim \int F\tilde{F} \in \mathbb{Z}$ under topo. sectors
    - ★ θ term: θQ (strong CP problem, axion physics, sign problem)
  - Discrete higher-form sym $\rightarrow$ background gauge field B: $Q \sim \int B\tilde{B} \in \frac{1}{N}\mathbb{Z}$,
    Then, 't Hooft anomaly $\mathcal{Z}_{\theta+2\pi}[B] = e^{-2\pi i Q}\mathcal{Z}_{\theta}[B]$
    [Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]

$\overset{\text{global desc.}}{\Rightarrow}$ 't Hooft twisted boundary condition $Q \sim \int F\tilde{F} \in \frac{1}{N}\mathbb{Z}$
[van Baal '82]

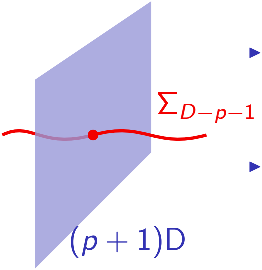
Generalization: higher-form symmetry

- (0-form) Symmetry



  - Charge (codim 1)
  $$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}$$
  - Symmetry operator
  $$U_\alpha(\Sigma_{D-1}) = e^{i\alpha Q}$$
  Topological under deformation of Σ_{D-1}

- Higher-form symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]



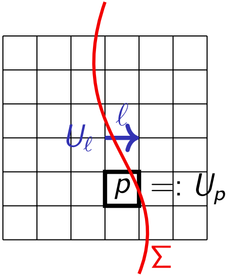
  - p-form symmetry $G^{[p]}$ (codim $p+1$)
  $$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$
  - Transforming a "loop" operator $W(C)$
  $$W(C) \mapsto U(\Sigma) W(C)$$
  $$= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \#$$

Center symmetry in YM theory

- Lattice $SU(N)$ YM theory
  - link variable $U_\ell \in SU(N)$
- Center symmetry: $\mathbb{Z}_N^{[1]}$

$e^{\frac{2\pi i}{N}k} \in \mathbb{Z}_N \subset SU(N); \quad U_\ell \mapsto e^{\frac{2\pi i}{N}k\#(\Sigma,\ell)} U_\ell$

Intersection # of Σ & link ℓ; $U_p \mapsto U_p$

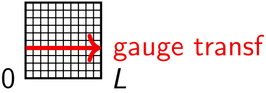


- Gauging the center symmetry

$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N}B_p} U_p \qquad B_p$: 2-form gauge field assoc. $\mathbb{Z}_N^{[1]}$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - Recall 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$

't Hooft flux $z_{\mu\nu} = \sum B_p \mod N$

Aiming at transparent understanding

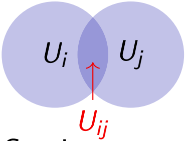
- Wise but *not transparent* understanding
  - Topological objects from lattice viewpoint as center sym
  - Formal discussion in continuum theory
    - ⋆ $\mathbb{Z}_N^{[q]}$ gauge field: $U(1)$ field $B^{(q)}$
      with constraint $NB^{(q)} = dB^{(q-1)}$ from charge-N Higgs
  - $Q \sim \frac{1}{N} \int B \wedge B? \xrightarrow[\text{cohomology op}]{\text{Swapping } \wedge \text{ w/}} \frac{1}{N} \int P_2(B) \sim -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N}$
    - ⋆ Global nature described by Čech cohomology (discrete group!)
  - Indicating mixed 't Hooft anomaly with chiral sym/θ-periodicity

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q} \mathcal{Z}_\theta[B_p] \quad \text{not } 2\pi \text{ periodic}$$

- Fully regularized framework: Lattice regularization
  - Lattice construction of fiber bundle and Q [Lüscher '84]
  - Coupled with *higher-form* lattice gauge fields

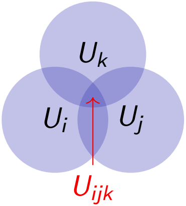
Principal fiber bundle

- Recall Dirac's discussion
  - Gauge fields cannot be defined globally in spacetime
  - Defined on each subspace: northern/southern hemisphere
- Manifold (spacetime) X; open covering (patches) $\{U_i\}$
  - Gauge group G, gauge field a_i on U_i
  - Relation between a_i and a_j?



Gauge transformation:
$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} d g_{ij}$ at U_{ij}

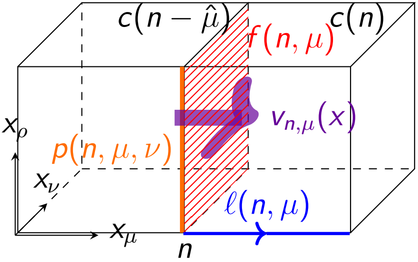
  - Consistency condition for transition function g_{ij}:



$g_{ii} = \mathrm{id}$, $g_{ji} = g_{ij}^{-1}$
Cocycle condition: $g_{ij} g_{jk} g_{ki} = 1$ at U_{ijk}

Bundle structure on lattice?

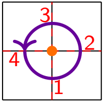
- No continuity for lattice fields?
  - Any configuration can be deformed continuously to others
  - We can observe topological structure even on lattice by Lüscher
- Setup/strategy: Lattice Λ divides X into 4D hypercubes



edge

4D cell $c(n)$
3D face $f(n,\mu)$
2D plaquette $p(n,\mu,\nu)$
1D link $\ell(n,\mu)$

1. Regard $\{c(n)\}$ as patches
2. Define transition function $v_{n,\mu}(x)$ at $f(n,\mu)$ from data as U_ℓ
   - Difficult to define it at $x \neq n$ s.t. cocycle condition is kept intact



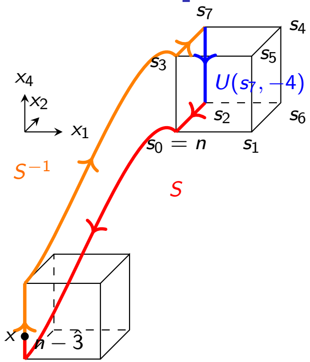
$v_1 v_2 v_3^{-1} v_4^{-1} = 1$
at $x \in p(n)$

Construction of topo. sectors on lattice [Lüscher]

- *Parallel transporter* (interpolation):
  $U_\ell \to v_f(n) \to v_f(x) \Leftarrow$ Cocycle

  $$v_{n,\mu}(n) = U(n - \hat{\mu}, \mu)$$

  $$v_{n,\mu}(x) \equiv S^{n-\hat{\mu}}_{n,\mu}(x)^{-1} v_{n,\mu}(n) S^{n}_{n,\mu}(x)$$



- Topo. sectors on lattice so that [Q in terms of $v_f(x)$] $\in \mathbb{Z}$

$$Q = \sum_{n \in \Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} \Big\{ 3 \int_{p_{n,\mu\nu}} d^2x \ \mathrm{Tr}\left[\left(v_{n,\mu} \partial_\rho v^{-1}_{n,\mu} \right) \left(v^{-1}_{n-\hat{\mu},\nu} \partial_\sigma v_{n-\hat{\mu},\nu} \right) \right]$$

$$+ \int_{f_{n,\mu}} d^3x \ \mathrm{Tr}\left[\left(v^{-1}_{n,\mu} \partial_\nu v_{n,\mu} \right) \left(v^{-1}_{n,\mu} \partial_\rho v_{n,\mu} \right) \left(v^{-1}_{n,\mu} \partial_\sigma v_{n,\mu} \right) \right] \Big\}$$

Construction of topo. sectors on lattice [Lüscher]

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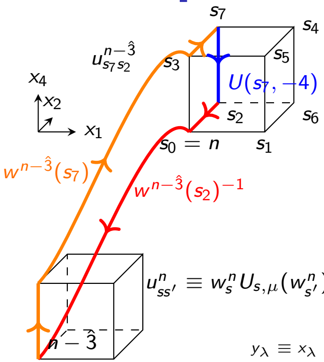


$$u^n_{ss'} \equiv w^n_s U_{s,\mu}(w^n_{s'})^{-1}$$

$$y_\lambda \equiv x_\lambda - n_\lambda$$

$$w^n(x) = U^{y_4}_{n,4} U^{y_3}_{n+y_4\hat{4},3} U^{y_2}_{n+y_4\hat{4}+y_3\hat{3},2} U^{y_1}_{n+y_4\hat{4}+y_3\hat{3}+y_2\hat{2},1}$$

$$f^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_7} u^m_{s_7 s_2} u^m_{s_2 s_0})^{y_\gamma} u^m_{s_0 s_2} (u^m_{s_2 s_7})^{y_\gamma},$$

$$g^m_{n,\mu}(x_\gamma) = (u^m_{s_5 s_1})^{y_\gamma} (u^m_{s_1 s_5} u^m_{s_5 s_4} u^m_{s_4 s_6} u^m_{s_6 s_1})^{y_\gamma} u^m_{s_1 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$h^m_{n,\mu}(x_\gamma) = (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_5} u^m_{s_5 s_1} u^m_{s_1 s_0})^{y_\gamma} u^m_{s_0 s_1} (u^m_{s_1 s_5})^{y_\gamma},$$

$$k^m_{n,\mu}(x_\gamma) = (u^m_{s_7 s_2})^{y_\gamma} (u^m_{s_2 s_7} u^m_{s_7 s_4} u^m_{s_4 s_6} u^m_{s_6 s_2})^{y_\gamma} u^m_{s_2 s_6} (u^m_{s_6 s_4})^{y_\gamma},$$

$$l^m_{n,\mu}(x_\beta, x_\gamma) = [f^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} [f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}]^{y_\beta} h^m_{n,\mu}(x_\gamma) [g^m_{n,\mu}(x_\gamma)]^{y_\beta},$$

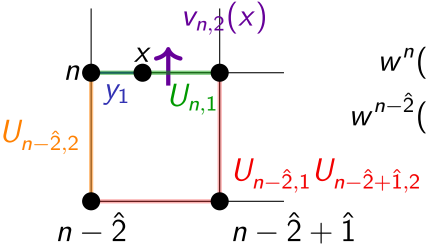
$$S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) = (u^m_{s_0 s_3})^{y_\gamma} [f^m_{n,\mu}(x_\gamma)]^{y_\beta} [l^m_{n,\mu}(x_\beta, x_\gamma)]^{y_\alpha}.$$

- Topo. sectors on lattice so that [Q in terms of $v_f(x)$] $\in \mathbb{Z}$

$$Q = \sum_{n\in\Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} \left\{ 3 \int_{p_{n,\mu\nu}} d^2x \, \mathrm{Tr}\left[\left(v_{n,\mu}\partial_\rho v^{-1}_{n,\mu}\right) \left(v^{-1}_{n-\hat{\mu},\nu} \partial_\sigma v_{n-\hat{\mu},\nu}\right)\right] + \int_{f_{n,\mu}} d^3x \, \mathrm{Tr}\left[\left(v^{-1}_{n,\mu}\partial_\nu v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\rho v_{n,\mu}\right) \left(v^{-1}_{n,\mu}\partial_\sigma v_{n,\mu}\right)\right] \right\}$$

Exercise: Lattice 2D $U(1)$ gauge theory

- $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$ [Lüscher '98, Fujiwara et al. '00]



$$w^n(x) = {U_{n,1}}^{y_1}$$
$$w^{n-\hat{2}}(x) = [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2}]^{y_1} {U_{n-\hat{2},2}}^{1-y_1}$$

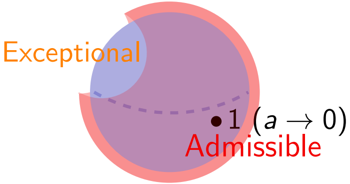
- Explicit expression of v:

$$v_{n,1}(x) = U_{n-\hat{1},1} \quad v_{n,2}(x) = U_{n-\hat{2},2} [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2} U_{n,1}^{-1} U_{n-\hat{2},2}^{-1}]^{y_1}$$
$$= U_{n-\hat{2},2} \exp[i y_1 F_{12}(n-\hat{2})]$$

  - Field strength: $F_p \equiv \frac{1}{i} \ln U_p$ for $-\pi < F_p \leq \pi$
  - (nD) To ensure Bianchi identity $dF_p = 0$, we should impose $\sup_p |F_p| < \epsilon$, $0 < \epsilon < \frac{\pi}{3}$ → Admissibility condition (see also §6)

Admissibility condition

- In general, admissibility = well-defined-ness of u^y $(0 \le y \le 1)$
  - ★ $U(1)$: $F_p = \frac{1}{i} \ln U_p$ for plaquette U_p
  - ★ $S^m_{n,\mu}(x)$ is written in terms of $(u^n_{ss'})^y$ where u is a loop $n \to s \to s' \to n$
  - ▸ E.g., u^y is ill-defined at $u = -1$; ill-def regions separate sectors
- Admissibility condition $\mathrm{tr}(1 - U_p) < \epsilon$ [Lüscher '84]



- ▸ Admissible lattice gauge fields: well-defined conf space ∼ disk
- ▸ Exceptional region
  - ★ Topological freezing
  - ★ Monopole as lattice artifact

- Under the admissibility condition, we can prove that $Q \in \mathbb{Z}$; we observe topo. sectors even on lattice!
- How about index theorem for finite a?

$$\mathrm{Index}(D) = \underbrace{-\frac{a}{2}\,\mathrm{Tr}\,\gamma_5 D_{\mathrm{ov}}}_{\text{Admissibility } \epsilon_{\mathrm{ov}}} = \underbrace{n_+ - n_-}_{\in\mathbb{Z}} \stackrel{?}{=} \frac{1}{32\pi^2}\int_x \varepsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}[F_{\mu\nu}F_{\rho\sigma}]$$

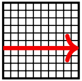
Cocycle condition relaxed by higher-form sym

- Lüscher's construction *Coupled* with higher-form gauge fields
  - $SU(N)$ YM theory coupled with $\mathbb{Z}_N$ 2-form gauge field

$$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N} B_p} U_p \qquad B_p\text{: 2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



0 gauge transf L

$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

- Cocycle condition can take a $\mathbb{Z}_N$ value:

$$\tilde{v}_{n-\hat{\nu},\mu}(x)\tilde{v}_{n,\nu}(x)\tilde{v}_{n,\mu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}$$

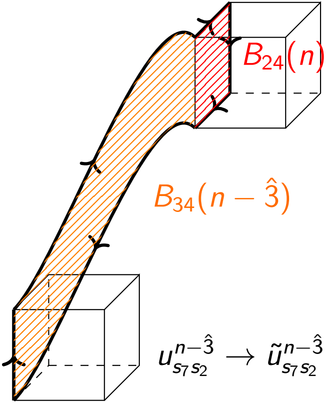
  - $\mathbb{Z}_N$ blind matters: adjoint repr.
  - $\mathbb{Z}_N^{[1]}$ gauge inv. if $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_{n-\hat{\mu},\mu}} \tilde{v}_{n,\mu}(x)$
- What is definition of $\tilde{v}_{n,\mu}(x)$?

$\mathbb{Z}_N^{[1]}$ gauge invariant construction

- Gauge invariant plaquette
  $\tilde{U}_p \equiv e^{-\frac{2\pi i}{N}B_p} U_p$
  - Recall $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell$
  - Admissibility $\mathrm{tr}(1 - \tilde{U}_p) < \epsilon$
- u: product of plaquettes $\to \tilde{u}$

$$\tilde{u}^{n-\hat{3}}_{s_7 s_2} = e^{\frac{2\pi i}{N}B_{34}(n-\hat{3})} e^{\frac{2\pi i}{N}B_{24}(n)} u^{n-\hat{3}}_{s_7 s_2}$$

- Similarly, $\tilde{v}$ is defined in terms of $\tilde{u}$
  - Gauge covariance
    $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N}\lambda_\mu(n-\hat{\mu})} \tilde{v}_{n,\mu}(x)$



- Fractional topological charge $Q = \sum_n q(n) \in -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N} + \mathbb{Z}$
  $\to$ Proof of 't Hooft anomaly in lattice regularization!

Numerical simulations of higher-form gauge fields

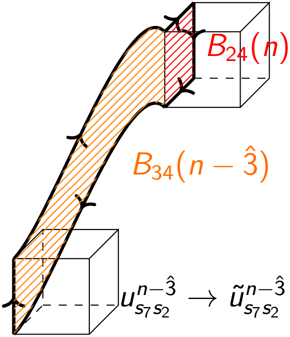
Public repository: `https://github.com/o-morikawa/Gaugefields.jl`

- Obviously, we can implement it on computer for numerical simulations
- $\forall u$, we have 1-form gauge invariant $\tilde{u}$

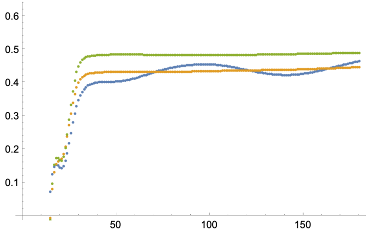
$$\tilde{u}^{n-\hat{3}}_{s_7 s_2} = e^{\frac{2\pi i}{N} B_{34}(n-\hat{3})} e^{\frac{2\pi i}{N} B_{24}(n)} u^{n-\hat{3}}_{s_7 s_2}$$

  even if missing a link (covariant)

- External/dynamical B to carry out HMC, gradient flow and MPI

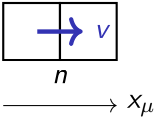


- You can see immediately fractional topo. charge
  - $SU(2)$: Green points (lattice improved Q) go to $1/2$!!!



Summary

- Generalized symmetries have been developed in this decade
  - Higher-form sym, higher-group sym, noninvertible sym, subsystem sym, . . .
  - Through the use of 't Hooft anomaly matching, new insights about *nontrivial dynamics & classification of phases*
- Standing on a fully regularized framework: lattice gauge theory
  - Generalization of Lüscher's construction of topology on lattice
  - Maintaining locality, $SU(N)$ gauge inv & higher-form gauge inv
  - There exists interpolation to smooth enough bundle structure
    - ⋆ Transition function $v_f(n) \to v_f(x)$
    - ⋆ Q is written in terms of $v_f(x)^{-1}\partial_\nu v_f(x)$



$$Q \in \mathbb{Z} \xrightarrow{\text{Gauging } \mathbb{Z}_N^{[1]}} \frac{1}{N}\mathbb{Z}$$

  - Mixed 't Hooft anomaly between $\mathbb{Z}_N^{[1]}$ & θ periodicity

Generalization: higher-group structure

- In general suppose $\otimes_{i,p} G_i^{[p]}$ global symmetry
- After gauging, a naive direct product of symmetry groups?
  - Can each symmetry be gauged *individually*?
- Gauging $G^{[0]} \times H^{[1]}$ global symmetry, then gauge transf.:

$$A \mapsto A + d\lambda^{(0)}, \qquad B \mapsto B + d\lambda^{(1)} + A d\lambda^{(0)}$$

  2-group symmetry (cf. superstring theory [Green–Schwarz '84])
  - p-group symmetry: $G_0^{[0]} \tilde{\times} \cdots \tilde{\times} G_{p-1}^{[p-1]}$
- E.g., 4D $SU(N)$ gauge theory with instanton number $p\mathbb{Z}$
  - For any $p \in \mathbb{Z}$, local and unitary [Seiberg '10]
  - Global symmetry: $\underbrace{\mathbb{Z}_N^{[1]} \text{ center sym}}_{\text{~~gauging~~}} \times \mathbb{Z}_p^{[3]}$ sym $\overset{\text{gauging}}{\to}$ 4-group [Tanizaki–Ünsal '19]

  cf. [Hidaka–Nitta–Yokokura '21]
- How to modify instanton sum & realize higher-group on lattice?

Modified instanton-sum: $\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ gauge sym?

- Inserting the delta function (or introducing Lagrange multiplier)

$$\delta\left(q_n - pc_n\right) \text{ or } \left[\sum_n \chi_n(q - \dots)\right] \quad \to \quad Q = p \underbrace{\sum_n c_n}_{\in\mathbb{Z}}$$

where $Q = \sum_n q_n$, $U(1)$ 4-form field strength c_n

  - $c = dc^{(3)}$; Charged object under $\mathbb{Z}_p^{[3]}$

$$V^{(3)} = e^{\int_{M_3} c^{(3)}} \to e^{i\chi(x)} V^{(3)} = e^{\frac{2\pi i}{p} \#(x, M_3)} V^{(3)}$$

  - θ term, $i\theta Q + i\hat{\theta}\sum_n c_n$, indicates the $2\pi/p$ periodicity of θ

- Just by counting numbers, obviously no nontrivial configurations for B_p:

$$\frac{1}{8N}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} = \sum_n \frac{1}{8N}\varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma} \text{ mod } 1 \in \mathbb{Z}$$

$\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ global symmetry $\to$ ~~gauge symmetry~~

Modified instanton-sum: higher-group symmetry

- Introducing new field Ω_n ($\Omega_n \in \mathbb{R}$ and $\sum_n \Omega_n \in \mathbb{Z}$)
  - Replacement: $c_n \to c_n - \frac{1}{Np}\Omega_n$: 3-form gauge inv

    $$q_n - pc_n + \frac{1}{N}\Omega_n = 0 \qquad : \text{fractionality allowed}$$

  - Redefine Ω_n as $\tilde{\Omega}_n \equiv \frac{1}{N}\Omega_n - \underbrace{\frac{1}{8N}\varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma}}_{\text{fractional part of } Q}$

    Again $\sum_n \tilde{\Omega}_n \in \mathbb{Z}$

    $$\check{q}_n - pc_n + \tilde{\Omega}_n = 0 \quad \text{where } \check{q}_n\text{: integral part of } Q$$

- 1-form and 3-form gauge transf with $\Omega_n^{(3)} \in \mathbb{R}$:

$$B_p \mapsto B_p + (d\lambda)_p, \quad c_n \mapsto c_n + \frac{1}{p} d\Omega_n^{(3)} \, (+\mathbb{Z}),$$

$$\tilde{\Omega}_n \mapsto \tilde{\Omega}_n + d\Omega_n^{(3)} \, (+p\mathbb{Z}) + \text{“}\frac{2}{N} B \wedge d\lambda + \frac{1}{N} d\lambda \wedge d\lambda\text{”} \, (+\mathbb{Z})$$

Finally, $w_n \equiv$ integral part of $\tilde{\Omega}_n$

- defined by using 1-form and *continuum* 3-form gauge transf
- Theory possesses “mixed 1-form” $\times$ *discrete* $\mathbb{Z}_p^{[3]}$ gauge sym

Admissibility $\to$ absence of monopole?

- Magnetic monopole
  - Magnetic defect operators provide nontrivial topology
  - Quite heavy but significant in nonperturbative dynamics
  - Maxwell equation w/ monopole current j_{m}: $d \star F = j_{\mathrm{e}}$, $dF = j_{\mathrm{m}}$.
- Admissibility $dF = 0$ to reinstate topo. structure on lattice fields
  - Exercise: $0 < \epsilon < \pi/3$?
    $F_p \equiv \frac{1}{i} \ln U_p$, $a_\ell \equiv \frac{1}{i} \ln U_\ell$ in $(-\pi, \pi]$; $F_p = (da)_p + 2\pi N_p$ since

$$(da)_p = \Delta_\mu a_{n,\nu} - \Delta_\nu a_{n,\mu} = [a_{n+\hat{\mu},\nu} - a_{n,\nu}] - [a_{n+\hat{\nu},\mu} - a_{n,\mu}]$$

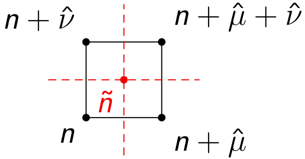
$$|da| \leq 4|a| \leq 4\pi; \qquad \text{Introduce } N \text{ as } |da + 2\pi N| \leq \pi.$$

Imposing $|F_p| < \epsilon$, $6\epsilon > |\underbrace{\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu F_{\rho\sigma}}_{\to 0}| = 2\pi |\underbrace{\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}}_{<1}|$; therefore $6\epsilon < 2\pi$

- How to discuss monopole properties on lattice w/ admissibility?
  - 2D compact bosons: θ term and magnetic operators
  - Observation of analogue of Witten effect
  - Future study: 4D gauge theory

Lattice formulation of θ angle and Witten effect

- Compact scalar fields: $\phi_1(n)$, $\phi_2(\tilde{n})$
  - Dual lattice $\tilde{n} = n + \frac{1}{2}\hat{1} + \frac{1}{2}\hat{2}$
  - $\partial\phi_a(s,\mu) \equiv \frac{1}{i} \ln e^{-i\phi_a(s)} e^{i\phi_a(s+\hat{\mu})}$
  - $\sup_\ell |\partial\phi_{a,\ell}| < \epsilon$, $0 < \epsilon < \pi/2$; then Bianchi identity $d\partial\phi = 0$

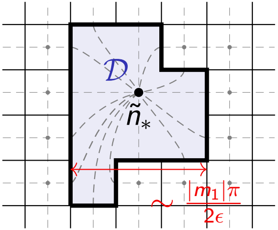


- Lattice action with θ angle

$$S = \frac{R^2}{2\pi} \sum [1 - \cos\partial\phi_{a,\ell}] - i\theta Q, \quad Q = -\frac{1}{4\pi^2} \sum \varepsilon_{\mu\nu} \partial\phi_{2,(\tilde{n},\mu)} \partial\phi_{1,(n+\hat{\mu},\nu)}$$

- Usually, $\mathcal{Q}_{\mathrm{m1}}(C) \equiv \frac{1}{2\pi} \sum_{\ell \in C} \partial\phi_{1,\ell} = 0$
- Excision method: sites&links eliminated inside $\mathcal{D}$; put one dual site $\tilde{n}_*$ in $\mathcal{D}$

$$m_1 = \mathcal{Q}_{\mathrm{m1}}(\partial\mathcal{D}) \in \mathbb{Z}$$



Continuum limit $\to$ point

- $\langle M_1(\tilde{n}_*)\dots\rangle_{\theta+2\pi} = \langle M_1(\tilde{n}_*) e^{im_1\phi_2(\tilde{n}_*)}\dots\rangle_\theta$ for magnetic defect operator $M_1(x)$

EM symmetry and 't Hooft anomaly

- $U(1)$ electric/magnetic global symmetries: $j_{\mathrm{e}} = \star d\phi$, $j_{\mathrm{m}} = d\phi$
  - Electric charged object $e^{i\phi}$;
    Monopole $M_1(x)$ with $(0, m_1) \overset{\text{Witten}}{\to}$ dyon with (m_1, m_1)
- Background gauging $U^{\mathrm{em},a}_{p,\tilde{p}}$:
  $F^{(\mathrm{e},1)}_p$ and $F^{(\mathrm{m},1)}_{\tilde{p}}$ for ϕ_1, $F^{(\mathrm{e},2)}_{\tilde{p}}$ and $F^{(\mathrm{m},2)}_p$ for ϕ_2
  - $\sup|F| < \delta$ with $0 < \delta < \min(\pi, 2\pi - 4\epsilon)$
  - $F^{\mathrm{e}}_{\mu\nu} = \Delta_\mu D\phi_{n,\nu} - \Delta_\nu D\phi_{n,\mu}$
- Analogue of θ shift is given by

$$\theta \to \theta + 2\pi, \quad F^{(\mathrm{m},1)}_{\tilde{p}} \to F^{(\mathrm{m},1)}_{\tilde{p}} - F^{(\mathrm{e},2)}_{\tilde{p}}, \quad F^{(\mathrm{m},2)}_p \to F^{(\mathrm{m},2)}_p + F^{(\mathrm{e},1)}_p$$

  - Mixing $\mathcal{Q}_{\mathrm{m1}}$ with $\mathcal{Q}_{\mathrm{e2}}$ on $\tilde{\Lambda}$; $\mathcal{Q}_{\mathrm{m2}}$ with $\mathcal{Q}_{\mathrm{e1}}$ on Λ (Witten effect)
- Under above shift, we observe mixed 't Hooft anomaly

$$\mathcal{Z}_{\theta+2\pi}[A^{(\mathrm{e},a)}, A^{(\mathrm{m},a)} - \varepsilon_{ab}A^{(\mathrm{e},b)}] = e^{-\frac{i}{2\pi}\sum \varepsilon_{\mu\nu} A^{(\mathrm{e},2)}_\mu(\tilde{n}) A^{(\mathrm{e},1)}_\nu(n+\hat{\mu})} \mathcal{Z}_\theta[A]$$

Summary

- Robust discussion on higher-group from lattice
  - [Continuum case] $SU(N) \to U(N)$ & many $U(1)$ fields: $F \to \tilde{F}$, $\mathrm{tr}\,\tilde{F} = B^{(2)}$, $NB^{(2)} = dB^{(1)}$, $pD^{(4)} = dD^{(3)} + \frac{N}{4\pi} B^{(2)} \wedge B^{(2)}$
    What is topological object? $\Rightarrow \int ND^{(4)} \in \frac{2\pi}{p}\mathbb{Z}$
  - [Lattice case] Counting integers and fractional numbers; mixture of symmetries
- Lattice construction of magnetic operators and observation of Witten effect
  - Lattice field theories with admissibility condition
  - Excision region $\mathcal{D} \to$ a point (monopole) in continuum limit
  - Arranging lattice/dual-lattice for mixing magnetic charges with electric ones
- Future works
  - Monopole, 't Hooft line in gauge theory
  - Other kinds: subsystem sym & non-invertible (categorical) sym