Massless Lifshitz Field Theory for Arbitrary z

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Overview

- Massless Lifshitz scalar field theory and ground state
- Entanglement in Quantum Information Theory
- Entanglement entropy and mutual information
- Reflected entropy and Markov gap
- Lifshitz Holography
- Summary

Lifshitz field theory

- LFTs are a class of non-relativistic field theories which are spatially isotropic, homogeneous and admits the scaling symmetry

$$t \to \lambda^z t, \quad x^i \to \lambda x^i, \quad \lambda > 0.$$

- Space-time translation (H, P_i) and rotation symmetry (M_{ij}) together with dilatation (D) forms a Lifshitz group.
- Other non-relativistic symmetry groups are Schrodinger and Galilean group.
- Lifshitz symmetry has found important applications in many physical systems.
- For example, the $z = 2$ Lifshitz scalar field theory in (2+1) dimensions also called Quantum Lifshitz model (QLM) describe the critical point of the well-known Rokhsar-Kivelson Quantum dimer model. [Moessner, Sondhi and Fradkin '01] [Ardonne, Fendley and Fradkin '04]
- Lifshitz holography which includes gravity duals for non-relativistic field theories admitting Lifshitz symmetry. [Kachru, Liu and Mulligan '08] [Taylor '08]

- In the literature, mostly Lifshitz scalar field theory with integer z has been considered.
- Various entanglement measures such as entanglement entropy [Fradkin, Moore, Hsu, Thorlacius....], entanglement negativity [Angel-Ramelli et al. '20], reflected entropy and Markov gap [Berthiere, Chen and Chen '23] also have been studied mostly for integer z.
- We employ the notion of fractional derivatives to study the massless Lifshitz theory for arbitrary values of z in any dimensions.
- We study the entanglement properties of the Lifshitz theory for arbitrary z by computing various entanglement measures.
- Note that for $z < 1$, the dispersion relation is acausal due to the existence of superluminal modes [Koroteev '11] and the violation of the null energy conditions of the holographic dual [Hoyos and Koroteev '10].
- We will focus on $z > 1$.

Massless Lifshitz scalar theory and Lifshitz ground state

- Consider the following action for the massless Lifshitz scalar field theory in (1+1)-dimensions for arbitrary $z > 1$ as

$$S = \frac{1}{2}\int dt dx \left[(\partial_t \phi)^2 - \kappa^2 (\nabla_x^z \phi)^2\right].$$

- In our work, we use the following definition of fractional derivative ∇_x^z

$$\nabla_x^z e^{ikx} \equiv (ik)^z e^{ikx}.$$

- Then, the fractional derivative of any arbitrary function can be obtained using the Fourier analysis with appropriate choice of integral contour

$$\nabla_x^z F(x) = \int_C \mathcal{F}(k)(ik)^z e^{ikx} dk.$$

- To construct the vacuum of the theory, it is convenient to consider the Hamiltonian of the theory which is given by

$$H = \frac{1}{2}\int dx \left(\Pi^2(x) + \kappa^2 (\nabla_x^z \phi)^2\right).$$

- Now, define the generalized annihilation and creation operators

$$A(x) = \frac{1}{\sqrt{2}} \left(i\Pi(x) + \kappa \nabla_x^z \phi(x) \right),$$
$$A^\dagger(x) = \frac{1}{\sqrt{2}} \left(-i\Pi(x) + \kappa \nabla_x^z \phi(x) \right).$$

- The Hamiltonian of the theory now takes the following form

$$H = \int dx A^\dagger(x) A(x) + E_{\text{vac}}.$$

- The ground state of the theory may be defined by using the position space annihilation operator as

$$A(x)|\Psi_0\rangle = 0, \quad \forall x.$$

- In the Schrodinger representation $\Pi(x) = -i\frac{\partial}{\partial\phi(x)}$, the above equation can be written as

$$\left[\frac{\delta}{\delta\phi} + \kappa \, \nabla_x^z \phi \right] |\Psi_0\rangle = 0.$$

- The ground state of the Lifshitz theory is then given by

$$|\Psi_0\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \int \mathcal{D}\phi \, e^{-S_{\rm cl}[\phi]/2} |\phi\rangle, \quad S_{\rm cl}[\phi] = \kappa \int \left(\nabla_x^{\frac{z}{2}} \phi \right)^2 dx.$$

- Here $\mathcal{Z} = \int \mathcal{D}\phi e^{-S_{\rm cl}[\phi]}$ is a normalization factor.
- This ground state wavefunctional takes the form of RK vacuum, it is given by a superposition of quantum states with a quantum mechanical amplitude $c[\phi] \propto e^{-S_{\rm cl}[\phi]/2}$.
- The propagator of the theory is given by

$$K(\phi_i, \phi_f; x_i, x_f) = \int_{\phi(x_i)=\phi_i}^{\phi(x_f)=\phi_f} \mathcal{D}\phi \exp\left(-\kappa \int_{x_i}^{x_f} \left(\nabla_x^{\frac{z}{2}} \phi \right)^2 dx \right).$$

- Consider the solution of

$$\nabla_x^z \phi = 0.$$

- With the Dirichlet boundary condition

$$\phi(x_i) = \phi_i, \quad \phi(x_f) = \phi_f.$$

- The general solution is given by

$$\phi = (\phi_f - \phi_i) \sum_{n=1}^{N_z} \frac{c_n}{l^{z-n}} (x - x_i)^{z-n} + \phi_i, \qquad \sum_n c_n = 1.$$

- Here we have used the following fractional derivative

$$\nabla_x^z x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta - z + 1)} x^{\beta - z} \quad \text{for any real } \beta \text{ and } z.$$

- The solution can be rewritten as

$$\phi = \phi_i + (\phi_f - \phi_i) g(t).$$

- The integral can be evaluated by integrating out the fluctuations around the classical solution ϕ_c and expressed as

$$K(\phi_i, \phi_f; l) = \sqrt{\frac{\gamma}{\pi l^{z-1}}} e^{-\gamma(\phi_f - \phi_i)^2 / l^{z-1}}.$$

Entanglement

- Consider a bipartite system $A \equiv A_1 \cup A_2$ in a state ρ_A with density matrices ρ_{A_1} and ρ_{A_2}. Such a state is called **separable** if it can be expressed as

$$\rho_{A_1 \cup A_2} = \sum_i p_i \left(\rho^i_{A_1} \otimes \rho^i_{A_2} \right), \quad \sum_i p_i = 1, \quad p_i \geq 0.$$

  Otherwise it is called **entangled**.
- If a state has density matrix ρ, then
  - $\mathrm{Tr}(\rho^2) = 1 \iff$ it is a pure state.
  - $\mathrm{Tr}(\rho^2) < 1 \iff$ it is a mixed state.
- An example of an entangled state for a two-spin system is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right).$$

- There are different measures to quantify the amount of entanglement such as entanglement entropy, mutual information, reflected entropy, etc.

Entanglement Entropy (EE)

- Consider a bipartite system made of subsystem A and remainder B in a state ρ. The reduced density matrix for the subsystem A is defined as

$$\rho_A = \mathrm{Tr}_B(\rho).$$

- The entanglement entropy for the subsystem A is defined by the von Neumann entropy of the reduced density matrix of A

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A).$$

- The Rényi entropy of order n is defined as

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}(\rho_A)^n.$$

- The entanglement entropy may also be expressed in terms of the Rényi entropy as

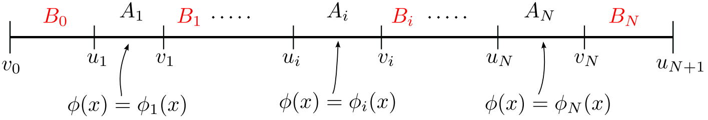
$$S_A = \lim_{n\to 1} S_A^{(n)} = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n.$$

Reduced density matrix and replica technique

- The density matrix corresponding to Lifshitz ground state is given by

$$\rho = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \mathcal{D}\phi' e^{-\frac{1}{2}(S_{\rm cl}[\phi]+S_{\rm cl}[\phi'])} \left|\phi\right\rangle \left\langle\phi'\right| .$$

- Consider a subsystem $A \equiv \bigcup_{i=1}^{N} A_i$



- The reduced density matrix $\rho_A = \mathrm{tr}_B\, \rho$ is obtained by tracing over $B := \bigcup_{i=0}^{N} B_i$ i.e. complement of A.

$$(\rho_A)_{\phi'_A,\phi''_A} = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi_B (\left\langle\phi_B\right| \left\langle\phi'_A\right|) \left|\Psi_0\right\rangle \left\langle\Psi_0\right| (\left|\phi''_A\right\rangle \left|\phi_B\right\rangle).$$

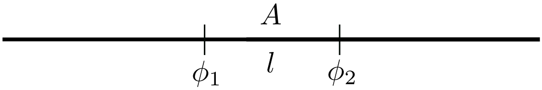
- The trace $\mathcal{Z}_n \equiv \int \mathcal{D}\phi_A (\rho_A^n)_{\phi_A,\phi_A}$ is given by

$$\mathcal{Z}_n = \frac{1}{\mathcal{Z}^n} \int_{-\infty}^{\infty} d\alpha_1 d\beta_1 \cdots d\alpha_N d\beta_N \prod_{i=1}^{N} K^n(u_i, v_i) \prod_{i=1}^{N} K^n(v_i, u_{i+1}).$$

Single interval

- For a finite subsystem A of length l in an infinite system, the trace $\mathcal{Z}_n$ is given by

$$\mathcal{Z}_n = \mathcal{Z}^{-n} \int d\phi_1 \int d\phi_2 K(\phi_1, \phi_2; l)^n.$$



- Now, using the form of the propagator, the Rényi entropy may be expressed as

$$S_n(A) = \frac{z-1}{2} \log \frac{l}{\epsilon} + \frac{c_n}{2}.$$

- Here ϵ is the UV cut-off and constant is non universal.

$$c_n = \log \frac{\pi}{\gamma} + \frac{\log n}{n-1}$$

- The entanglement entropy is given by taking the replica limit $n \to 1$

$$S(A) = \frac{z-1}{2} \log \frac{l}{\epsilon} + \frac{c_1}{2}.$$

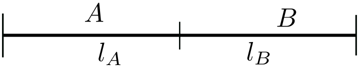
- The Rényi entropy and the entanglement entropy share the same universal UV part and differs only in their constant terms.
- This is different from the usual case of a conformal vacuum where the UV parts are proportional with a nontrivial n-dependent coefficient:

$$[S_n(A)]_{\text{UV}} = \frac{1}{2}(1 + 1/n)[S(A)]_{\text{UV}}$$

- It shows that the Lifshitz vacuum is different from the vacuum of the CFT.

Adjacent intervals in a finite system

- Consider the case of a finite system divided into two adjacent intervals A and B with length l_A and l_B respectively.



- The junction point between A and B contains a free field ϕ and the endpoints of the theory have the Dirichlet boundary conditions satisfying $\phi(0) = 0$ and $\phi(l_A + l_B) = 0$.
- In this case, the trace $\mathcal{Z}_n = \mathrm{Tr}\,\rho_A^n$ is given by

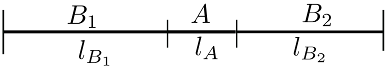
$$\mathcal{Z}_n = \mathcal{Z}^{-n} \int \mathcal{D}\phi\, K(0,\phi;l_A)^n K(\phi,0;l_B)^n = \mathcal{Z}^{-n} \int \mathcal{D}\phi\, e^{-n\gamma\left(\frac{1}{l_A^{z-1}} + \frac{1}{l_B^{z-1}}\right)\phi^2}.$$

- Using this, we obtain the entanglement entropy as

$$S(A) = \frac{1}{2} \log \frac{(l_A l_B)^{z-1}}{\left(l_A^{z-1} + l_B^{z-1}\right)\epsilon^{z-1}} + \frac{1}{2} c_1.$$

Two disjoint intervals in a finite system

- The configuration for disjoint intervals is shown as below.



- For this case, the path integral $\mathcal{Z}_n$ is given by

$$\mathcal{Z}_n = \mathcal{Z}^{-n} \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 K(0, \phi_1, l_{B_1})^n K(\phi_1, \phi_2, l_A)^n K(\phi_2, 0, l_{B_2})^n.$$

- The corresponding entanglement entropy is given by

$$S(A) = \frac{1}{2} \log \frac{l_{B_1}^{z-1} l_A^{z-1} l_{B_2}^{z-1}}{\left(l_{B_1}^{z-1} + l_A^{z-1} + l_{B_2}^{z-1}\right) \epsilon^{2(z-1)}} + c_1.$$

- Note that when $l_A \to \epsilon$, this configuration reduces to a union of two adjacent intervals.

Mutual information

- The mutual information between two subsystems A and$B$ is defined by

$$I(A:B) = S(A) + S(B) - S(A \cup B).$$

- For the case of a single interval and two adjacent intervals, we have $I(A:B) = 2S(A)$.
- In case of two disjoint intervals, the mutual information between two subsystem B_1 and B_2 whose density matrix is given by $\rho_{B_1 \cup B_2} = \mathrm{Tr}_A \, \rho$.
- To determine the entanglement entropy from this state, we consider the moment of the reduced matrix $\rho_{B_1} = \mathrm{Tr}_{B_2} \, \rho_{B_1 \cup B_2}$.
- We have

$$\mathcal{Z}_n = \mathrm{Tr}(\rho_{B_1})^n = \mathcal{Z}^{-n} \int \mathcal{D}\phi \; K(0, \phi; l_{B_1})^n K(\phi, 0; l_{AB_2})^n.$$

- Here $K(\phi, 0; l_{AB_2})$ is the propagator

$$K(\phi, 0; l_{AB_2}) = \int \mathcal{D}\phi' K(\phi, \phi'; l_A) K(\phi', 0; l_{B_2}).$$

- Using this, we obtain the entanglement entropy for the subsystem B_1 as

$$S(B_1) = \frac{1}{2} \log \frac{l_{B_1}^{z-1}(l_A^{z-1} + l_{B_2}^{z-1})}{\left(l_{B_1}^{z-1} + l_A^{z-1} + l_{B_2}^{z-1}\right) \epsilon^{z-1}} + \frac{c_1}{2}.$$

- Note that instead of $K(\phi, 0; l_A + l_{B_2})$ which might be expected naively, it is the propagator $K(\phi, 0; l_{AB_2})$ which appears in the trace $\mathcal{Z}_n$.
- On using these results, the mutual information between B_1 and B_2 is given by

$$I(B_1 : B_2) = \frac{1}{2} \log \frac{\left(l_{B_1}^{z-1} + l_A^{z-1}\right)\left(l_{B_2}^{z-1} + l_A^{z-1}\right)}{l_A^{z-1}\left(l_{B_1}^{z-1} + l_A^{z-1} + l_{B_2}^{z-1}\right)} = \frac{1}{2} \log \frac{1}{1 - \tilde{\eta}} \ ,$$

- Here the cross-ratio $\tilde{\eta}(z)$ is given by

$$\tilde{\eta}(z) := \frac{\left(l_{B_1} l_{B_2}\right)^{z-1}}{\left(l_{B_1}^{z-1} + l_A^{z-1}\right)\left(l_{B_2}^{z-1} + l_A^{z-1}\right)}.$$

- When $l_A \ll l_{B_i}$, then $\tilde{\eta}(z) \to 1$. It happens same for $l_A < l_{B_i}$ and $z \gg 1$.
- The mutual information maximizes in these cases which is expected since the interactions of the theory have increasing range while the length l_A is small compared to the rest subsystems sizes.

Reflected entropy and Markov gap

- Purification: It is always possible to purify the mixed state ρ_{AB} by embedding the system $A \cup B$ in a larger tripartite system $A \cup B \cup C$ which is in a pure state.
- Consider a bipartite system in an arbitrary mixed state ρ_{AB} on a finite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.
- A canonical purification is defined by a a pure state $|\sqrt{\rho_{AB}}\rangle$ in a doubled Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{A^*} \otimes \mathcal{H}_{B^*}$ where A^* and B^* are dual copies of A and B respectively such that

$$\rho_{AB} = \mathrm{Tr}_{A^* B^*}(|\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}|).$$

- The reflected entropy is defined as the von Neumann entropy of the reduced density matrix $\rho_{AA^*} = \mathrm{Tr}_{BB^*}(|\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}|)$ [Dutta and Faulkner '19]

$$S^R(A:B) = -\,\mathrm{Tr}_{AA^*}\left(\rho_{AA^*}\log\rho_{AA^*}\right).$$

- The replica method for reflected entropy consists two replica indices m and n, where the former is related to the purification $|\rho_{AB}^{m/2}\rangle$ of ρ_{AB}^{m} for positive even integer m, and the latter denotes the usual Rényi index.
- The reflected density matrix is then given by

$$\rho_{AA^*}^{(m)} = \mathrm{Tr}_{BB^*}\left(|\rho_{AB}^{m/2}\rangle\langle\rho_{AB}^{m/2}|\right).$$

- The (m,n)-Rényi reflected entropy is defined as,

$$S_{m,n}^{R}(A:B) = \frac{1}{1-n}\log\left[\frac{\mathrm{Tr}\left(\rho_{AA^*}^{(m)}\right)^{n}}{\left(\mathrm{Tr}\rho_{AA^*}^{(m)}\right)^{n}}\right].$$

- Markov gap has been proposed as a measure of tripartite entanglement [Zou et al. '21].
- It is defined as the difference between the reflected entropy $S_R(A:B)$ and the mutual information $I(A:B)$

$$h(A:B) = S_R(A:B) - I(A:B).$$

- As the reflected entropy is lower bounded by the mutual information, Markov gap is non-negative.

Two adjacent intervals

- In this case, the trace $\mathcal{Z}_{m,n} = \text{Tr}\left(\rho_{AA^*}^{(m)}\right)^n$ is given by

$$\mathcal{Z}_{m,n} = \mathcal{Z}^{-(m-2)n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi_1 d\phi_2 K(0,\phi_1;l_A)^n K(0,\phi_1;l_B)^n K(0,\phi_2;l_A)^n K(0,\phi_2;l_B)^n,$$

- On using the expression for the kernel, the reflected entropy is given by

$$S^R(A:B) = \log\left[\frac{(l_A l_B)^{z-1}}{(l_A^{z-1} + l_B^{z-1})\epsilon^{z-1}}\right] + c_1.$$

- The reflected entropy for adjacent interval is twice the entanglement entropy $S(A)$ which is expected since $A \cup B$ is in a pure state.
- The Markov gap is zero for this configuration since the two adjacent subsystems constituting the whole system is a pure state.
- As a result no tripartite entanglement should be detected from the study of this bipartite state.
- Markov gap being zero correctly serves as a consistency check of our results.

Two disjoint intervals

- For this configuration, the trace is given by

$$\mathcal{Z}_{m,n} = \mathcal{Z}^{-(m-2)n} \int d\phi_1 d\phi_2 \ldots . d\phi_{2n} K(0,\phi_1;l_{B_1})^m K(\phi_1,\phi_2;l_A)^{\frac{m}{2}} K(0,\phi_2;l_{B_2})^m K(\phi_2,\phi_3;l_A)^{\frac{m}{2}}$$
$$\times K(0,\phi_3;l_{B_1})^m \ldots .. K(\phi_{2n-1},\phi_{2n};l_A)^{\frac{m}{2}} K(0,\phi_{2n};l_{B_2})^m K(\phi_{2n},\phi_1;l_A)^{\frac{m}{2}} .$$

- Using the kernel, the Rényi reflected entropy is

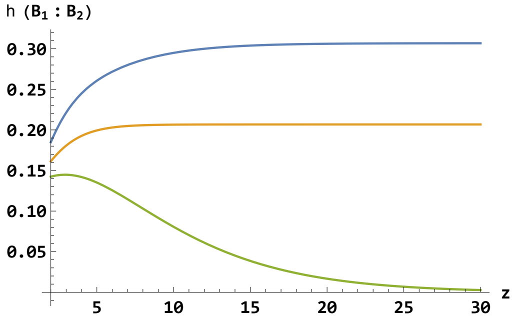
$$S^R_{m,n}(B_1 : B_2) = \frac{1}{n-1} \log \frac{(\sqrt{1-\tilde{\eta}}+1)^{2n} - \tilde{\eta}^n}{((\sqrt{1-\tilde{\eta}}+1)^2 - \tilde{\eta})^n}.$$

- The reflected entropy can be obtained by taking the replica limit as

$$S^R(B_1 : B_2) = \frac{1}{\sqrt{1-\tilde{\eta}}} \log\left(\frac{1+\sqrt{1-\tilde{\eta}}}{\sqrt{\tilde{\eta}}}\right) - \log\left(2\sqrt{\frac{1-\tilde{\eta}}{\tilde{\eta}}}\right).$$

- The Markov gap for the configuration of disjoint intervals is given by

$$h(B_1 : B_2) = \frac{1}{\sqrt{1-\tilde{\eta}}} \log\left(\frac{1+\sqrt{1-\tilde{\eta}}}{\sqrt{\tilde{\eta}}}\right) - \log\left(\frac{2(1-\tilde{\eta})}{\sqrt{\tilde{\eta}}}\right).$$



- For $l_A \leq \min\{l_{B_1}, l_{B_2}\}$, $h(B_1 : B_2)$ increases up to a constant value whereas for $l_A > \min\{l_{B_1}, l_{B_2}\}$, $h(B_1 : B_2)$ decays to zero.
- We observe that with increasing degrees of anisotropy of the Lifshitz field theory, the tripartite entanglement can be enhanced or completely destroyed depending on the sizes of the partitions.

Lifshitz Holography

- The standard form of the (2+1)-dimensions Lifshitz metric with one-direction anisotropy is given by

$$ds^2 = L^2 \left[-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx^2}{r^2} \right].$$

- The above metric is not Lorentz-invariant and supports non-relativistic Lifshitz scaling invariance given by

$$t \to \lambda^z t, \quad x \to \lambda x, \quad r \to \lambda r.$$

- This metric appears as solution of the equations of motion of the bulk action given by [Taylor '08]

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(R + \frac{z^2+1}{2L^2} - \frac{F^2}{4} - \frac{1}{2} M^2 A^2 \right).$$

- At constant time slice, the above metric is the same as that for the (2+1)-dimensional AdS. This would imply the same form of holographic entanglement entropy for space-like interval which is well known

$$S(A) = \frac{L}{2l_P} \log \frac{l}{\epsilon}.$$

- On matching it with the field theory results, we expect the following relation

$$L = (z-1) l_P.$$

- This relation is different from the well-known Brown-Henneaux relation for the CFT vacuum (at $z = 1$).

$$c = \frac{3L}{2 l_P}.$$

- Away from $z = 1$, holographic analysis inspired by cMERA gives [He, Magan and Vandoren '17]

$$S(A) = \frac{z}{3} \ln \frac{l}{z\epsilon}.$$

- This suggests that the bulk dual of the RK vacuum is different from the usual Fock vacuum for the Lifshitz theory.
- The z-dependence in L is consistent with the fact the RK vacuum respects a z-dependent Lifshitz symmetry.

Summary

- We have used fractional derivative to propose a definition of the massless Lifshitz theory with arbitrary dynamical exponent z.
- In (1+1)-dimensions, the massless Lifshitz theory admits a Lifshitz scaling invariant ground state having the form of RK vacuum.
- We showed that there is a 2d/1d correspondence between the (1+1)-dimensional Lifshitz field theory and a dual quantum mechanical system defined with a fractional derivative.
- We then computed various bipartite and tripartite entanglement measures in the Lifshitz field theory and determined their z-dependence respectively.
- Finally, we considered a gravity dual corresponding to the Lifshitz vacuum of the Lifshitz field theory.
- We showed that in order to reproduce the field theory result for the entanglement entropy, the previously considered Lifshitz bulk geometry has to be supplemented by a Lifshitz radius scale that is dependent on z.

THANK YOU!