Higher-form symmetry and eigenstate thermalization hypothesis

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at National Taiwan University

Based on

[O.F., Ryusuke Hamazaki, Phys. Rev. Lett. 131 (2023), arXiv:2305.04984]

[O.F., PTEP 2024, no.4, 043B03 (2024), arXiv:2310.11425]

Introduction

- **Thermalization (Relaxation to thermal equilibrium):**

  Transition from atypical states to typical states

  $\langle\psi|\mathcal{O}|\psi\rangle \neq \mathrm{tr}(\mathcal{O}\,\rho_{\mathrm{thermal}})$ $\langle\psi|\mathcal{O}|\psi\rangle = \mathrm{tr}(\mathcal{O}\,\rho_{\mathrm{thermal}})$

➡ Universal property in (quantum/classical) many-body systems

However, it is quite nontrivial:

- what kind of systems and which initial states relax to equilibrium,
- what statistical ensembles are realized, if the system thermalizes.

Thermalization process highly depends on datailed data in general.

Introduction

Motivation in this talk

Nontrivial thermalization process in quantum field theories

- How does higher-form symmetry affect thermalization?

  Consequence for real-time evolution

- What kind of observables distinguish nontrivial thermal equilibriums in the presence of non-local conserved quantities?

  It is well-known that local conserved quatities leads to nontrivial thermal ensembles.

Introduction

Thermalization in isloated quantum systems:

Quantum systems without diffusion exhibit unitary time evolutions.

$\underset{?}{\Longrightarrow}$ Non-thermal states remain non-thermal after time evolution?

- A way to understand this point: consider expectation values of operators

$$\lim_{t\to\infty} \overline{\langle \psi(t) | \mathcal{O} | \psi(t) \rangle} = \mathrm{tr}(\, \mathcal{O}\, \rho_{\mathrm{thermal}}\,)$$

Time average

Statistical average
at thermal equilibrium

Sufficient condition:

(strong) Eigenstate Thermalization Hypothesis (ETH)

[Deutsch, Phys. Rev. A 43(1991); Srednicki, Phys. Rev. E 50(1994)…]

⇐ *All eigenstates are thermal.*

Introduction

Our work

- If a $(d+1)$-dimensional QFT with a p-form symmetry satisfies some reasonable conditions(*), we **showed** (rather than expect): $(d$-$p)$-dimensional observables detect the ETH-violation.

  ➡ The system does not necessarily relax to the standard canonical ensemble.

- We numerically demonstrated the argument above in the case of the (2+1)-dimensional $\mathbb{Z}_2$ lattice gauge theory

  In particular, the resulting thermal equilibrium is described by a Generalized Gibbs ensemble(GGE) taking account of the $\mathbb{Z}_2$ 1-form symmetry.

Note: The (2+1)-dimensional $\mathbb{Z}_2$ gauge theory enjoys the electric $\mathbb{Z}_2$ 1-form symmetry, and satisfies the conditions(*).

Contents

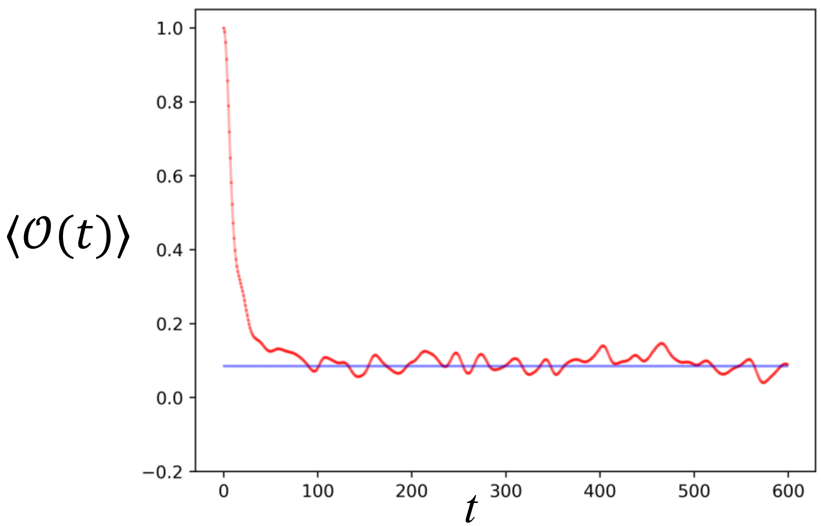
Eigenstate thermalization hypothesis

In general, the followings depend on the detail of the system and initial states:
①Does every initial state relax to some stationary state?

$$\delta\mathcal{O}^2 := \overline{\langle\mathcal{O}\rangle^2} - \left(\overline{\langle\mathcal{O}\rangle}\right)^2 \to 0 \;\text{ as }\; t \to \infty$$

②If so, is the long-time average described by a thermal ensemble?

$$\overline{\langle\mathcal{O}\rangle} \simeq \langle\mathcal{O}\rangle_{\mathrm{eq}}$$



① ⇐ off-diagonal ETH
② ⇐ diagonal ETH

* We use the terminology "ETH", referring to diagonal ETH in this talk.

Once you assume the ETH, thermalization occurs regardless of the initial conditions…

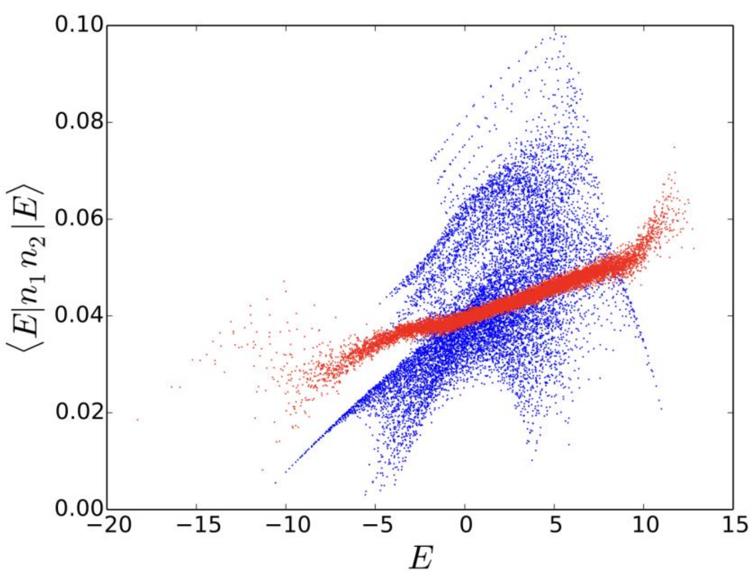
Eigenstate thermalization hypothesis

Let $|\psi(0)\rangle = \sum_\alpha c_\alpha\, |E_\alpha\rangle$ ($H|E_\alpha\rangle = E_\alpha|E_\alpha\rangle$: energy eigenstates),
(No degeneracy for the Hamiltonian is assumed.)

$$\overline{\langle \mathcal{O} \rangle} = \sum_\alpha |c_\alpha|^2 \langle E_\alpha|\mathcal{O}|E_\alpha\rangle$$

(Diagonal) ETH

$$\langle E_\alpha|\mathcal{O}|E_\alpha\rangle(E_\alpha) \simeq \mathrm{tr}(\,\mathcal{O}\,\rho_{\text{micro canonical}}(E_\alpha))$$



[Deutsch, 1805.01616]

e.g.) Hard core boson:

blue: integrable ⇒ ETH ✗

red: non-integrable ⇒ ETH ✓

Higher-form symmetry

Topological operator

- p-form symmetry is characterized by

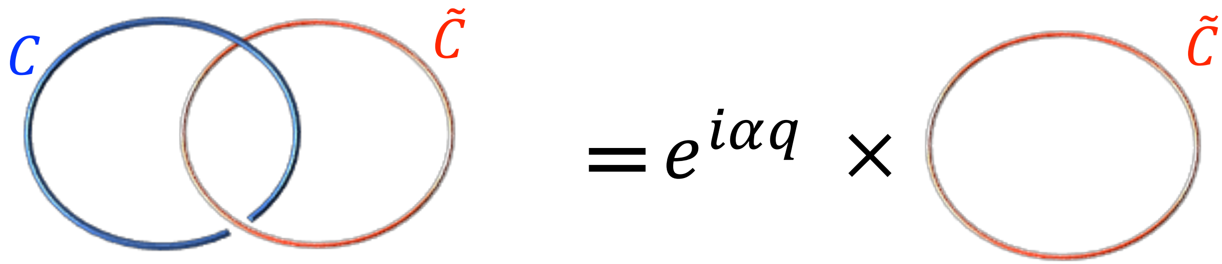
  a codimension-$(p+1)$ (unitary) symmetry operator.

  In $(d+1)$-dimensional QFTs,

$$\langle U_\alpha(C) W(\tilde{C}) \ldots \rangle = e^{i\alpha q} \langle W(\tilde{C}) \ldots \rangle$$

Symmetry operator: $(d-p)$-dimensional

Charged operator: p-dimensional



Correlation functions are invariant under the continuous deformation of C.

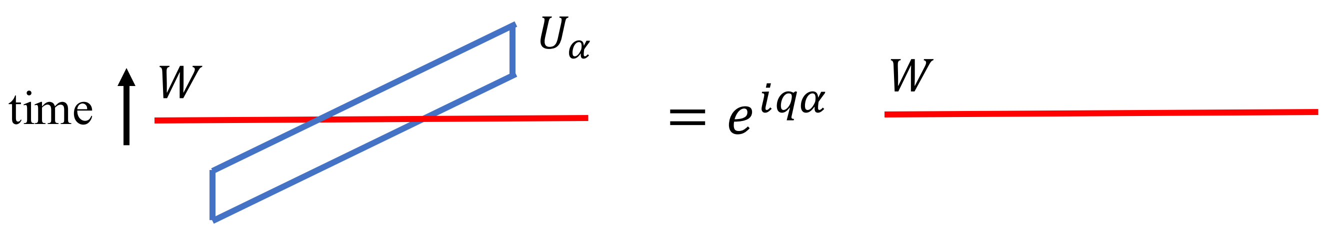
- Higher-form symmetry has a G group structure (G: abelian group)

$$U_\alpha(C) U_\beta(C) = U_{\alpha+\beta}(C)$$

Higher-form symmetry

Action on Hilbert space

- We consider actions of $U_\alpha(C)$ and $W(\tilde{C})$ on the Hilbert space (space-like symmetry [Gorantla-Lam-Seiberg-Shao, 2201.10589]):



$$U_\alpha^\dagger(C_x)W(C_y)U_\alpha(C_x) = e^{iq\alpha}W(C_y) \qquad C_x, C_y \subset \mathcal{M}\text{: space manifold}$$

Topological nature of U_α $\Rightarrow$ $[H, U_\alpha] = 0$

ETH breaking by higher-form symmetry

[O.F., Ryusuke Hamazaki, Phys. Rev. Lett. 131 (2023), arXiv:2305.04984]

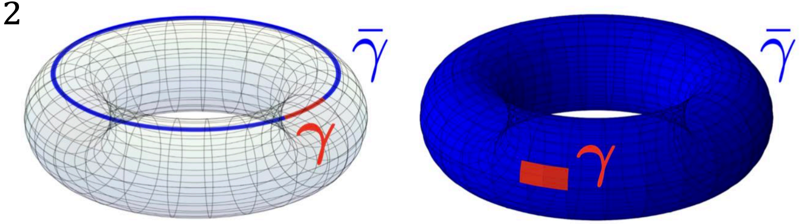
Setup

We consider a $(d+1)$-dimensional QFT on a spacetime manifold $\mathcal{M} \times \mathbb{R}$.
The Hamiltonian H is non-degenerate.
The system exhibits a p-form symmetry with the symmetry operator $U_\alpha(C)$.

Main claim

The operator $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ (or both) breaks the ETH if some reasonable assumptions(*) are satisfied.
$\gamma, \bar{\gamma}$: $(d-p)$-d manifold with boundary, s.t. $\gamma \cup \bar{\gamma} = C \subset \mathcal{M}$

e.g.) $\mathcal{M} = T^2$



$p = 1$ $p = 0$

The result implies that the operator $U(\gamma)/U(\bar{\gamma})$ may not relax to the standard canonical ensemble.

ETH breaking by higher-form symmetry

Main claim

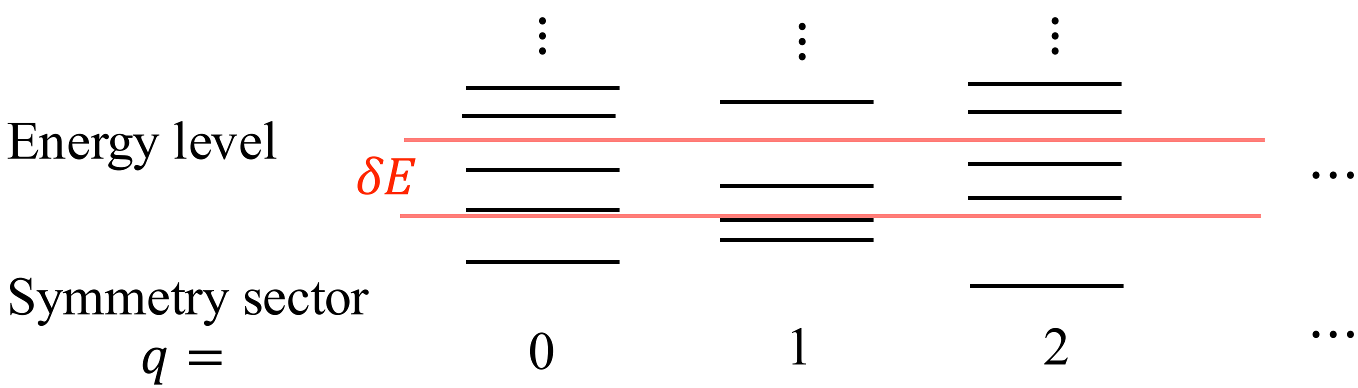
The operator $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ (or both) breaks the ETH if some reasonable assumptions(*) are satisfied. $(\gamma \cup \bar{\gamma} = \tilde{C} \subset \mathcal{M})$

Assumptions:

i) The operator $U_\alpha(\tilde{C})$ can be decomposed as $U_\alpha(\tilde{C}) = U_\alpha(\gamma)U_\alpha(\bar{\gamma})$.

ii) There exist energy eigenstates $|E_n\rangle, |E_m\rangle$, with $E_n, E_m \in [E, E + \delta E]$,
s.t. $\langle E_n | U_\alpha(\tilde{C}) | E_n \rangle \neq \langle E_m | U_\alpha(\tilde{C}) | E_m \rangle$.

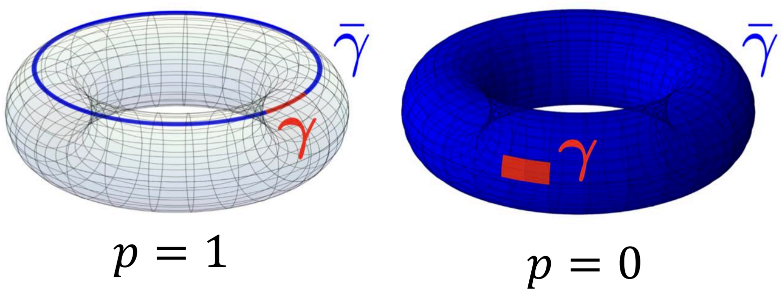
iii) The microcanonical average $\langle U_\alpha(\tilde{C}) \rangle_{\mathrm{mc}}^{\delta E} \neq 0$.



ETH breaking by higher-form symmetry

Comments

- The result is especially nontrivial for $p \geq 1$, since the ETH-violation is not due to the smallness of "bath."



- Generalization is possible:
  $A(g)U_\alpha(\gamma)$ or $A(g)^\dagger U_\alpha(\bar{\gamma})$ violates the ETH.

  $A(g)$: operator defined on a region $g(\subset \mathcal{M}\ )$ with $g \cap \bar{\gamma} = \phi$

  ➡ Many ETH-violating operators for a fixed γ.

ETH breaking in $\mathbb{Z}_2$ lattice gauge theory

Model (2+1)-dimensional $\mathbb{Z}_2$ lattice gauge theory ← With $\mathbb{Z}_2$ electric 1-form symmetry

The space manifold $\mathcal{M}$ is 2-torus T^2

Hamiltonian [Fradkin-Susskind, Phys. Rev. D, 17(1978)]

$$H_{\mathbb{Z}_2} = -\sum_{p\in \text{plaquette}} \lambda_p B_p - \sum_{b\in\text{link}} \lambda_b \sigma_b^x$$

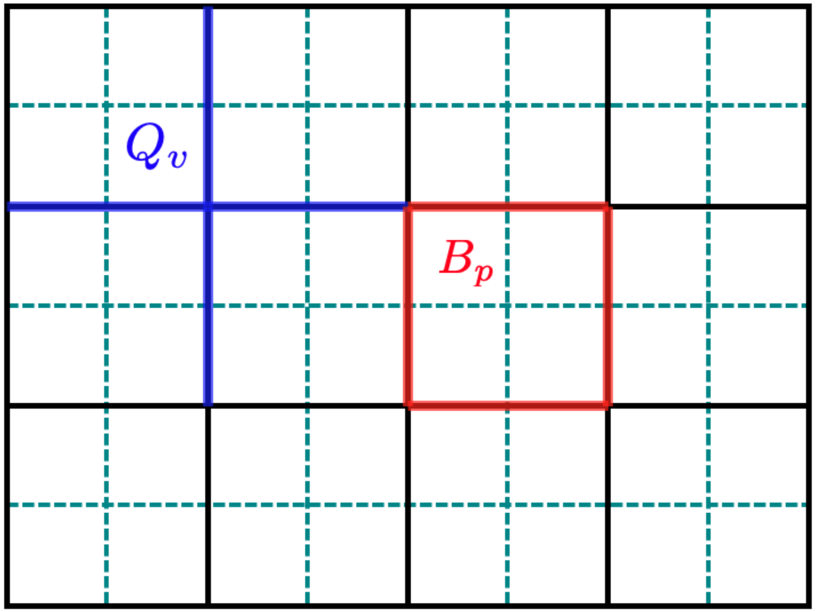
$$B_p := \prod_{b\in\text{plaquette } p} \sigma_b^z$$

Physical Hilbert space

Gauss law constraint:

$$Q_v := \prod_{\substack{b:\ \text{spatial link} \\ b\ni v}} \sigma_b^x = 1$$

Constraint on physical Hilbert space



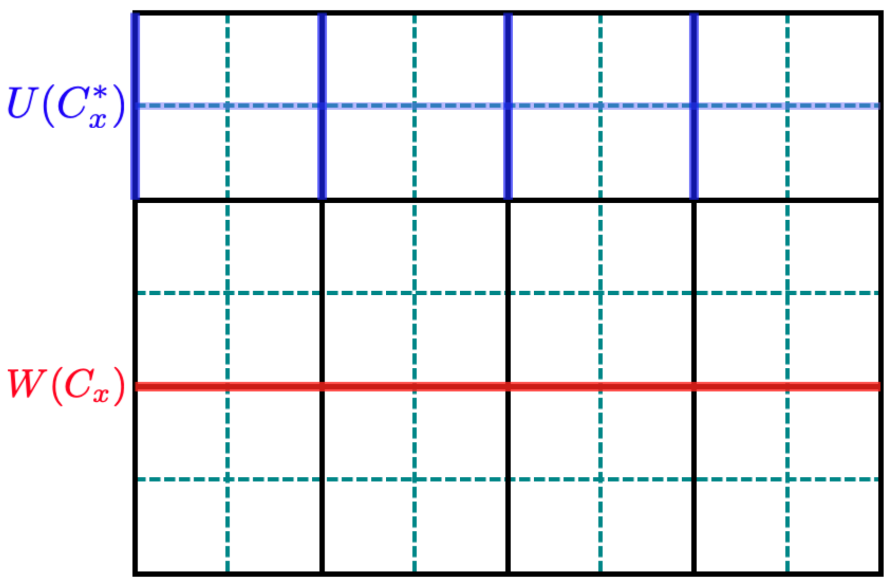
ETH breaking in $\mathbb{Z}_2$ lattice gauge theory

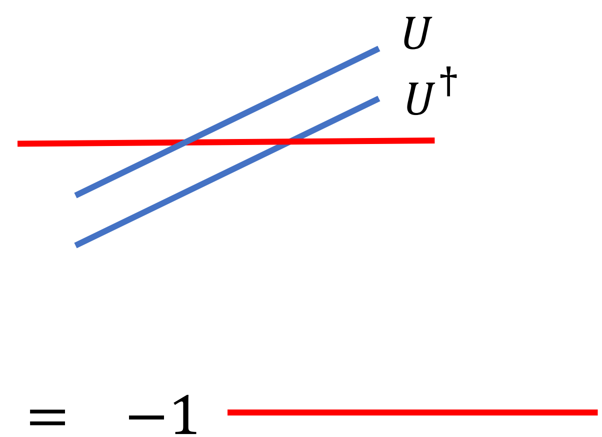
- Wilson line ⇐ Charged object under the $\mathbb{Z}_2$ 1-form symmetry

$$W(C) := \prod_{b \in C} \sigma_b^z \qquad \Rightarrow W(C)^2 = 1$$

- Symmetry operator ('t Hooft operator)

$$U(C^*) := \prod_{b^* \in C^*} \sigma_{b^*}^x \qquad \Rightarrow U(C^*)^2 = 1$$





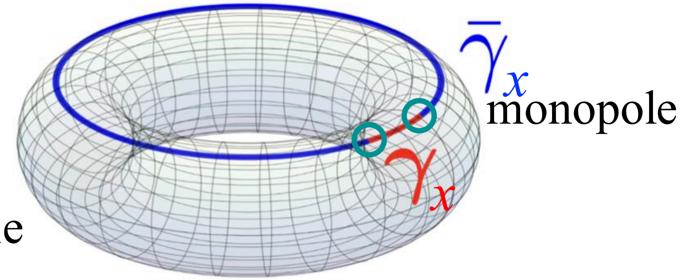
ETH breaking in $\mathbb{Z}_2$ lattice gauge theory

<u>Symmetry operator can have endpoints</u>

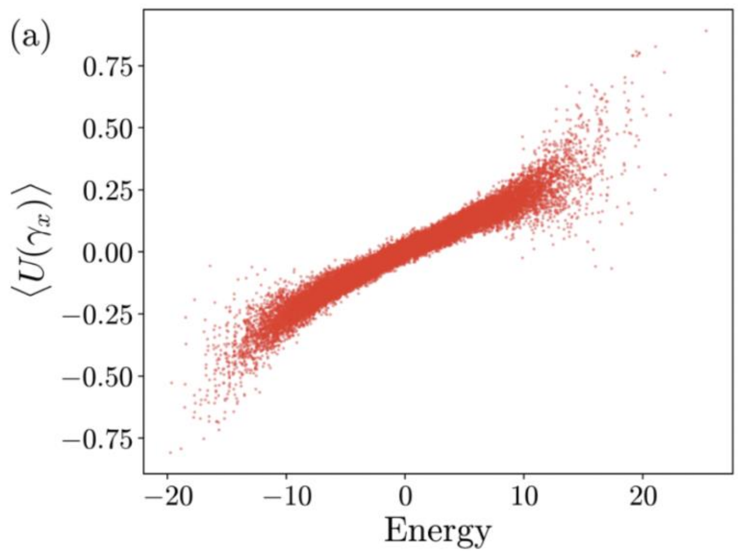
$$U(\overline{\gamma_x}) \coloneqq \prod_{b^* \in \overline{\gamma_x}} \sigma^x_{b^*}$$

$\overline{\gamma_x}$: open curve

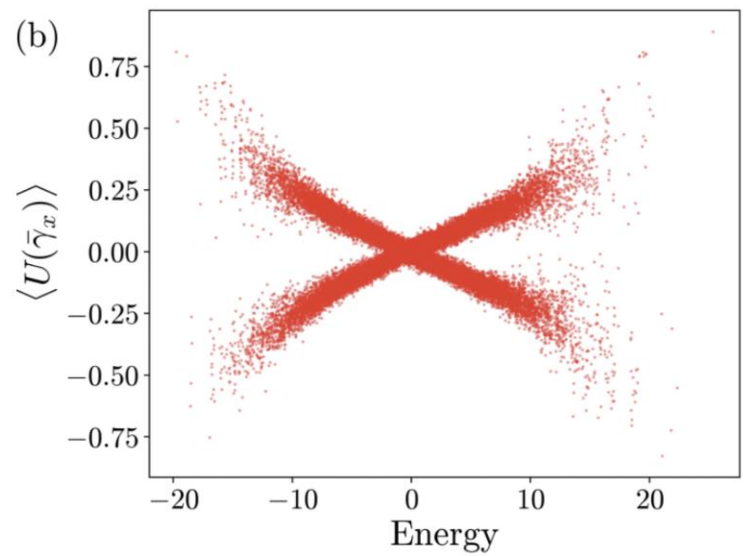
$\gamma_x \cup \overline{\gamma_x} = C_x$: x-cycle



Numrtical calculation for 5×3 lattice



$U(\gamma)$ (local operator)

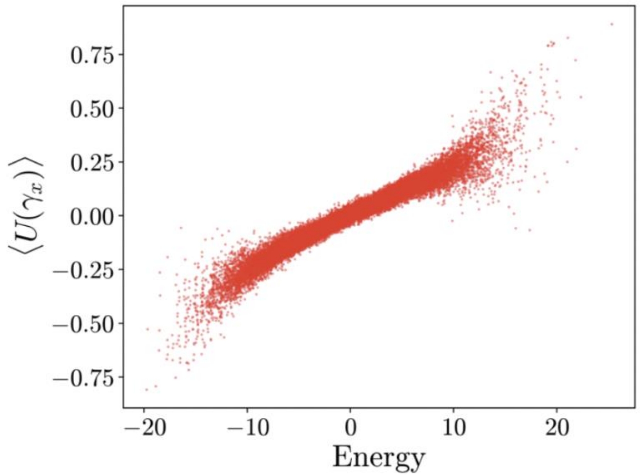


$U(\bar{\gamma})$ (non-local operator)

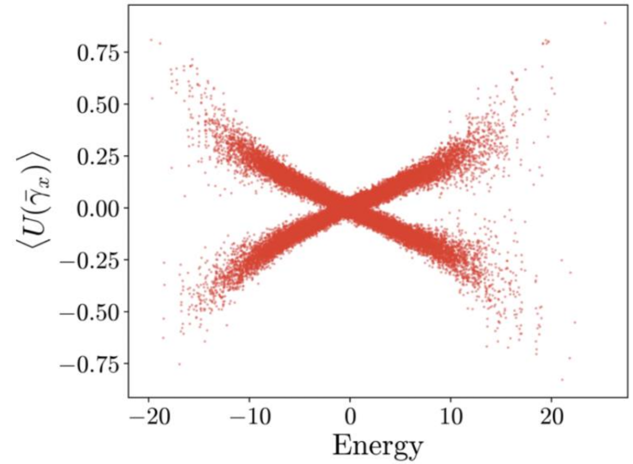
The ETH for the 1-dimensional operator is broken.

ETH breaking in $\mathbb{Z}_2$ lattice gauge theory

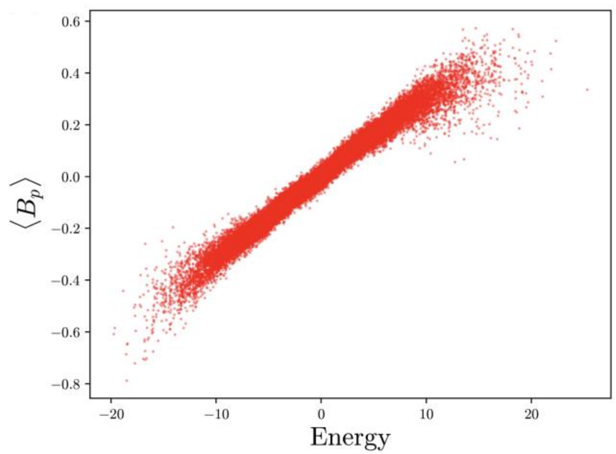
Other operators



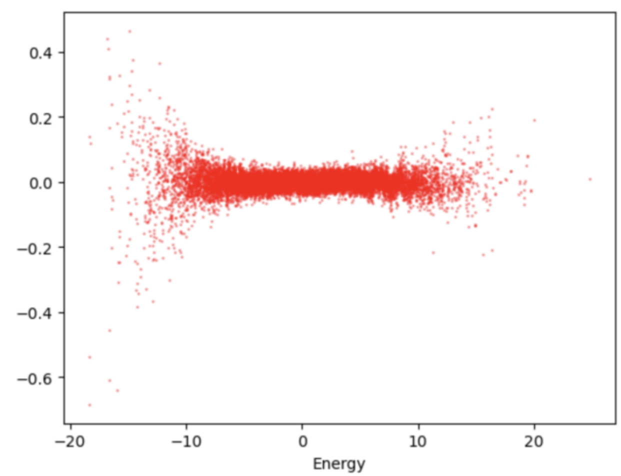
$U(\gamma)$ (local operator)



$U(\bar{\gamma})$ (non-local operator)



Plaquette operator B_p



Double Wilson lines $W(C_x^1)W(C_x^2)$

Generalized Gibbs ensemble for higher-form symmetry

- What is Generalized Gibbs ensemble(GGE)?

GGE is originally introduced for integrable spin chain.

[Pozsgay, 1304.5374; Ilievski et al., 1507.02993;…]

$$\langle \mathcal{O} \rangle_{\mathrm{GGE}} = \mathrm{tr}\, \mathcal{O}\, \rho(\beta, \{\lambda_i\})$$

$$\rho(\beta, \{\lambda_i\}) \coloneqq e^{-\beta H - \sum_i \lambda_i Q_i} / Z(\beta, \{\lambda_i\}),$$
$$Z(\beta, \{\lambda_i\}) \coloneqq \mathrm{tr}\, e^{-\beta H - \sum_i \lambda_i Q_i}$$

Q_i: (quasi-)local conserved quantity
λ_i: "chemical potential"

For integrable systems, GGE is realized as a thermal equilibrium.

Generalized Gibbs ensemble for higher-form symmetry

- The GGE for $\mathbb{Z}_2$ gauge theory

$$\langle \mathcal{O} \rangle_{\mathrm{GGE}} = \mathrm{tr}\, \mathcal{O}\, \rho(\beta, \{\lambda_i\})_{\mathbb{Z}_2}$$

$$\rho(\beta, \{\lambda_i\})_{\mathbb{Z}_2} := e^{-\beta H_{\mathbb{Z}_2} - \lambda_1 P_x - \lambda_2 P_x H_{\mathbb{Z}_2}} / Z(\beta, \{\lambda_i\})_{\mathbb{Z}_2},$$
$$Z(\beta, \{\lambda_i\})_{\mathbb{Z}_2} := \mathrm{tr}\, e^{-\beta H_{\mathbb{Z}_2} - \lambda_1 P_x - \lambda_2 P_x H_{\mathbb{Z}_2}}$$

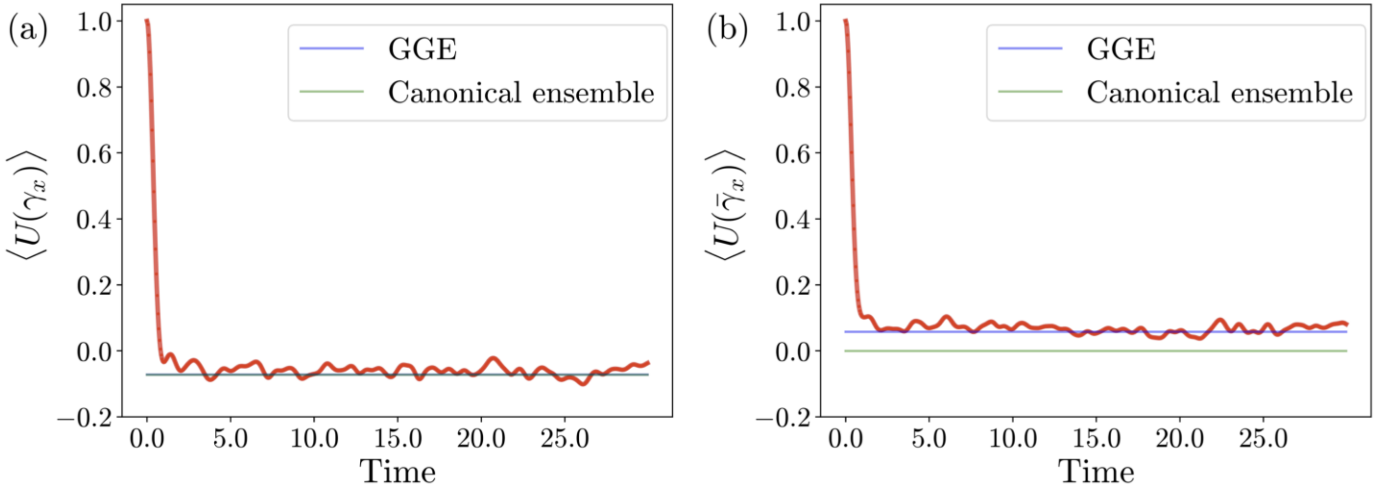
$P_x = \frac{1 - U(C_x)}{2}$: projection to the sector $U(C_x) = 1$

To specify $\langle P_x H_{\mathbb{Z}_2} P_x \rangle$, $\langle (1 - P_x) H_{\mathbb{Z}_2} (1 - P_x) \rangle$, and $\langle P_x \rangle$ for a given state, three chemical potentials are needed.

Generalized Gibbs ensemble for higher-form symmetry

- Numerical analysis of time-evolution

The thermal ensemble for $\mathbb{Z}_2$ gauge theory is given by the GGE.



Initial state: eigenstate of $U(\gamma_x)$ / $U(\overline{\gamma_x}) = 1$,with $E \in [-5.0, -3.0]$.

Summary and outlook

We analytically **showed**

- Higher-form symmetry affects thermalization.
- p-form symmetry leads to $(d-p)$-dimensional ETH-violating operators.

In the case of $\mathbb{Z}_2$ lattice gauge theory

- The ETH for dipole-exciting operator $U(\bar{\gamma})$ (1-dimensional) is broken.
- Thermal equilibrium is given by the GGE taking account of $\mathbb{Z}_2$ 1-form symmtry rather than the canonical ensemble.

Outlook

- Effect on entanglement spectrum
- Implication to finite-temperature phase transition
- Demonstration for other QFTs

  $\mathbb{Z}_N$ gauge theory, U(1) gauge theory, SU(N) gauge theory…

  superconducter, super fluid…

Etc…

Backups

Introduction

Higher-form symmetry

A generalized concept of conventional global symmetry:

In $(d+1)$-dimensional QFTs, p-form symmetry is characterized by $(d-p)$-dimensional topological operator.

[Kapustin-Seiberg-Gaiotto-Willett, 1412.5148…]

It has similar properties as conventional symmetry
e.g.)

- Selection rule
- SSB ➡ Nambu-Goldstone's theorem
- Anomaly ➡ Identification of low-energy effective theories

➡ Identification of phase structure

Eigenstate thermalization hypothesis

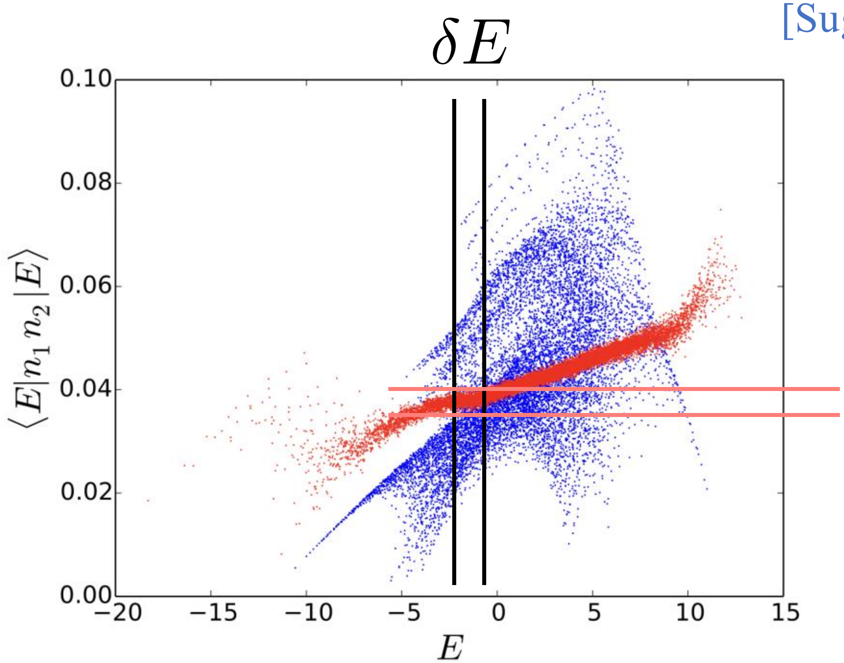
Finite-size scaling analysis:

To evaluate the validity of the ETH quantitatively, we consider

$$\Delta_\infty(\mathcal{O}) \coloneqq \max_{n;\, E_n \in [E, \delta E]} \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\mathrm{mc}}^{\delta E}(E_n) \right|$$

$\Delta_\infty(\mathcal{O}) \to 0$ as $V \to \infty$ $\Longleftrightarrow$ The ETH for $\mathcal{O}$ is satisfied

[Sugimoto-Hamazaki-Ueda, 2005.06379]



$\sim \Delta_\infty(\mathcal{O} = n_1 n_2) \to 0$

[Deutsch, 1805.01616]

Eigenstate thermalization hypothesis

ETH and local/non-local conserved quantities

Conserved quantity Q: $[H, Q] = 0$, (H: Hamiltonian)

- Local conserved quantity: sum of local operators

$$Q = \sum_{s:\text{site}} \prod_{|\tilde{s}-s|\leq k} \mathcal{O}_{\tilde{s}} \quad :k\text{-local} \quad (k \text{ is independent of the system size.})$$

$$\simeq \int dx\, \mathfrak{O}(x) \quad \text{in continuum}$$

- Non-local conserved quantity: otherwise

$$Q \simeq \int dx_1 dx_2 \ldots \mathfrak{O}(x_1, x_2, \ldots)$$

In the system size $V \to \infty$ limit, local conserved quanties typically behaves as extensive variable: $Q = O(V)$.

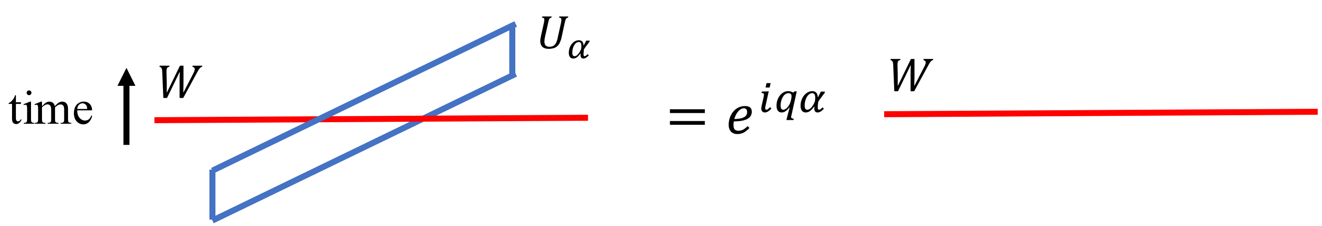
➡ Violation of the ETH

Treatment of Non-local conserved quantities is quite subtle, i.e., it requires going beyond conventional statistical mechanics.

Higher-form symmetry

Action on Hilbert space

- We consider actions of $U_\alpha(C)$ and $W(\tilde{C})$ on the Hilbert space (space-like symmetry [Goranta-Lam-Seiberg-Shao, 2201.10589]):



$$U_\alpha^\dagger(C_x)W(C_y)U_\alpha(C_x) = e^{iq\alpha}W(C_y) \qquad C_x, C_y \subset \mathcal{M}: \text{space manifold}$$

Topological nature of U_α $\Rightarrow$ $[H, U_\alpha] = 0$

These topological operators typically lead to the following:

Continuous symmetry ➡ Local conserved quantity

Discrete symmetry ➡ Non-local conserved quantity

ETH breaking by higher-form symmetry

Main claim

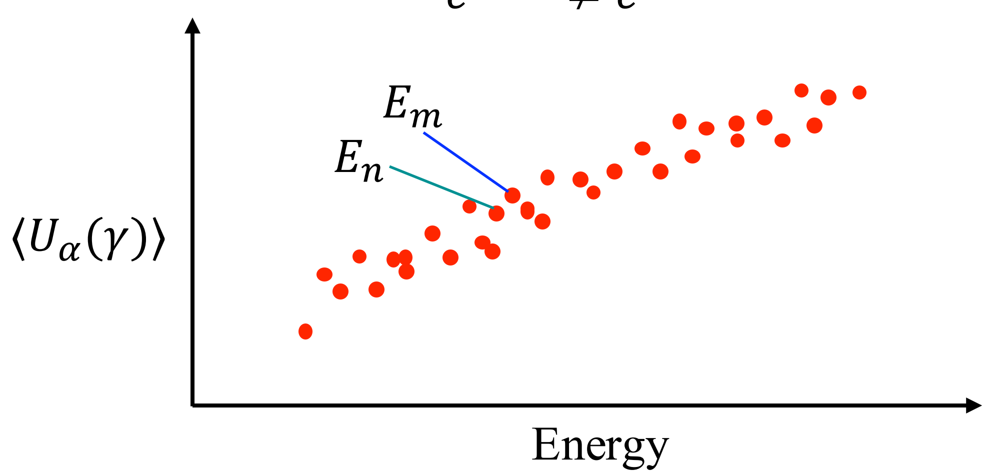
The operator $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ (or both) breaks the ETH if some reasonable assumptions(*) are satisfied. $(\gamma \cup \bar{\gamma} = \tilde{C} \subset \mathcal{M})$

Sketch of the proof

- If the ETH for $U_\alpha(\gamma)$ is violated, the claim holds.
- We consider the case $\langle E_n | U_\alpha(\gamma) | E_n \rangle \simeq \langle E_m | U_\alpha(\gamma) | E_m \rangle \simeq \langle U_\alpha(\gamma) \rangle_{\mathrm{mc}}^{\delta E}$.
- By ii), there exist eigenstates $|E_n\rangle$ and $|E_m\rangle$ of H, s.t., $E_n, E_m \in [E, E + \delta E]$ and

$$U_\alpha(\tilde{C})|E_n\rangle = e^{i\alpha q_n}|E_n\rangle, \qquad U_\alpha(\tilde{C})|E_m\rangle = e^{i\alpha q_m}|E_m\rangle$$

$$e^{i\alpha q_n} \neq e^{i\alpha q_m}$$



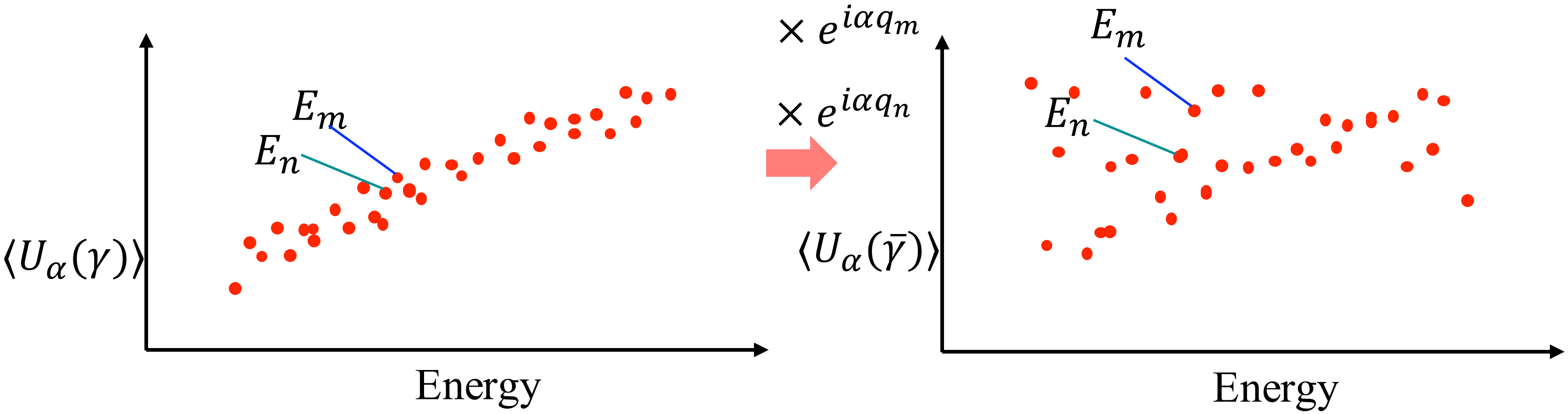
ETH breaking by higher-form symmetry

Main claim

The operator $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ (or both) breaks the ETH if some reasonable assumptions(*) are satisfied. $(\gamma \cup \bar{\gamma} = \tilde{C} \subset \mathcal{M})$

Sketch of the proof

- Noting $U_\alpha^{-1}(\bar{\gamma}) = U_\alpha(\gamma) U_\alpha^{-1}(\tilde{C})$



The ETH for $U_\alpha(\bar{\gamma})$ is broken.

ETH breaking by higher-form symmetry

Comments

- For charged operators, the ETH always holds.

$$\langle W(C_1)\rangle = e^{i\alpha q\cdot \mathrm{link}(C_1,C_2)}\langle U_\alpha^\dagger(C_2)W(C_1)U_\alpha(C_2)\rangle = e^{i\alpha q\cdot \mathrm{link}(C_1,C_2)}\langle W(C_1)\rangle$$

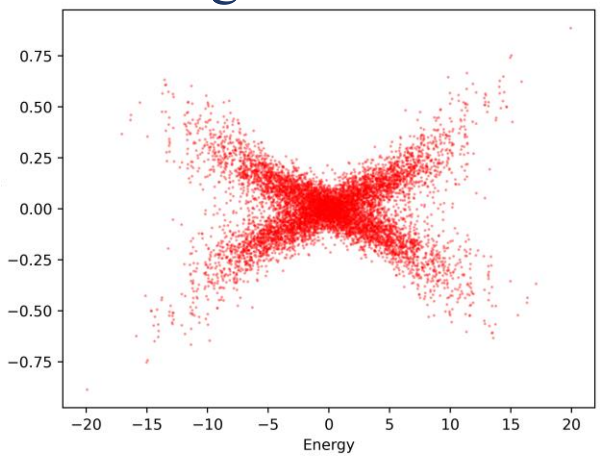
$$e^{i\alpha q\cdot \mathrm{link}(C_1,C_2)} \neq 1 \;\;\Rightarrow\;\; \langle W(C_1)\rangle = 0$$

- Since the discrete symmetry typically leads to non-local conserved quantities, it means the ETH-violation caused by non-local conserved quantities.

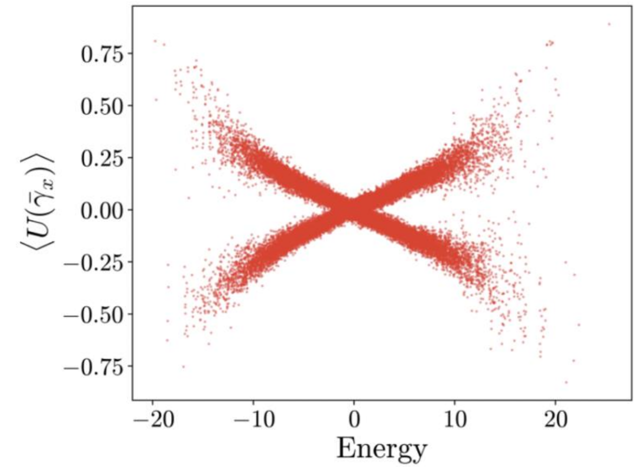
➡ Beyond the conventional statistical mechanics

ETH breaking in $\mathbb{Z}_2$ lattice gauge theory

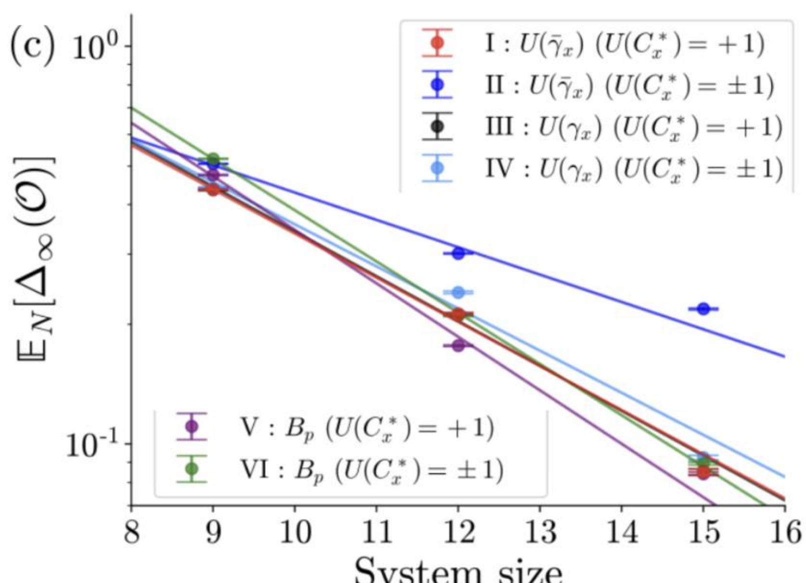
Finite-size scaling



4×3 lattice



5×3 lattice



Blue line:
Deviation $\Delta_\infty\big(U(\overline{\gamma_x})\big)$
for total symmetry sector

$$\Delta_\infty(\mathcal{O}) := \max_{n;\, E_n \in [E, \delta E]} \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle_{\mathrm{mc}}^{\delta E}(E_n) \right|$$

Generalized Gibbs ensemble for higher-form symmetry

- GGE for $\mathbb{Z}_2$ gauge theory

$$\langle \mathcal{O} \rangle_{\mathrm{GGE}} = \mathrm{tr}\, \mathcal{O}\, \rho(\beta, \{\lambda_i\})_{\mathbb{Z}_2}$$

$$\rho(\beta, \{\lambda_i\})_{\mathbb{Z}_2} := e^{-\beta H_{\mathbb{Z}_2} - \lambda_1 P_x - \lambda_2 P_x H_{\mathbb{Z}_2}} / Z(\beta, \{\lambda_i\})_{\mathbb{Z}_2},$$
$$Z(\beta, \{\lambda_i\})_{\mathbb{Z}_2} := \mathrm{tr}\, e^{-\beta H_{\mathbb{Z}_2} - \lambda_1 P_x - \lambda_2 P_x H_{\mathbb{Z}_2}}$$

$P_x = \frac{1 - U(C_x)}{2}$: projection to the sector $U(C_x) = 1$

To specify $\langle P_x H_{\mathbb{Z}_2} P_x \rangle$, $\langle (1 - P_x) H_{\mathbb{Z}_2} (1 - P_x) \rangle$, and $\langle P_x \rangle$ for a given state, three chemical potentials are needed.

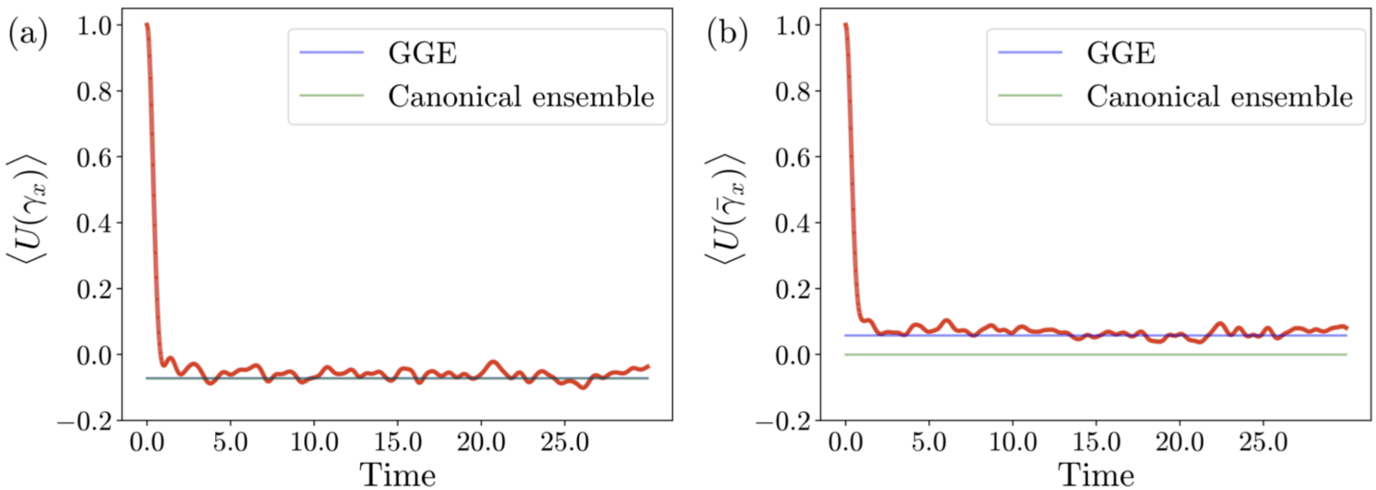
Redefining the chemical potentials,

$$\Rightarrow \quad \rho(\beta, \lambda, \mu)_{\mathbb{Z}_2} = e^{-\beta H_{\mathbb{Z}_2} - \lambda U(C_x) - \mu U(C_x) H_{\mathbb{Z}_2}} / Z(\beta, \lambda, \mu)_{\mathbb{Z}_2},$$
$$Z(\beta, \lambda, \mu)_{\mathbb{Z}_2} = \mathrm{tr}\, e^{-\beta H_{\mathbb{Z}_2} - \lambda U(C_x) - \mu U(C_x) H_{\mathbb{Z}_2}}$$

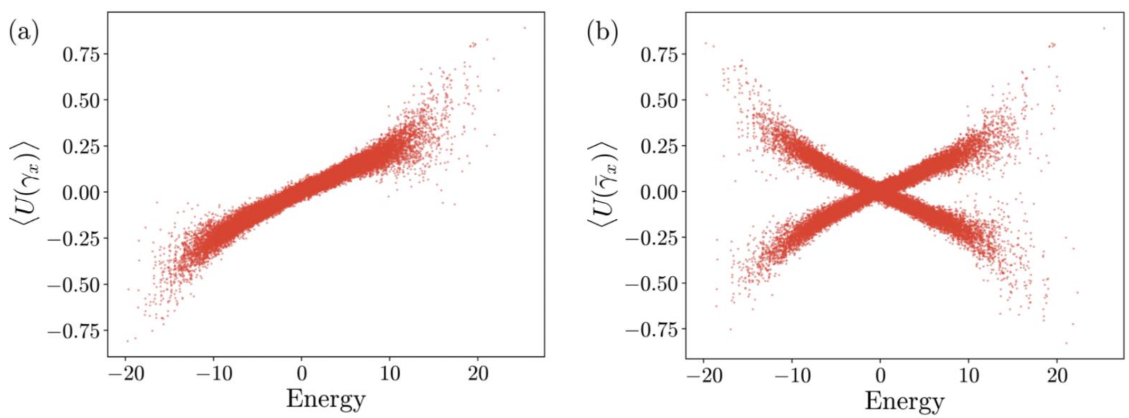
Generalized Gibbs ensemble for higher-form symmetry

- Numerical analysis of time-evolution

The thermal ensemble for $\mathbb{Z}_2$ gauge theory is given by GGE.



Initial state: eigenstate of $\langle U(\gamma_x) \rangle$ / $\langle U(\overline{\gamma_x}) \rangle = 1$,with $E \in [-5.0, -3.0]$.



Generalized Gibbs ensemble for higher-form symmetry

- GGE for general discrete (abelian) group G

$$\langle \mathcal{O} \rangle_{\mathrm{GGE}} = \mathrm{tr}\, \mathcal{O}\, \rho(\beta, \{\lambda_i\}, \{\mu_i\})$$

$$\rho(\beta, \{\lambda_i\}, \{\mu_i\}) := e^{-\beta H - \sum_{i=1}^{N-1} \lambda_i P_i - \sum_{i=1}^{N-1} \mu_i P_i H} / Z(\beta, \{\lambda_i\}, \{\mu_i\}),$$

$$Z(\beta, \{\lambda_i\}, \{\mu_i\}) := \mathrm{tr}\, e^{-\beta H - \sum_{i=1}^{N-1} \lambda_i P_i - \sum_{i=1}^{N-1} \mu_i P_i H}$$

P_i: projection to each symmetry sector

N: number of symmetry sector $|H_{d-p}(\mathcal{M}, G)|$

Assume that the canonical ensemble is realized for each symmetry sector.

➡

At least for the operator $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$, the thermal ensemble is given by this GGE for the total symmetry sector.