A Proposal for Quantum Gravity and Quantum Mechanics of Black Hole

NCTS-iTHEMS Joint Workshop on Matters to Spacetime: Symmetries and Geometry

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based on 2406.01466, 2406.12704
and also 2307.06164, 2307.06176, 2209.03610, in collab. with Rong-Xin Miao

Outline

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Problems with Black Hole

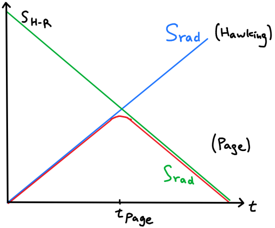
1. Bekenstein-Hawking entropy:

$$S_{\mathrm{B-H}} = \frac{A}{4G\hbar}.$$

   What is the interpretation? entanglement? microstates?
   Why area dependence?

2. Black hole interior: existence of singularity
   What replace the continuum spacetime description inside the black hole?

3. Information problem: non-unitarity evolution appears in 2 ways.
   - Hawking radiation is a "mixed thermal state", while the initial state can be a pure state.
   - Hawking curve behaviour of S_{rad} violates unitarity.



What should be quantum degrees of freedom of black hole so that one can evolve them quantum mechanically?

Top-Down Approach to QG

- Obviously, all these problems are due to a lack of understanding of the d.o.f. of quantum gravity.
- The best thing to do is to start from a theory X of quantum gravity, construct the classical black hole as a solution of it, then the properties of quantum black hole should follow.
  $X =$ e.g. string theory, branes, AdS/CFT, matrix model etc

This top-down approach has been quite successful in string theory.

- Duality in QFT e.g. Montone-Olive S-duality
- Non-commutative geometry can be derived from first principle
- For BH, some of the progress made are:
    1. *Bekerstein-Hawking entropy*: Microstate counting (Strominger-Vafa), fuzzy ball proposal (Mathur), and progress in index computation.
    2. *Page curve:* AdS/CFT Island proposal (Engelhardt, Wall, Penington, Almheiri, Marlof, Maxfield, Mahajan, Maldacena, Zhao, ...)

Yet we still do not know: e.g.

- it hasn't been possible to study an ordinary Schwarzschild BH
- what fundamental degrees of freedom of BH are being counted?
- how are they related to the Hawking radiation?

Bottom up approach to QG

- Historically, QM was built bottom-up. Various models (e.g. Planck model of blackbody radiation, Bohr model of atom) helped in identifying fundamental features of QM, which eventually got built in to the formulation of QM.
- Current mainstream construction of string theory: AdS/CFT, M(atrix) theory etc have captured truth about quantum gravity, but not complete.
- We initiated an bottom-up approach to QG by taking 2 steps:
  1. Fermi model of BH (2209.03610, 2307.06164, 2307.06176)
  2. QM of spacetime (2406.01466, 2406.12704)

Outline

- General relativity is a classical theory of the continuum spacetime $\mathcal{M}$. The spacetime is endrosed with a metric $g_{\mu\nu}$ which became dynamical. The theory is then shown to contain gravity.
- Quantum gravity should be a theory of quantum spacetime $\hat{\mathcal{M}}$. One may model model quantum space by operator coordinates (noncommutative geometry). (Synder; Yang (1947)).
- We propose to construct a theory of quantum gravity as a quantum mechanics of noncommutating coordinates.

Fermi model of BH

- In 2307.06164, a phenomenological model of quantum BH was constructed. as a matrix model of fermionic degrees of freedom coupled with bosonic degrees of freedom. Schematically,

$$L = i\psi^\dagger\dot{\psi} + h(X)\psi^\dagger\psi - V(X), \quad \text{e.g. BFSS}$$

  where $h(X)$ denotes some Yukawa coupling and $V(X)$ the self-interaction. Presumably spacetime/gravity will emerge in some "effective" way.
- Instead of committing to a specific Hamiltonian, e.g. BFSS matrix model, we derived necessary properties of the matrix model so that the box of fermi system resembles a BH.

- We found that
  - if the model admits a constant density of energy eigenstates and
  - if the Fermi sea of the system is filled up to a Fermi energy level that is inversely proportional to the system size,

  then
  - the Schwarzschild radius of BH is reproduced for the system.
  - Moreover, a counting of microstates gives precisely the Bekenstein-Hawking entropy.
- The model works also for charged BH, BH with a cosmological constant. And also for Kerr BH and higher dimensional BH (unpublished).

Theory of Quantum Space

- We consider the large N quantum mechanics of quantum space

$$L = \mathrm{tr}\left(\frac{\dot{X}^{a2}}{2M_P} + \frac{M_P}{N^2}[X^a, X^b]^2 + \frac{4M_P}{N^2}X^{a2} + i\dot{\psi}^\dagger\psi + \frac{M_P}{N^2}\psi^\dagger\sigma^a X^a\psi - M_P\mathbf{1}\right),$$

where $a = 1, 2, 3$, $X^a_{mn}, \psi_{mn}, \psi^\dagger_{mn}$ in the adjoint of $SU(N)$, and

$$M_P := \sqrt{2/\pi G} \quad = \text{Planck Mass}.$$

- Remarks:
  1. The necessity of fermionic geometry in quantum gravity arises from the previous analysis of Fermi model of BH
  2. SUSY is needed for consistency in popular top down approaches of QG, however, it is not required for the well-definedness of QM. As a result, our model of boson/fermi d.o.f. is OK. Also, a Higgs like potential $-\phi^2 + \phi^4$ is allowed.
  3. The theory is invariant under global $SU(N)$ transformation

$$X^a \to UX^aU^\dagger, \quad \psi \to U\psi U^\dagger, \quad \psi^\dagger \to U\psi^\dagger U^\dagger,$$

which replaces diffeomorphism.

The Proposal

We propose that the large N quantum mechanics gives a fundamental formulation of quantum gravity in 3 space dimensions.

- We will show that the theory admits quantum space solutions that describe quantum Schwarzschild BH and quantum Kerr BH.
- We will also show that the interaction energy between two fuzzy spheres has the correct dependence: GM_1M_2 as required by gravity. We conjectured that Newton gravity will be reproduced in the large distance limit.

Outline

Quantum Schwarzschild as Fuzzy Sphere

- The classical EOM for bosonic matrix configuration is given by

$$-\frac{1}{M_P}\ddot{X}^a + \frac{4M_P}{N^2}\left([X^b,[X^a,X^b]] + 2X^a\right) = 0.$$

  For static configuration, this becomes $[[X^a,X^b],X^b] = 2X^a$.

- This can be solved by the spin $j = (N-1)/2$ reps. of $SU(2)$:

$$[X^a,X^b] = i\epsilon^{abc}X^c.$$

  Due to the Casimir $\sum_a X^{a2} = \frac{N^2-1}{4}\mathbf{1}$, the config is a fuzzy sphere.

- Or, introduce dimensional coords $Y^a = 2l_P X^a$, we get a fuzzy sphere of radius R,

$$[Y^a,Y^b] = \frac{2iR}{\sqrt{N^2-1}}\epsilon_{abc}Y^c, \quad \sum_a Y^{a2} = R^2\mathbf{1},$$

  where $R^2 = (N^2-1)l_P^2$ and $l_P = 2/\pi M_P$.

Energy

- Over the fuzzy sphere, the Hamiltonian of the system reads

$$H = H_B + H_F, \quad H_B = \frac{NM_P}{2}(1 + \frac{1}{N^2}), \quad H_F = -\frac{M_P}{N^2}\psi^\dagger \sigma^a X^a \psi$$

- H_F describes a collection of fermionic oscillators with equal frequency since $K := \sigma^a X^a$ satisfies $K^2 + K - \frac{N^2-1}{4}\mathbf{1} = 0$ and so it has eigenvalues $(N-1)/2$ or $-(N+1)/2$. In large N, K has the eigendecomposition

$$K_{(m\alpha)(n\beta)} = \frac{N}{2}\sum_{p=1}^{N}\left(\mathcal{U}^p_{m\alpha}\mathcal{U}^{p\dagger}_{n\beta} - \mathcal{V}^p_{m\alpha}\mathcal{V}^{p\dagger}_{n\beta}\right).$$

- Introduce the $2N^2$ oscillators $\xi^p_k := \mathcal{U}^{p\dagger}_{n\beta}\psi_{nk\beta}, \chi^{p\dagger}_k := \mathcal{V}^{p\dagger}_{n\beta}\psi_{nk\beta}$, then

$$H_F = \frac{M_P}{2N}\left(\sum_{p,k=1}^{N}\xi^{p\dagger}_k\xi^p_k + \chi^{p\dagger}_k\chi^p_k - N^2\right).$$

- Energy of the fermi system is

$$H_F = \frac{M_P}{2N} n, \quad n := N^2 - r - s,$$

where $n = -N^2, \cdots, 0, \cdots, N^2$ specifies the energy level within the Fermi sea. The lowest level (highest weight) of energy $-NM_P/2$ $(+NM_P/2)$ corresponds to a completely filled (empty) Fermi sea.

- Let us consider the $n = 0$ energy state which corresponds to a half-filled Fermi sea. We have the total energy $(R = Nl_P)$,

$$E = \frac{NM_P}{2} = \frac{R}{2G}.$$

- This is precisely the Schwarzschild mass-radius relation if equivalence principle holds

$$E = M \quad \text{(internal = grav energy)}$$

Entropy

- Let us consider the microstates counting. The level n eigenvalue has a degeneracy of

$$\Omega_n = \binom{2N^2}{N^2 - n}.$$

- For the $n = 0$ energy state, we have $\Omega_0 = 2^{2N^2}$ in large N. These microstates give rises to the entropy $S = \log_2 \Omega_0$:

$$S = 2N^2 = \frac{A}{4G},$$

  i.e. precisely the Bekenstein-Hawking entropy!

- For the first time, we have a fundamental theory where both the geometry and the Berkenstein-Hawking entropy of an ordinary non-SUSY BH can be accounted for consistently.
- We thus propose that Schwarzschild black hole in GR is described by a fuzzy sphere quantum geometry with a half-filled Fermi sea in QG.

Remarks

- 't Hooft (85): 't Hooft has proposed that BH entropy is an entanglement entropy. In our construction, the BH entropy is a coarse grained entropy rather than an entanglement entropy.
- Bekenstein entropy bound (81):

$$S \leq 2\pi RE$$

  is satisfied for $n \geq 0$. Negative n states (more than half filled) are probbaly excluded because of stability reason.
- Holography(94): Our description of BH is naturally holographic: (1) The BH is described by a 2-dimensional quantum geometry. (2) If we divide the fuzzy sphere into N^2 cells, each cell of a Planck size area $\Delta A = 4\pi l_P^2$, then there is precisely one pair of fermionic oscillators ξ_a, χ_a $(a = 1, \cdots N^2)$ in each cell of the fuzzy sphere to describe the quantum fluctuations over the fuzzy sphere.

Outline

Kerr metric in General Relativity

- In the Boyer-Lindquist coordinates, the Kerr metric of a BH of mass M and angular momentum J reads

$$ds^2 = -\left(1 - \frac{2M}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dtd\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,$$

$$\rho^2 := r^2 + a^2\cos^2\theta, \Delta := r^2 - 2Mr + a^2, \Sigma := (r^2 + a^2)^2 - a^2\Delta\sin^2\theta.$$

- In the asymptotic region $r \to \infty$, we have

$$ds^2 = -\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right] dt^2 - \left[\frac{4Ma\sin^2\theta}{r} + O\left(\frac{1}{r^3}\right)\right] d\phi dt + \left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right]\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

This shows that M is the mass and $J = aM$ is the angular momentum of the black hole spacetime.

- For $M = 0$, we obtain

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2.$$

  This is Minkowski metric written in the "oblate spherical" coordinates, which is related to the Cartesian coordinates as

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi,\ y = \sqrt{r^2 + a^2}\sin\theta\sin\phi,\ z = r\cos\theta.$$

- For $a \neq 0$, the oblate spherical coordinates (r, θ, ϕ) is different from the usual polar coordinates. e.g. $r = \mathit{constant}$ describes an ellipsoid.

- The Kerr BH has a horizon at $r_+ = M + \sqrt{M^2 - a^2}$. Viewed in the Cartesian coordinates, the horizon is an elliptical surface

$$\frac{x^2 + y^2}{r_+^2 + a^2} + \frac{z^2}{r_+^2} = 1.$$

- The area of the horizon is $A = 4\pi(r_+^2 + a^2)$ and give rises to the Bekenstein-Hawking entropy

$$S = \frac{A}{4G} = \frac{\pi(r_+^2 + a^2)}{G}.$$

- The mass of the Kerr black hole written in terms of r_+ is

$$M = \frac{r_+^2 + a^2}{2Gr_+}.$$

We will show below a noncommutative geomeytry solution to the matrix QM that precisely reproduces the shape, mass and entropy of the Kerr BH.

Rotating Fuzzy Ellipsoid Solution

- The axial symmetry of the Kerr BH suggests to consider a rotating solution in the QM. Introduce new basis (X^+, X^-, X^3), $X^\pm := \frac{1}{\sqrt{2}}(X^1 \pm iX^2)$ and consider the ansatz

$$X^\pm(t) = e^{\pm i\omega t} X^\pm(0), \quad X^3 \text{ independent of } t$$

The equation of motion becomes

$$[X^+, [X^+, X^-]] + [X^3, [X^+, X^3]] + 2c^2 X^+ = 0,$$

$$[X^-, [X^3, X^+]] + [X^+, [X^3, X^-]] + 2X^3 = 0$$

where $c^2 := 1 + N^2\omega^2/(8M_P^2)$.

- This has the soln

$$X^\pm = e^{\pm i\omega t} T^\pm, \quad X^3 = c_3 T^3, \quad c_3 = \sqrt{1 + \frac{N^2\omega^2}{4M_P^2}},$$

or, in terms of dimensionful coord

$$Y^{1,2} = Nl_P \ \hat{T}^{1,2}, \quad Y^3 = Nl_P \cos\beta \ \hat{T}^3$$

where $\hat{T}^a := \frac{2}{N} T^a$ and $\cos\beta := 1/c_3$.

- Identify $\hat{T}^a$ with the directional cosines and R and a as

$$R = Nl_P \cos\beta, \quad a = Nl_P \sin\beta,$$

we have

$$\frac{Y_1^2 + Y_2^2}{R^2 + a^2} + \frac{Y_3^2}{R^2} = 1,$$

which is precisely the shape of the horizon of the Kerr BH.

Entropy

- Over the fuzzy ellipsoid geometry, the bosonic Hamiltonian is

$$H_B = \frac{NM_P}{2} + \frac{N^3\omega^2}{12M_P}.$$

- The fermionic Hamiltonian $H_F = -\frac{M_P}{N^2}\psi^\dagger \sigma^a X^a \psi = H_F^0 + h_F$,

$$H_F^0 = -\frac{M_P}{N^2}\psi^\dagger \sigma^a T^a(t)\psi, \quad h_F = -(c_3 - 1)\frac{M_P}{N^2}\psi^\dagger \sigma^3 T^3 \psi$$

  where $T^\pm(t) = e^{\pm i\omega t}T^\pm$, $T^3(t) = T^3$. H_F^0 is an isotropic part, h_F is of order a^2/R^2 and can be considered a perturbation.

- Consider the $n = 0$ level, this is a degenerate level with $\Omega_0 = 2^{2N^2}$ states. These microstates give rise to the entropy $S = \log_2 \Omega_0$:

$$S = 2N^2 = \frac{\pi(R^2 + a^2)}{G},$$

  which is precisely the Bekenstein-Hawking entropy of a Kerr BH if we identify R with r_+ of the outer horizon radius of the Kerr BH.

Energy

- Rather than computing the corrections for individual states of the level $n = 0$, it is more meaningful to compute the ensemble average of the corrections

$$\overline{h_F} := \frac{1}{\Omega_0} \sum_p \lambda_p$$

for the set of microstates. We find $\overline{h_F} = -\frac{c_3 - 1}{6} N M_P$ and

$$E = \frac{R^2 + a^2}{2GR}\left(1 + \frac{1}{6}\sin^4\beta + \cdots\right), \quad \sin\beta = \frac{a}{\sqrt{R^2 + a^2}}.$$

- The quantum energy agrees precisely with that of general relativity, with a correction of the order $O(a^4/R^4)$.
- This result relies on an exact cancellation between the contributions of the bosonic and fermionic Hamiltonian at order $O(a^2/R^2)$. This is a nontrivial check of our proposal.

Outline

- Let us consider block diagonal configuration

$$X^a = \left(\begin{array}{c|c} X_1^a & 0 \\ \hline 0 & X_2^a \end{array} \right),$$

Define COM coordinates for each block $x_{\rm i}^a := \frac{1}{N_{\rm i}} {\rm tr} X_{\rm i}^a$. We can interpret $x_{\rm i}^a$ as the location of the BH with respect to the COM of the system (since $N_1 x_1^a + N_2 x_2^a = 0$ and BH masses $M_{\rm i} = \frac{N_{\rm i} M_P}{2}$.).

- The energy of the block config can be computed. We obtain

$$E = M_1 + M_2 + V_{int}$$

where the interaction energy is a function of $\Delta x := \sqrt{(x_1^a - x_2^a)^2}$.

- In terms of the dimensionful distance $r := \Delta x l_P$, we have

$$V_{int} = \text{const} - \frac{G M_1 M_2}{r} g(r)$$

where $g(r) = -4r/R + \cdots$ for small $r/R \ll 1$. Note that the structure of Newtonian gravity is reproduced.

Remarks:

- Note that V_{int} is finite at $r = 0$. No singularity!
- Newtonian limit is for $r/R \gg 1$. It is important to devise method to reliably compute the potential $V(r)$ between the fuzzy spheres in order to confirm that gravity does emerge from our proposed theory of quantized spacetime. large N resummation? contributions from off-diagonal blocks?

Outline

- We have made a proposal of quantum gravity as a large N quantum mechanics of non-abelian bosonic and fermionic coordinates with a Higgs like potential.
- The QM admits solutions with quantum fuzzy geometries and half filled fermi sea. These solutions reproduce the properties of quantum Schwarzschild BH and quantum Kerr BH. A number of crucial properties of black hole and quantum gravity that people have conjectured before appear naturally in our theory. e.g. noncommutative geometry, holography, membrane paradigm, etc.

To do:

1. Cosmological solution? Dark energy? Inflation?
2. It is important to understand how Einstein gravity and geodesic equation arise in the GR limit (probe analysis).
3. We have now a theory where one can resolve other puzzling properties of BH in GR+QM: e.g. Hawking radiation? Page curve? information loss?
4. Note that unlike BH in GR, the quantum BH in our theory is described by a quantum noncommutative geometry and is non-singular. It is interesting to understanding how a singularity would arise in the GR limit.
5. Is it possible to derive AdS/CFT? holography?