Proving SSB

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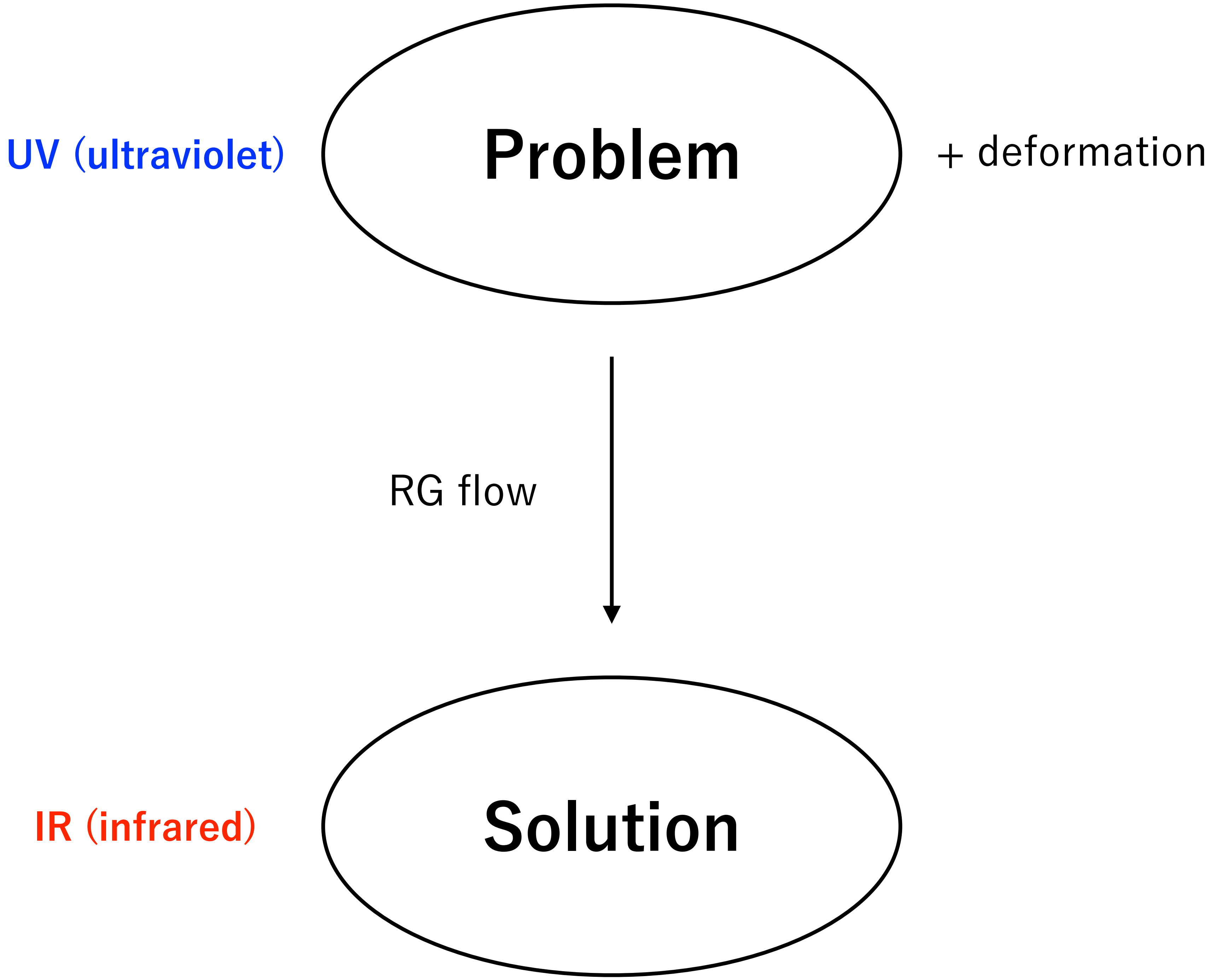
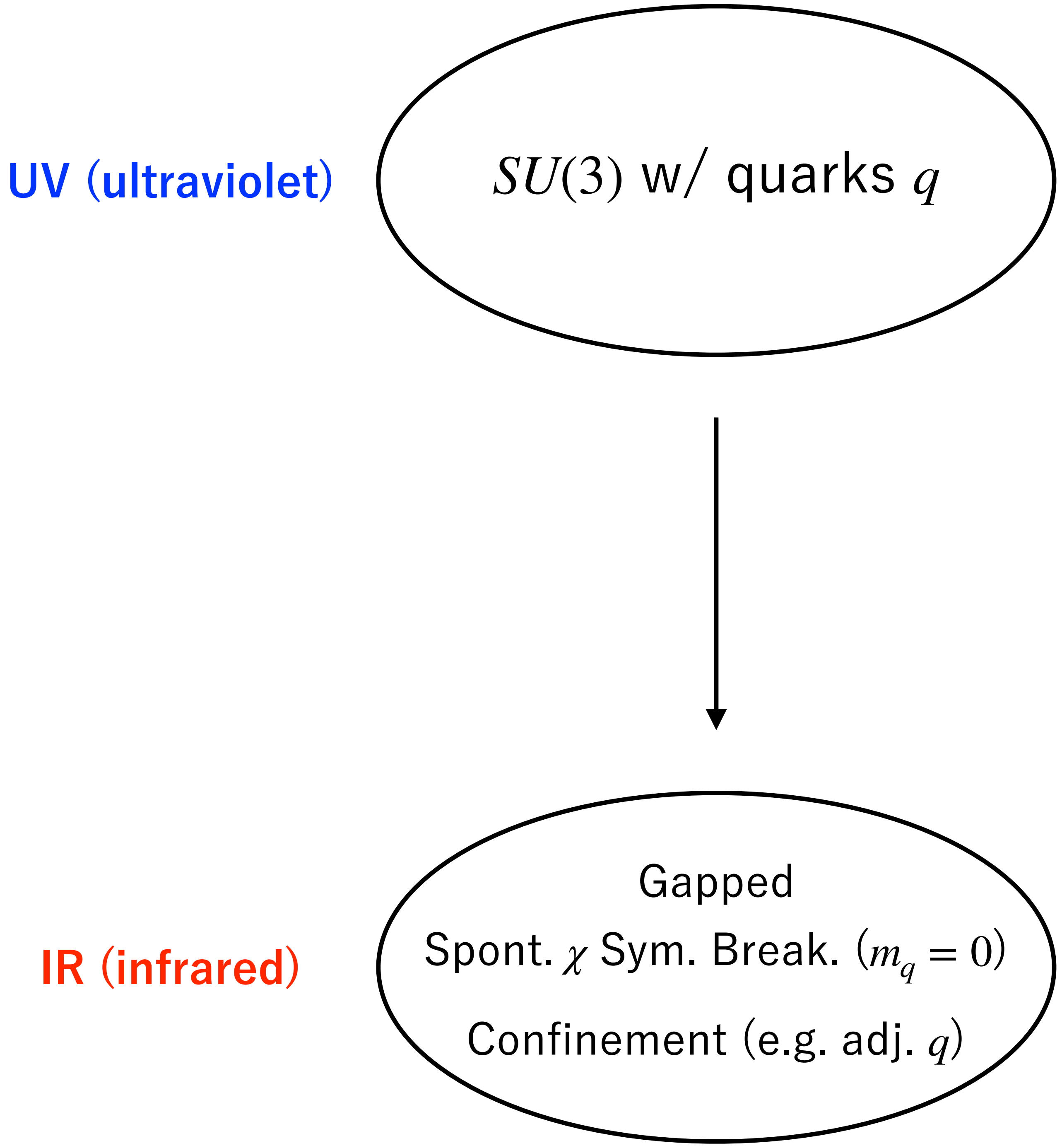
Based on
2312.13353 [math.QA] (KK)

Renormalization Group (RG) flow



Example: Quantum ChromoDynamics



Possible answers

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved		
Spont. broken		

Example: QCD

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved	Confinement	
Spont. broken	SχSB	

This talk

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved		
Spont. broken		

Main result (abstract)

We **prove** certain **SSB**.

Main result (concrete)

Theorem. [2312.13353 (KK)]

A symmetry C is spontaneously broken if C is an MFC w/ rank(C) > 1.

Content

1. **Modular fusion category (MFC)**

2. **Proof**

3. **Examples**

Content

1. **Modular fusion category (MFC)**
2. **Proof**
3. **Examples**

Modular fusion category

Generalized symmetry

= fusion category

Modular fusion category

fusion category

=analogue of **representation**

Modular fusion category

2-dim. irrep. of $SU(2)$ obeys

$$2 \otimes 2 = 1 \oplus 3.$$

Modular fusion category

Analogously, fusion category C has

- simple objects $c_i \in C$,
- fusion product $\otimes$,
- direct sum $\oplus$.

Modular fusion category: FPdim

Example: Ising fusion category (FC)

Ising FC$=\{1, \eta, N\}$

i.e., rank=3

It has fusion product

$$\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta \,.$$

Modular fusion category: FPdim

Example: Ising FC

The product $\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta$

is described by fusion matrices (in the basis $\{1, \eta, N\}$)

Definition. $i \otimes j = \oplus_{k \in C} (N_i)_{jk} k$

$$N_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad N_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Modular fusion category: FPdim

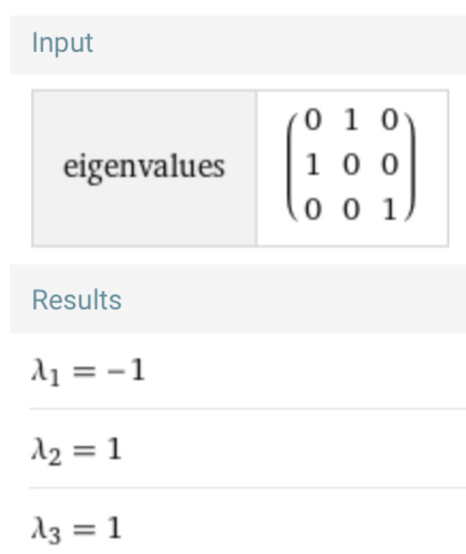
Frobenius-Perron dimension $\mathrm{FPdim}_C(j)$ of N_j (or j)

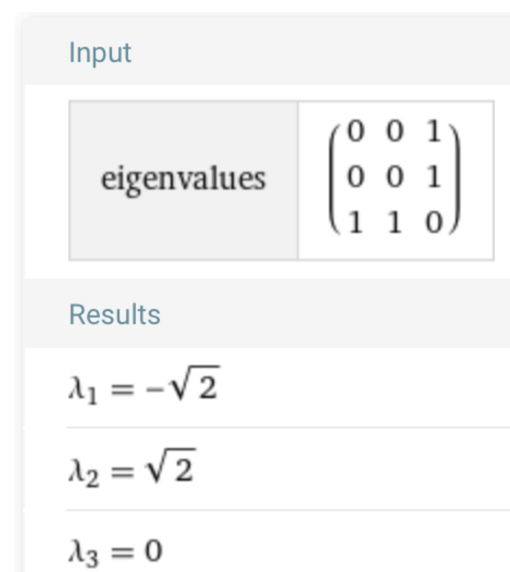
:=largest eigenvalue of N_j

Modular fusion category: FPdim

Example: Ising FC

$$N_1=\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix},\quad N_\eta=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad N_N=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$$ have





$\text{FPdim}_{\mathcal{C}}(1)=1=\text{FPdim}_{\mathcal{C}}(\eta)$, $\text{FPdim}_{\mathcal{C}}(N)=\sqrt{2}$.

Modular fusion category: FPdim

Frobenius-Perron dimension of an FC C

$$\mathrm{FPdim}(C) := \sum_{j \in C} \mathrm{FPdim}_C(j)^2.$$

Modular fusion category: FPdim

Frobenius-Perron dimension of an FC C

$$\mathrm{FPdim}(C) := \sum_j \mathrm{FPdim}_C(j)^2.$$

Example: Ising FC

$$\mathrm{FPdim}(\text{Ising FC}) = 1^2 + 1^2 + \sqrt{2}^2 = 4.$$

Modular fusion category

- Fusion category C may have **braiding** $c_{c_1,c_2} : c_1 \otimes c_2 \to c_2 \otimes c_1$.
  $(c_1, c_2 \in C)$

- **Braided fusion category** (BFC):=fusion category w/ braiding c.
  (w/ consistency conditions)

Modular fusion category

- A BFC C is called **modular** or **modular fusion category** (MFC)

  if $c_{c_2,c_1}c_{c_1,c_2} = id_{c_1 \otimes c_2}$ only for $c_1 = 1_C$ or $c_2 = 1_C$.

Modular fusion category

MFCs are **anomalous**.

Content

1. Modular fusion category (MFC)
2. Proof
3. Examples

Proof

We prove

(nontrivial) MFCs do **not** have

``trivial irrep.''

Proof

``**rep.** of fusion category C''

$=C$**-module category**

Module category: definition

Representation:

Let G be a group. A rep. of G is a homomorphism

$$\rho : G \to \text{actions on vector sp. } V.$$

Concretely, $\rho(g) \in GL(V)$ sends $v \in V$ to $\rho(g)v \in V$.

Module category: definition

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Module category:

Let C be a fusion category. A (left) module category M has

$$\triangleright : C \times M \to M.$$

Concretely, $c \in C$ sends $m \in M$ to $c \triangleright m \in M$.

Module category: definition

An important module category:

Any C has C itself as a C-module category defined by

$$\triangleright := \otimes : C \times C \to C.$$

It is called the regular module category $M = C$.

Module category

Group symmetry G	Categorical symmetry C
group element $g \in G$	object $c \in C$
group multiplication $g_1 g_2 \in G$	fusion product $c_1 \otimes c_2 \in C$
$\nexists$	direct sum $c_1 \oplus c_2 \in C$
representation space V	C-module category M
action $\rho(g)$ sends $v \in V$ to $\rho(g)v \in V$	action $c \in C$ sends $m \in M$ to $c \triangleright m \in M$

Proof: Definition of SSB

[2311.00746 (KK)]

<u>Definition</u>:

Let C be a fusion category and M a C-module category. An object $c \in C$ is called **spontaneously broken** if it acts faithfully (i.e., $\exists m \in M$ such that $c \triangleright m \neq m$). C is **spontaneously broken** if $\exists$ spontaneously broken $c \in C$, and **preserved** otherwise.

Proof: Lemma

Lemma: [2311.00746 (KK)]

Let M be an indecomposable C-module category.

$$\mathrm{rank}(M) > 1$$

$\Rightarrow C$ is spont. broken.

Proof: speciality of 2d

In 2d space(time), we have 1-to-1 correspondence

$$\{C\text{-symmetric gapped phases}\}$$

$$\cong \{C\text{-module categories } M\}.$$

[Thorngren-Wang ’19]
[Huang-Lin-Seifnashri ’21]

Proof: speciality of 2d

In 2d space(time), we have 1-to-1 correspondence

$$\{C\text{-symmetric gapped phases}\}$$

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[Thorngren-Wang '19]
[Huang-Lin-Seifnashri '21]

This especially implies

$$\mathrm{GSD} = \mathrm{rank}(M).$$

$\mathrm{rank}(M)$:= # of (simple) objects in M

Main result (precise)

Theorem. [2312.13353 (KK)]

Assume 2d C-symmetric gapped phases are described by indecomposable C-module category C_A w/ $A \in C$. The C symmetry is spontaneously broken if C is an MFC w/ rank(C) > 1.

Proof: SSB of MFC $\mathcal{C}$

Proof.

[2312.13353 (KK)]

- Assume the opposite $\text{rank}(\mathcal{C}_A)=1$.
- This means $\mathcal{C}_A=\mathcal{C}_{\text{rank }1}$, which has $\text{FPdim}(\mathcal{C}_{\text{rank }1})=1$.
- Formula $\text{FPdim}(\mathcal{C}_A)=\text{FPdim}(\mathcal{C})/\text{FPdim}_{\mathcal{C}}(A)$ demands

$$\text{FPdim}_{\mathcal{C}}(A) = \text{FPdim}(\mathcal{C}).$$

- By formula

$$\text{FPdim}(\mathcal{C}_A^0)=\text{FPdim}(\mathcal{C})/(\text{FPdim}_{\mathcal{C}}(A))^2=1/\text{FPdim}(\mathcal{C}),$$

but there does not exist such $\mathcal{C}_A^0$ for $\text{rank}(\mathcal{C})>1$. □

Proof: Remark

Not all $\mathcal{C}$-module categories are given by $\mathcal{C}_A$ w/ $A \in \mathcal{C}$.

But, a little more work can relax the technical assumption thanks to

[Davydov-Muger-Nikshych-Ostrik '10]

$$\{\text{indecomp. } \mathcal{C}\text{-module cats.}\}$$

$$\cong \{(\text{special})\ A \in \mathcal{C} \boxtimes \widetilde{\mathcal{C}}\}.$$

Content

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Examples

Example 1. Tricritical Ising model $+\phi_{1,3}$ [2311.00746 (KK)]

The deformation preserves Ising MFC

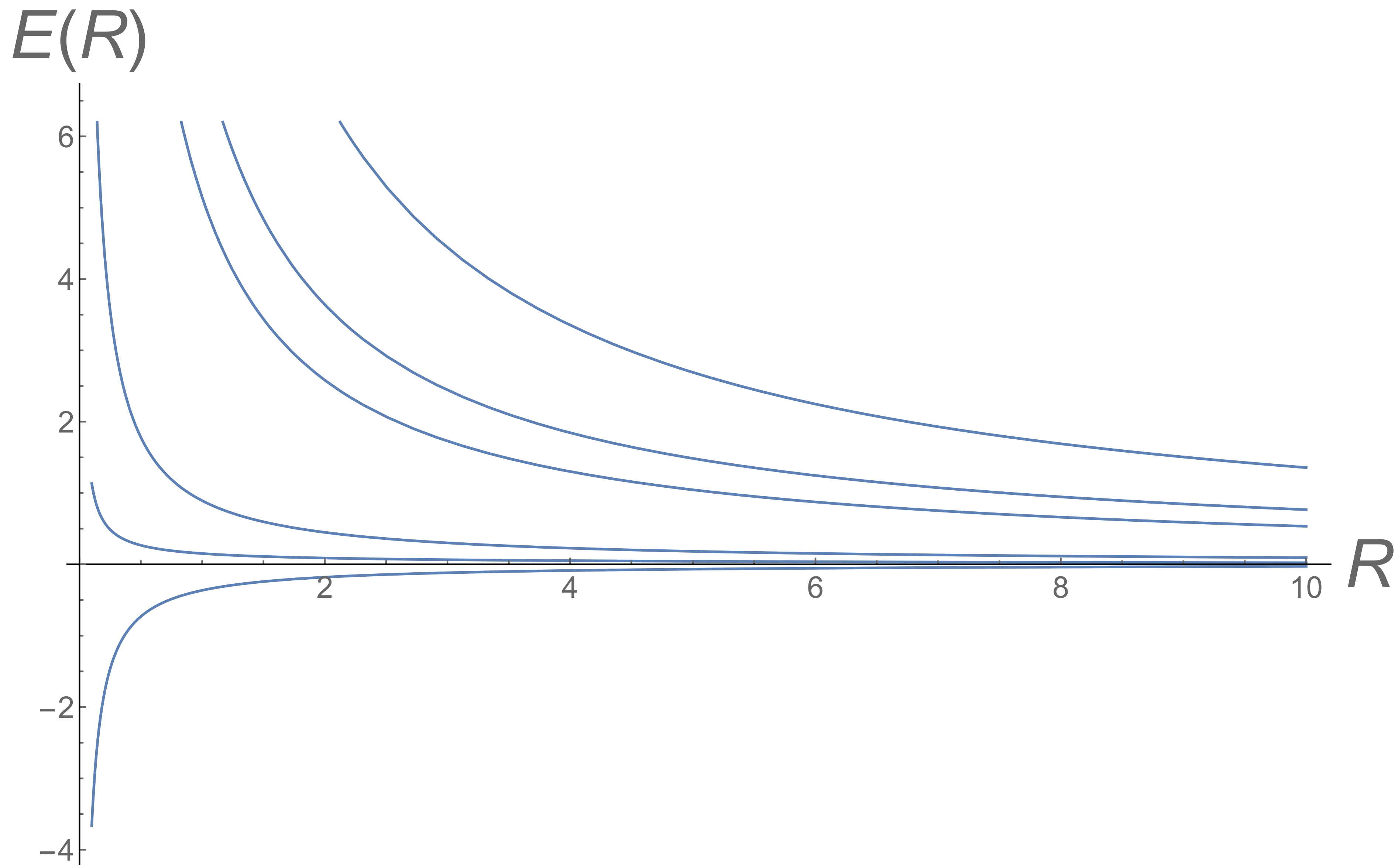
$$C = \{1, \eta, N\}.$$

C is an MFC with rank(C)=3 $\Rightarrow$ C is spontaneously broken.

Classifying $A \in C$, we find GSD=3, reproducing known result.

Examples

Example 1. Tricritical Ising model $+\phi_{1,3}$



Consistent with numerical computation.

Examples

[2312.13353 (KK)]

Example 2. Non-unitary minimal model $M(5,9) + \phi_{1,2}$

The deformation preserves rank 4 MFC

$$C = \{1, X, Y, Z\}.$$

rank(C)=4 $\Rightarrow$ C is spontaneously broken.

Classifying $A \in C$, we find GSD=4, giving prediction.

Examples

[2312.13353 (KK)]

Example 3. Non-unitary minimal model $M(6,11) + \phi_{1,2}$

The deformation preserves rank 5 MFC

$$C = \{1, X, Y, Z, W\}.$$

rank(C)=5 ⇒ C is spontaneously broken.

Classifying $A \in C$, we find GSD=5, giving prediction.

Examples

[2312.13353 (KK)]

Example 4. Non-unitary minimal model $M(6,13) + \phi_{5,1}$

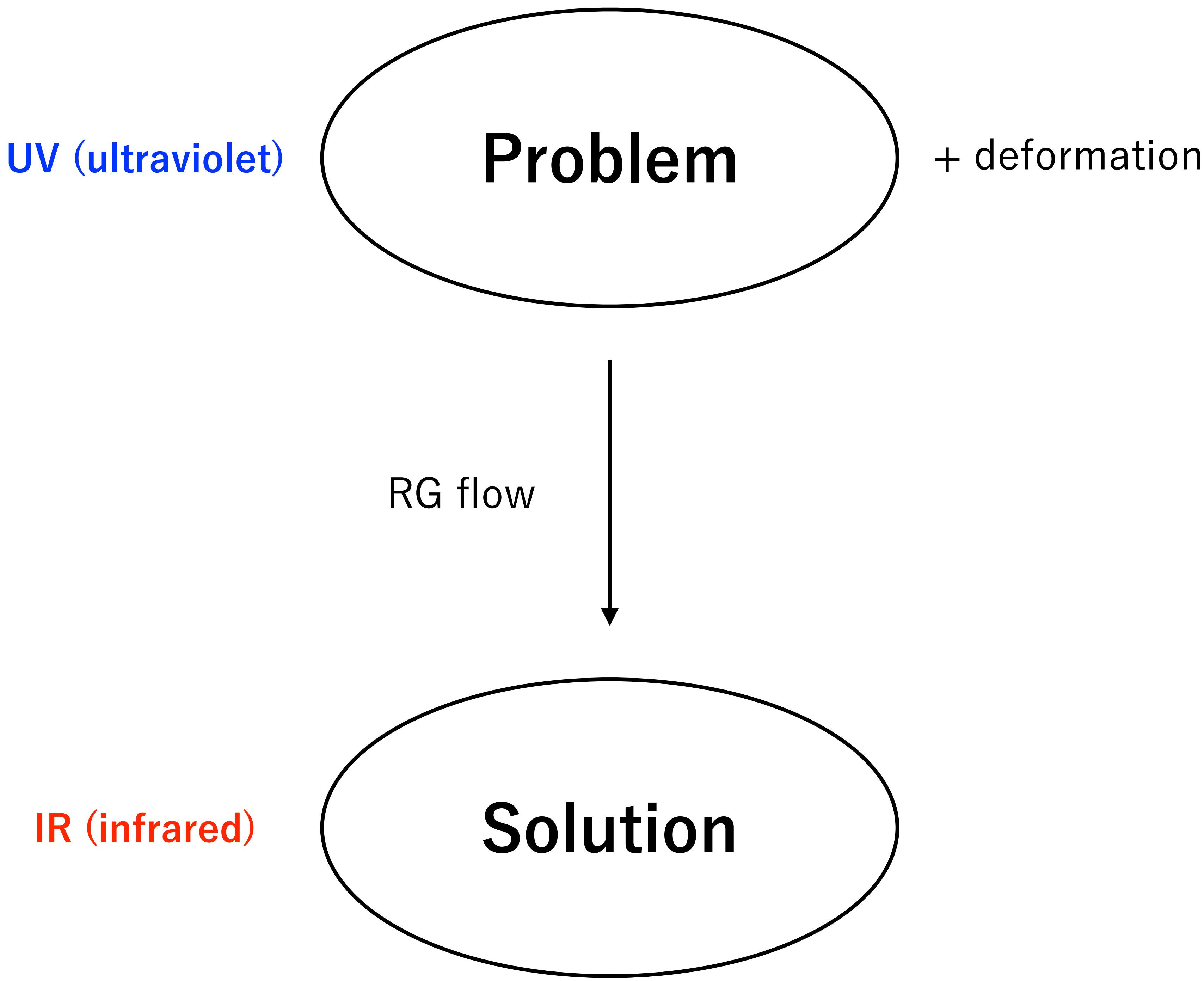
The deformation preserves rank 5 MFC

$$C = \{1, X, Y, Z, W\}.$$

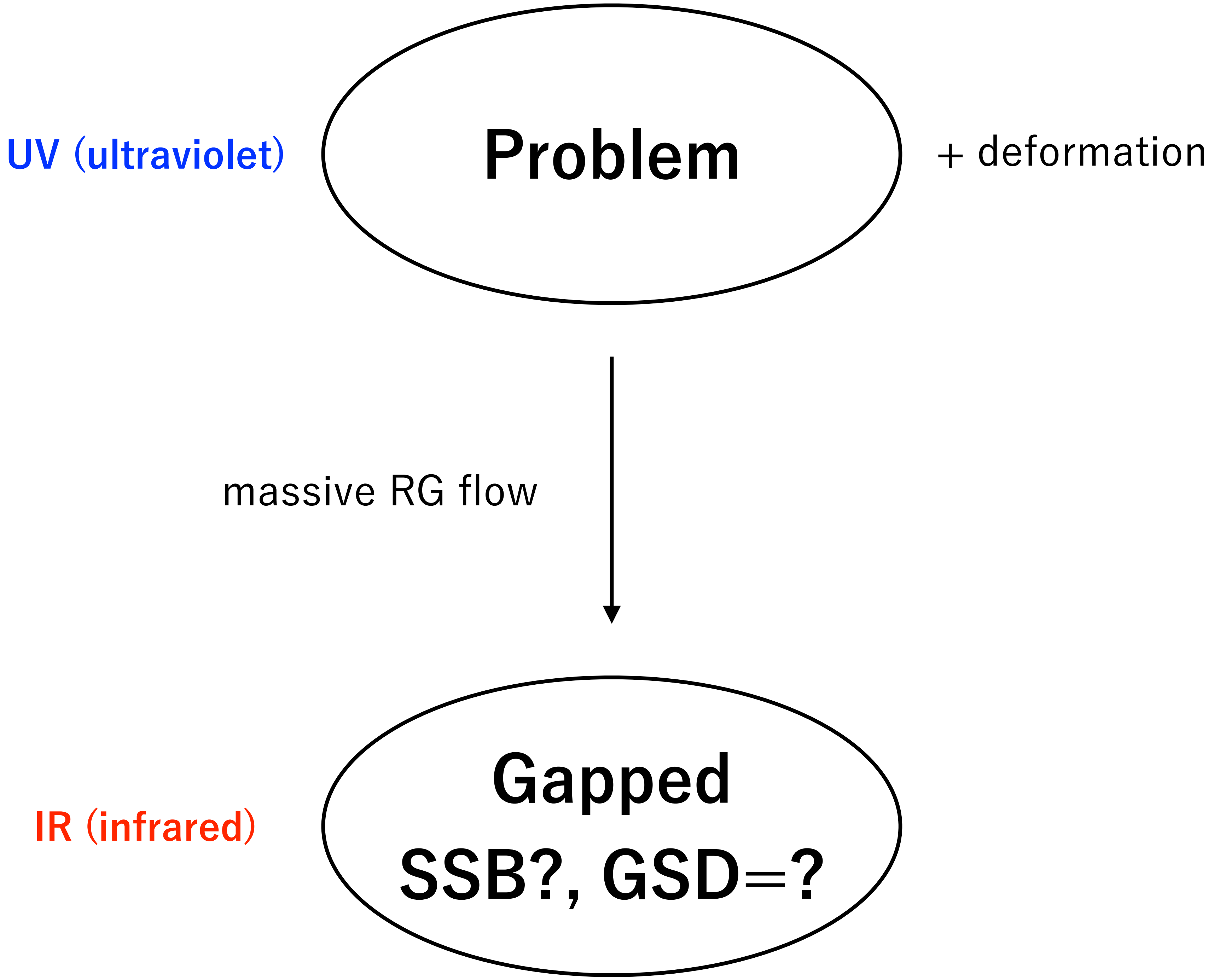
rank(C)=5 $\Rightarrow$ C is spontaneously broken.

Classifying $A \in C$, we find GSD $\in \{4,5\}$, giving prediction.

Summary


UV (ultraviolet)
Problem
+ deformation
RG flow
IR (infrared)
Solution

Summary



UV (ultraviolet)
Problem
+ deformation
massive RG flow
IR (infrared)
Gapped
SSB?, GSD=?

Summary

- Generalized symmetry = fusion category $\mathcal{C}$.
- \`Representation' of $\mathcal{C}$ = $\mathcal{C}$-module category $\mathcal{M}$.
- rank$(\mathcal{M})>1 \Rightarrow$ SSB of $\mathcal{C}$. [2311.00746 (KK)]
- Proved SSB of MFC $\mathcal{C}$ w/ rank$(\mathcal{C})>1$. [2312.13353 (KK)]
- Constrained GSD by classifying $A \in \mathcal{C}$.

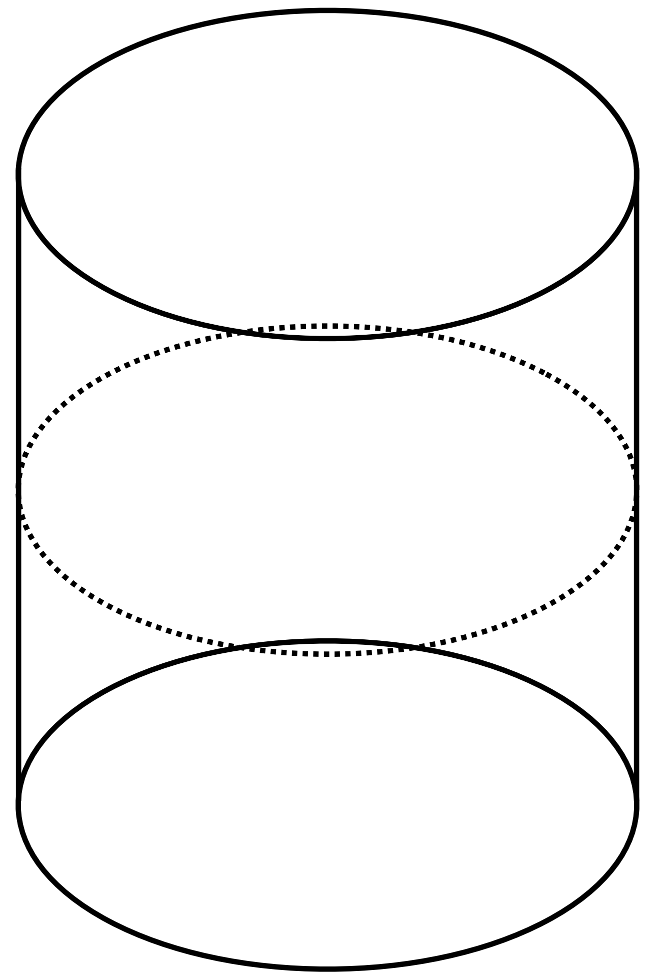
[2311.00746 (KK)]
[2311.15631 (KK)]
[2312.13353 (KK)]
[2402.00403 (KK-Kam-Huang)]
[2404.13353 (KK)]

Appendix

Numerical check

Truncated Conformal Space Approach

[Yurov-Zamolodchikov '89]



$$H_{deform} = H_{CFT_{UV}} - \lambda \int_{circle} O$$