Baby Universes and Branching Singularities in Euclidean Quantum Gravity

Christy Kelly

From Matter to Spacetime: Symmetries and Geometry
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Overview

Section 1

Euclidean Path Integrals

Definition (Heuristic)

A Euclidean field theory (in the path integral formalism) is a triple $(\Omega, \mathcal{A}, \mu)$ such that:

- Ω is a space of *Euclidean histories*.
- $\mathcal{A} : \Omega \to \mathbb{K}$ is an *action functional* controlling the dynamics of the field.
- μ is a *path integral measure* on Ω, i.e. formally we have:

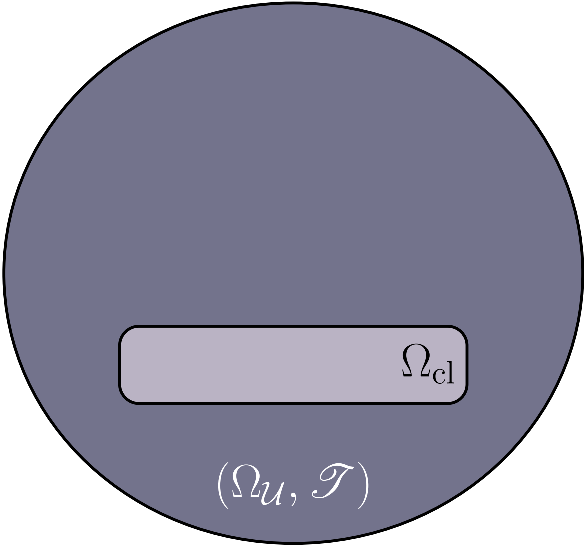
$$\mathrm{d}\mu(\omega) = \frac{1}{\mathcal{Z}[\omega]} \exp\left(-\mathcal{A}(\omega)\right) \tag{1}$$

  where the *partition function* $\mathcal{Z}$ is a normalisation factor.

1. Start with a classical theory $(\Omega_{\text{cl}}, \mathcal{A}_{\text{cl}}[\lambda])$. λ is some family of parameters.
2. Discretise the classical theory to obtain a family of theories $(\Omega_a, \mathcal{A}_a[\lambda])$ depending on the discretisation parameter a.
3. Consider the (well-defined) finite-dimensional statistical 'field' theories $(\Omega_a, \mathcal{A}_a[\lambda], \mu_a^\lambda)$. These are often actual statistical mechanical models. We will denote $\{ (\Omega_a, \mathcal{A}_a[\lambda], \mu_a^\lambda) \}_{a,\lambda}$.
4. A *scaling limit* of the family $\{ (\Omega_a, \mathcal{A}_a[\lambda], \mu_a^\lambda) \}_{a,\lambda}$ is any cluster point $(\Omega, \mathcal{A}[\lambda_0], \mu_{\lambda_0})$ of the family such that $a \to 0$ and $\lambda \to \lambda_0$ for any net of points in $\{ (\Omega_a, \mathcal{A}_a[\lambda], \mu_a^\lambda) \}_{a,\lambda}$ converging to $(\Omega, \mathcal{A}[\lambda_0], \mu_{\lambda_0})$.

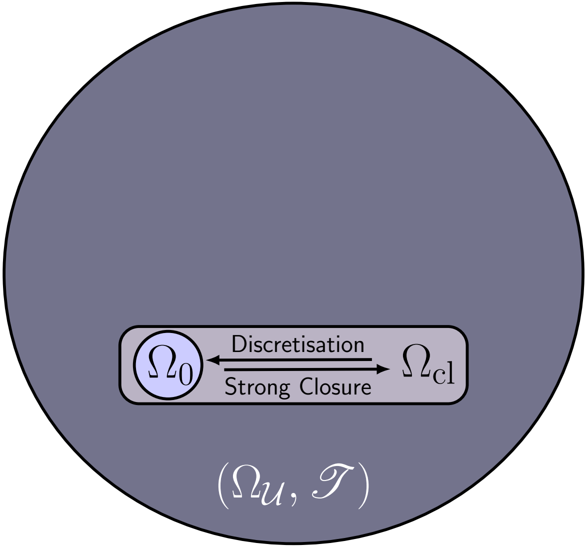
- The notion of scaling limit given here is flexible enough to take into account of more standard physical conceptualisations of scaling limits (critical conformal field theories) and also can encompass the process of renormalisation.
- Discretisation appears to be necessary to deal with various nonuniqueness problems associated to quantisation procedures.
- For instance there is a systematic connection between operator-ordering prescriptions and precise discretisations of the path integral.
- The notion of scaling limit as defined here involves a notion of *limit*; this means we need to specify a notion of convergence for both path-integral measures and histories.

Quantum Histories and Topology



- Ω_{cl}: space of classical histories.
- $\Omega_{\mathcal{U}}$: space of mathematical histories.
- $\mathscr{T}$: topology in $\Omega_{\mathcal{U}}$.

Quantum Histories and Topology



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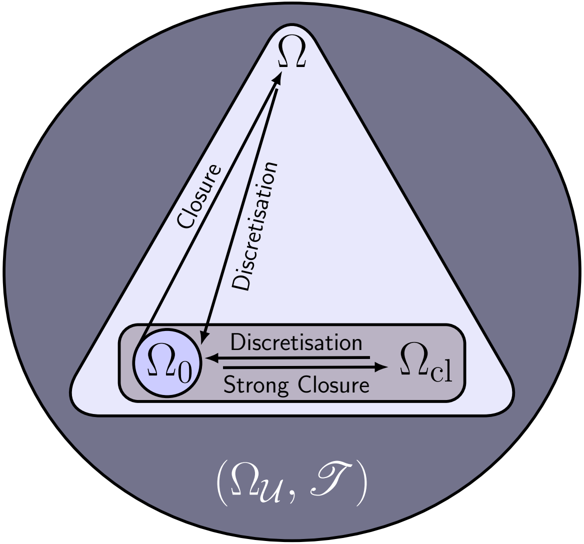

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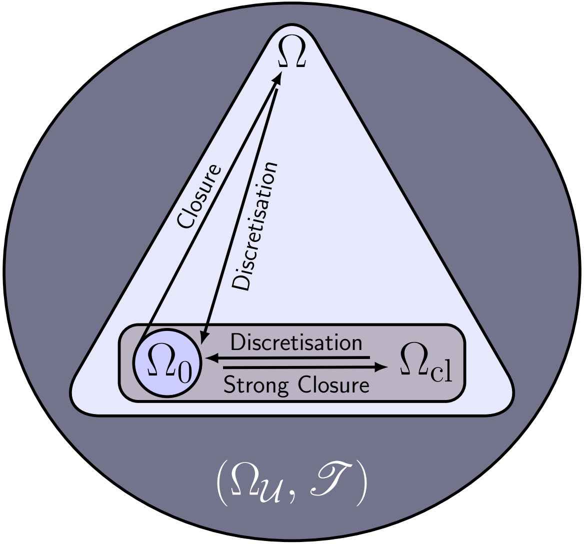

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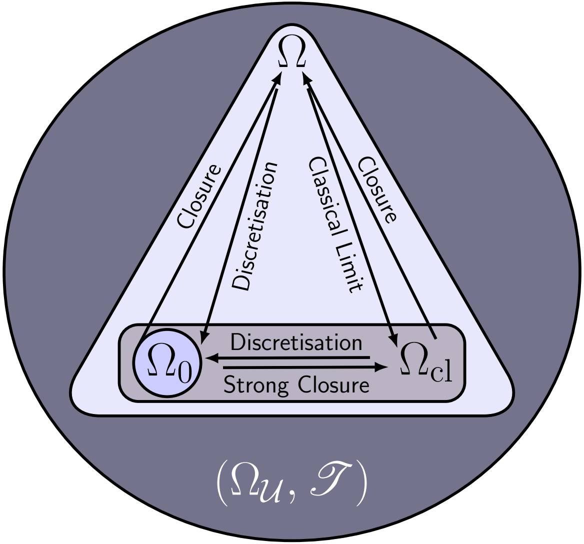

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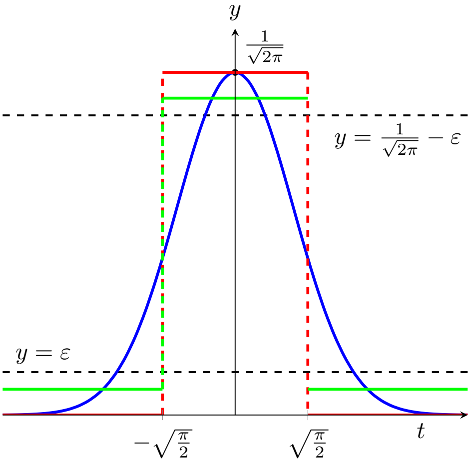
- The problem of specifying the support of the path-integral measure becomes the problem of choosing a topology $\mathscr{T}$ in the space of mathematical histories.
- Traditionally this has been seen as (largely) a technical mathematical problem.
- In fact a topology in the space of fields is simply a set of standards for evaluating when two fields 'look alike'.
- As such a good topology $\mathscr{T}$ should encode important physical properties of the fields such as symmetry, localisability and asymptotic (decay) properties of the fields.
- It also turns out that a topology on the space of histories is essentially determined by a choice of discretisation of the action.



Unit area symm. rectangle of height $\frac{1}{\sqrt{2\pi}}$.

Deformation of . Unit area Gaussian.

Fix $f, g : [a,b] \to [0,\infty)$ and $\varepsilon > 0$.

- f and g are *weakly ε-close* iff:

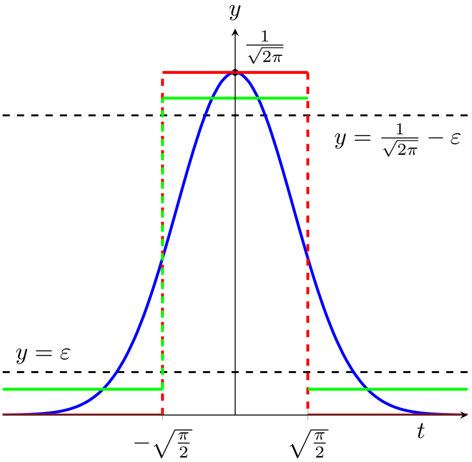
$$\left| \int_a^b \mathrm{d}t\, f(t) - \int_a^b \mathrm{d}t\, g(t) \right| < \varepsilon. \quad (2)$$

- f and g are *uniformly ε-close* iff:

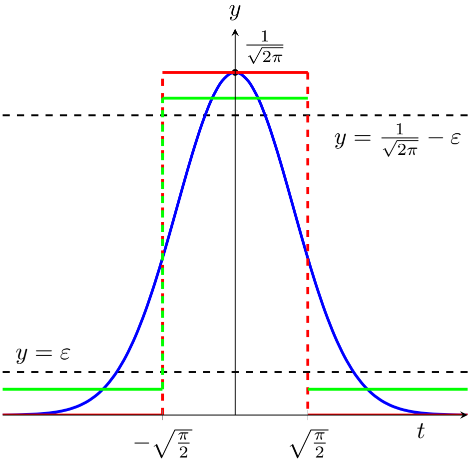
$$\sup_{t\in[a,b]} |f(t) - g(t)| < \varepsilon. \quad (3)$$

- f and g are L^2 *ε-close* iff:

$$\int_a^b \mathrm{d}t (f(t) - g(t))^2 < \varepsilon. \quad (4)$$



- – and – are WεC.
- – is WεC to any translate of itself, *even if the supports do not overlap.*
- – and – are not WεC to – for any $\varepsilon < \infty$.
- – and – are not UεC to – but – and – are UεC.
- – and – are not $L^2\varepsilon$C to – while – and – may be $L^2\varepsilon$C depending on ε.
- – is not $L^2\varepsilon$C to any translate of itself with disjoint support for $\varepsilon < 2$.



- Unit area symm. rectangle of height $\frac{1}{\sqrt{2\pi}}$.
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- WεC does not say anything about local observables; it is thus appropriate for diffeomorphism invariant theories.
- UεC gives good control over local observables independently of the location. This notion is good for comparing theories with many continuous local observables.
- $L^2\varepsilon$C requires similarity of local observables *and* convergent asymptotic behaviour. This topology is suitable if we are interested in infrared structure.

Action and Topology

- It is natural to assume that the topology $\mathscr{T}$ ensures the stability of the action at the discrete and classical level:

$$\mathcal{A}_0(\gamma_0) \to \mathcal{A}_{\text{cl}}(\gamma_{\text{cl}}) \text{ as } \gamma_0 \xrightarrow{\mathscr{T}} \gamma_{\text{cl}}. \tag{2}$$

- $\mathcal{A}$ can then be extended to Ω via stability:

$$\mathcal{A}_0(\gamma_0) \to \mathcal{A}(\gamma) \text{ as } \gamma_0 \xrightarrow{\mathscr{T}} \gamma \qquad \mathcal{A}_{\text{cl}}(\gamma_{\text{cl}}) \to \mathcal{A}(\gamma) \text{ as } \gamma_{\text{cl}} \xrightarrow{\mathscr{T}} \gamma \tag{3}$$

- Thus for a given discretisation of the action we can choose $\mathscr{T}$ to be the weakest topology such that the stability criteria 2 and 3 hold, such that all *a priori* physical constraints on the theory are imposed and such that

$$\mathcal{A}(\Omega) \subseteq \mathbb{R}. \tag{4}$$

- Choosing the *weakest* topology gives the biggest possible space of quantum histories Ω. (Note that if stability holds for one topology then it holds for all stronger topologies.)

Section 2

Euclidean Quantum Gravity

Definition (Heuristic)

A *EQG theory* is a Euclidean Field Theory $(\Omega, \mathcal{A}, \mu)$ such that:

- Ω is a configuration space of quantum Euclidean spacetime structures.
- $\mathcal{A} : \Omega \to \mathbb{K}$ is a gravitational action functional.
- μ is a path integral measure on Ω.

In particular we obtain Ω and $\mathcal{A}$ from a classical theory $(\Omega_{\mathrm{cl}}, \mathcal{A}_{\mathrm{cl}})$ where Ω_{cl} consists of Riemannian manifolds and $\mathcal{A}_{\mathrm{cl}}$ is the Euclidean Einstein-Hilbert action:

$$\mathcal{A}_{EH}^{\kappa,\Lambda}(\omega) = \frac{1}{\kappa} \int_{\omega} \mathrm{d}\,\mathrm{vol}_{\omega}(x)(2\Lambda - R(x)) \tag{5}$$

where R is the scalar curvature in ω and Λ is the *cosmological constant*.

- How do we discretise Riemannian manifolds and the Einstein-Hilbert action on manifolds?
- What topology do we put on the space of 'mathematical Euclidean spacetimes' $\Omega_{\mathcal{U}}$?
- More generally what is an element of $\Omega_{\mathcal{U}}$?
- Can such an approach avoid renormalisability problems?

- Classical gravity in 2D is trivial by the Gauss-Bonnet theorem.
- *Trivial* here means that there are no locally propagating modes and that the equations of motion are satisfied automatically.
- 2D EQG is power-counting renormalisable and in fact finite so it may be interesting to try and quantise.
- *Quantum mechanically* we expect the theory to be nontrivial for *two* reasons.
- Firstly large quantum fluctuations may lead to a change in the topology of space(time) and so a quantum theory of EQG in 2D may exhibit nontrivial topological dynamics.
- 2D EQG may also allow us to resolve the nontrivial *kinematic* challenges described above, allowing us to specify a reasonable notion of quantum Euclidean spacetime.

- By the Gauss-Bonnet theorem the classical action becomes

$$\mathcal{A}_{\mathrm{cl}}(\omega) = \beta A(\omega) - \gamma\chi(\omega) \tag{6}$$

where $A(\omega)$ denotes the area and χ the Euler characteristic of the surface ω.

- The coupling constant β is related to the cosmological constant Λ while γ is determined by the gravitational coupling κ. In particular the Euler characteristic term contains all the *geometric* dynamics associated to the curvature of the surface.
- Closed (compact without boundary) surfaces are classified up to diffeomorphism by the *genus* $g(\omega) \in \mathbb{N}$ of the surface. Thus it is convenient to take Ω_{cl} to be (diffeomorphism classes) of closed surfaces.
- We have the famous formula

$$\chi(\omega) = 2 - 2g(\omega). \tag{7}$$

- Consider a triangulated surface $\triangle$ with V vertices, E edges and F faces.
- The Euler formula states

$$\chi(\triangle) = V - E + F. \tag{8}$$

- Simple combinatorics implies:

$$3F = 2E. \tag{9}$$

- Let $f(v)$ denote the number of faces incident to the vertex v. Then

$$V = \sum_{v \in V(\triangle)} \qquad F = \frac{1}{3} \sum_{v \in V(\triangle)} f(v) \tag{10}$$

- Finally taking all triangles to be equilateral with unit are we have $A(\triangle) = F$.
- Thus a discretisation of the classical action becomes

$$\mathcal{A}_0(\triangle) = \sum_{v \in V(\triangle)} \left(\frac{1}{3}\beta f(v) - \gamma \left(1 - \frac{1}{6} f(v) \right) \right). \tag{11}$$

- From the above it is natural to consider discrete histories

$$\Omega_{N,g} = \{ \triangle : |V(\triangle)| = N \text{ and } g(\triangle) = g \} \tag{12}$$

and the path-integral measures associated to the partition functions

$$\mathcal{Z}_{N,g} = \sum_{\triangle \in \Omega_{N,g}} \exp(-\mathcal{A}_0(\triangle)). \tag{13}$$

- The general partition function on N-vertices allowing topological dynamics is then written

$$\mathcal{Z} = \sum_{g \in \mathbb{N}} N^{2-2g} \mathcal{Z}_{N,g} \tag{14}$$

after some rescaling of the action.

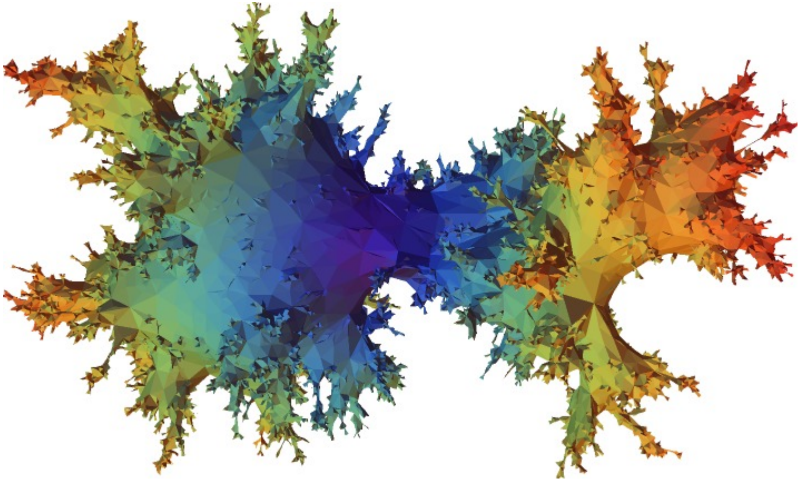
- *In the large-N limit* higher genus contributions to the partition function are suppressed.

- $\mathcal{Z}_{N,0}$ admits a perturbative expansion in some coupling $\lambda = \lambda(\beta)$ which diverges at some critical λ_c. In particular we have

$$\mathcal{Z}_{N,0} \sim (\lambda - \lambda_c)^{2-\gamma}. \tag{15}$$

- The continuum limit partition function is obtained by tuning $\lambda \to \lambda_c$ as $N \to \infty$.
- The measure associated to this partition function defines a random continuum geometry known as the *Brownian sphere* $\mathbb{BS}$.

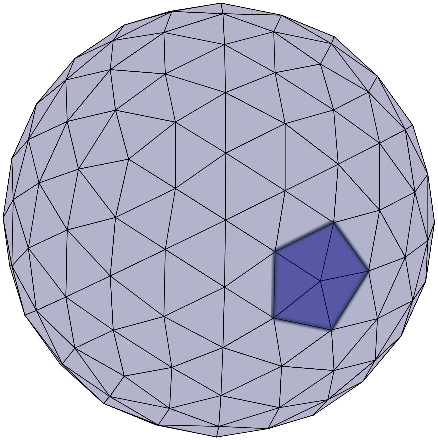
The Brownian Sphere: Emergent 2D Quantum Spacetime



- $\mathbb{BS} \cong \mathbb{S}$
- $D_S = 2$
- $D_H = 4$

Timothy Budd, `https://hef.ru.nl/~tbudd/gallery/`

- $\mathbb{BS}$ describes the effective geometry of $2D$ quantum spacetime.
- $\mathbb{BS}$ can also be shown to be equivalent in a precise sense to a geometric model of $c = 0$ Liouville field theory on the sphere
- $\implies$ the continuum limit is really *a* correct continuum limit.
- The Brownian sphere is a highly fractal topological 2-sphere.
- Thus even 'small' quantum fluctuations ruin smooth structure.
- 2D EQG is *entropic*: the Brownian sphere is also the scaling limit of uniformly selected random triangulations of the sphere.
- This is to be expected for the scaling limit of $\mathcal{Z}_{N,0}$ since the classical gravitational dynamics is trivial and there is no topological dynamics by definition.
- In the large-N scaling limit on the other hand the gravitational dynamics in fact only contributes to suppress topological dynamics.



- Hinge: *(D-2)*-simplex (corner).
- Deficit angle:

$$\omega_h = \frac{1}{3}\pi(6 - n_h) \qquad (16)$$

where n_h is the number of $(D-1)$-simplices at the hinge h.

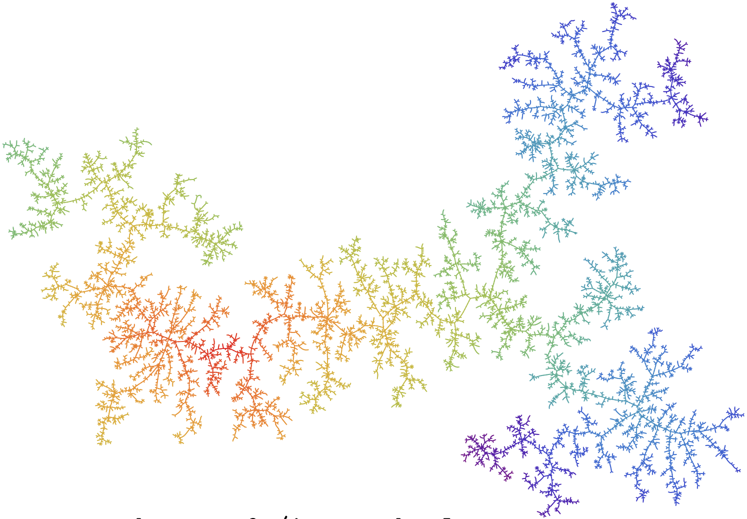
- The *Regge action*:

$$\mathcal{A}_R = \frac{1}{8\pi}\sum_h A_h\omega_h \qquad (17)$$

where A_h is the 'area' of the hinge h.

- $\mathcal{A}_R \to \mathcal{A}_{EH}^{\Lambda=0}$ as $N \to \infty$.

- $D = 1$
- $D_S = \frac{4}{3}$
- $D_H = 2$



Igor Kortchemski,
https://igor-kortchemski.perso.math.cnrs.fr/images.html

- The continuum random tree is a *pathology*: the appearance of this universal scaling limit (for all high dimensions) is a defect of the theory.
- The theory is again *entropic*: the Brownian continuum random tree appears as the scaling limit of *uniform* triangulations of higher dimensional (closed) manifolds.
- In 3D, since the theory is renormalisable this is perhaps again an expression of the triviality of the theory (no local propagating degrees of freedom).
- In dimensions $D > 3$ it appears to be a defect of nonrenormalisability: we only obtain a well-defined scaling limit by 'turning off' the gravitational interaction.
- The Brownian continuum random tree can be obtained as the universal scaling limit of suitable *branching processes*.

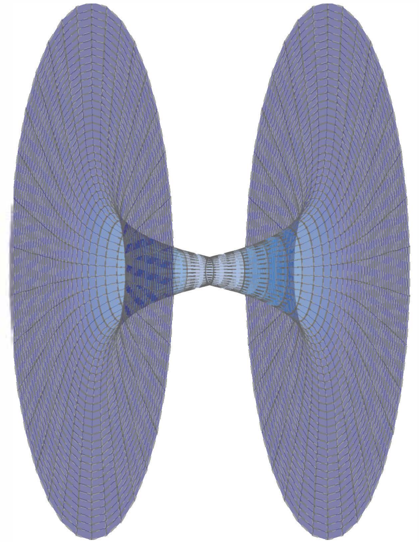
Section 3

Baby Universes in Quantum Gravity

Topology Change in Quantum Gravity

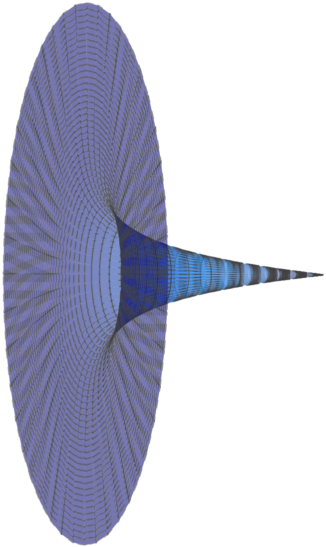
In a reasonable theory of quantum gravity the topology of spacetime must be able to be different from that of flat space. Otherwise, the theory would not be able to describe closed universes or black holes. Presumably, the theory should allow all possible spacetime topologies. In particular, it should allow closed universes to branch off, or join onto, our asymptotic flat region of spacetime. ... If it is possible for a closed universe the size of a blackhole to branch off, it is also presumably possible for little Planck-size closed universes to branch off and join on.

Stephen Hawking, 'Wormholes in spacetime', PRD, 37(4), 1988.

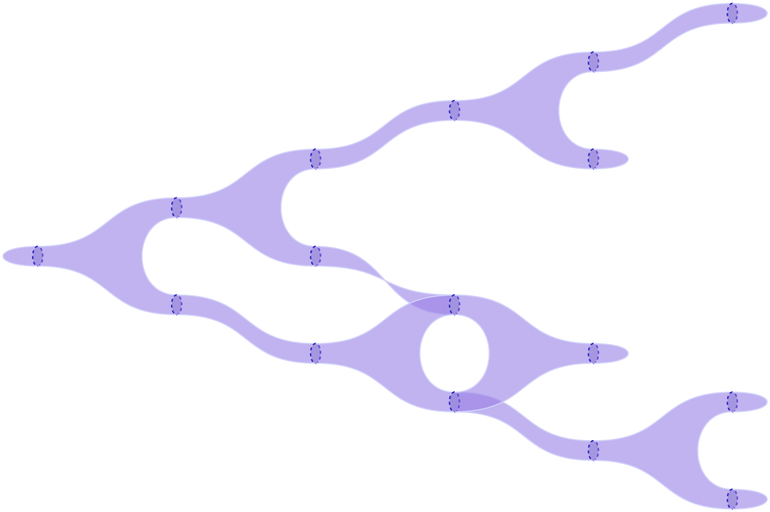


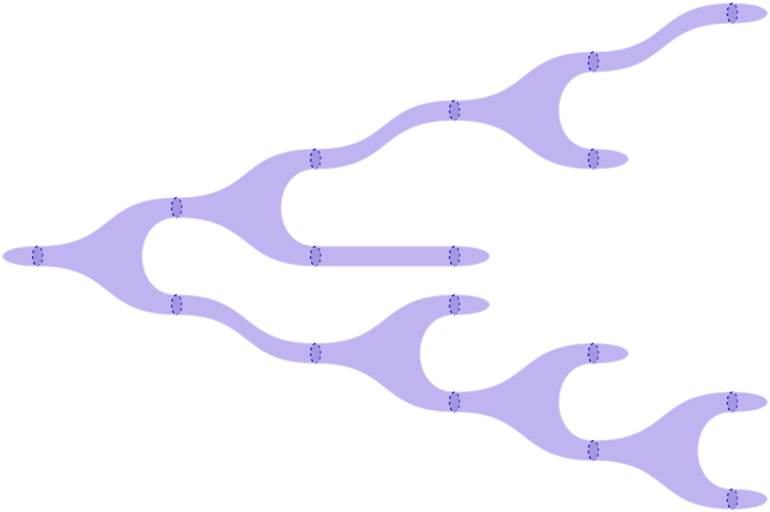
- A Euclidean wormhole is an instance of tunnelling between two extended (noncompact) regions of Euclidean spacetime.
- The 'interior' of the wormhole has nontrivial topology (is not simply connected).
- If the regions of Euclidean spacetime are connected even without the wormhole, the wormhole is effectively an additional handle for the Euclidean spacetime manifold.
- Wormholes themselves can thus be detected through the specification of suitable Morse functions on the spacetime manifold.

Wormholes and Baby Universes



- A baby universe is an aborted wormhole, i.e. an event where the universe tunnels to nothing.
- *Baby universe nucleation* describes the process whereby a little Planck sized closed universe branches off from a parent spacetime universe.
- *Any branch of spacetime created by baby universe nucleation can support further nucleation of baby universes.*
- Baby universe nucleation may generate a branch of spacetime described by a relatively complex bordism before the universe completely tunnels to nothing.





1. At 'time' $t = 0$ there is a single baby universe nucleation event.
2. At the time $t = \tau$ the baby universe either tunnels to nothing, continues unchanged or nucleates another baby universe with probabilities p_0, p_1 and p_2 respectively, $1 = p_0 + p_1 + p_2$.
3. Each baby universe evolves independently preserving its geometry for a duration τ.
4. At time $t = 2\tau$ the process in step 2 repeats; more generally at each time $t = n\tau$, $n \in \mathbb{N}$, the connected components of space independently undergo random baby universe nucleation as described in step 2.

1. Baby universes only have a single possible ‘spatial’ geometry.
2. Baby universe nucleation is an entirely quantum process that is independent of the geometric structure of each connected ‘spatial’ geometry.
3. Disconnected spatial slices are not communicating during nucleation.
4. Nucleation only occurs at given discrete timesteps.
5. Nucleation of more than two universes in a single instant is a very low probability procedure.
6. Different branches of spacetime never reattach.
7. When the quantum nucleation events are not taking place the spatial geometry evolves in a trivially classical (in fact Lorentzian) manner.

- Some of these assumptions (e.g. 3 and 5) are rather reasonable. In particular 5 can probably be derived heuristically in theories of quantum gravity. Recent thinking in holography appears to be that assumption 3 fails (factorisation problem) but the precise ramifidations of this fact are not entirely clear.
- Other assumptions such as 1, 2, 4 and 7 are essentially only made for convenience; modifying these assumptions will not change our subsequent conclusions.
- Assumption 6 (that reattachment never occurs) is not obvious; for our purposes it can be replaced by the weaker (but still mysterious) assumption that 'wormholes are never too long'.

- The bordism described by this model is essentially described by a binary branching process ancestry tree (as well as information about the initial spatial geometry and the classical evolution duration τ).
- Since the scaling limit of such processes describes a (rooted) Brownian continuum random tree we see that the this 'phenomenological' model of baby universe nucleation describes the naive scaling limit of high-dimensional quantum gravity.
- Because of the universality of the Brownian continuum random tree scaling limit our phenomenological model is robust to several changes in the assumptions.
- For instance the same conclusion holds if we have a finite family of (diffeomorphism classes) of spatial geometries describing baby universes or if we assume that nucleation time is Poisson distributed.

The Upshot

- Recall that in the large-N limit in 2D, topological dynamics was suppressed by the geometric dynamics of the theory.
- In $D > 2$ on the other hand, topological dynamics as exhibited in the nucleation of baby universes is strongly entropically favoured.
- Since gravity is not contributing in the Euclidean dynamical triangulations framework, a scaling limit in which gravitational dynamics remains relevant might allow for the suppression of baby universe nucleation.

Section 4

Regularisations of Curvature

The Main Argument

- We wish to suppress baby universe nucleation by taking a discretisation of the curvature and a scaling limit such that gravitational interactions remain relevant.
- The idea will be to take a different regularisation of the Ricci curvature to the one traditionally employed in the Regge calculus and Euclidean Dynamical Triangultions.
- This also amounts to considering an unusual topology in the space of discrete histories.
- The topology encodes two key facts: firstly that branching singularities associated with baby universe nucleation cannot be localised at corners of the manifold and secondly that classical symmetries cannot completely decouple distance from volume.

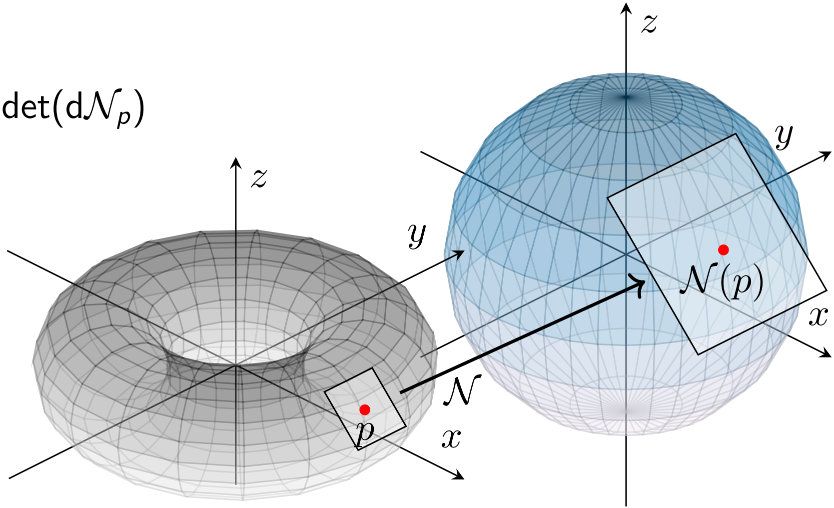
The Concrete Task

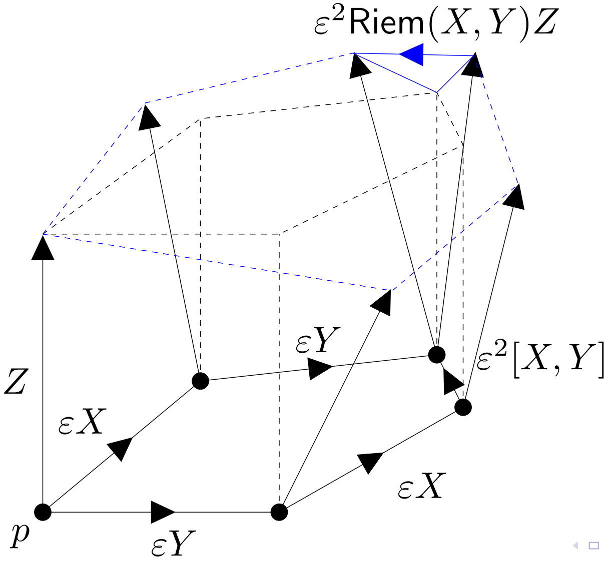
- It is too difficult to completely replicate the Dynamical Triangulation/Matrix Model analysis using different discretisations of the curvature.
- We begin by specifying coarse notions of curvature that apply to rough fractal spaces like the continuum random tree.
- Since the Gibbs factor behaves as

$$\exp\left(\frac{1}{\kappa}\int R\right) \tag{18}$$

with R a generalised scalar curvature, we hope to show that the pathological 'branching singularity' described by the continuum random tree is *infinitely negatively curved* ('*infinitely hyperbolic*').

$$K(p) = \det(d\mathcal{N}_p)$$





ε²Riem(X, Y)Z
εY
ε²[X, Y]
Z
εX
εX
p
εY

- Sectional (Riemannian) curvature controls the second-order behaviour of geodesics:

$$\rho_{\mathcal{M}}(\gamma_1(s), \gamma_2(t))^2 = s^2 + t^2 - 2st\cos\theta - \frac{1}{3}K(\dot{\gamma}_1(0), \dot{\gamma}_2(0))s^2t^2\sin^2\theta + \cdots \quad (19)$$

- The Ricci curvature controls infinitesimal volume corrections:

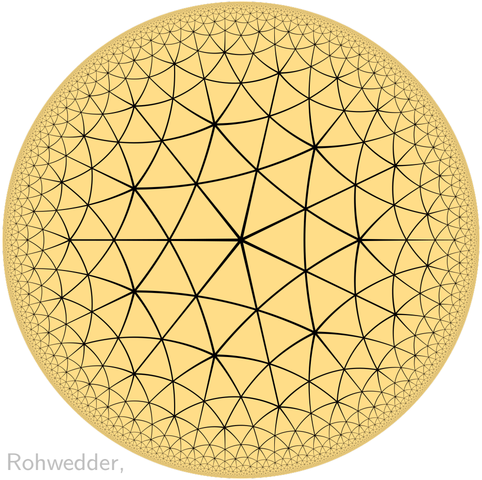
$$\mathrm{d}\,\mathrm{vol}_{\mathcal{M}}(x) = \mathrm{d}\,\mathrm{vol}_{\mathbb{R}^D}\left(1 - \frac{1}{6}\mathrm{Ric}(u_x, u_x)|x|^2 + \mathcal{O}(|x|^3)\right) \quad (20)$$

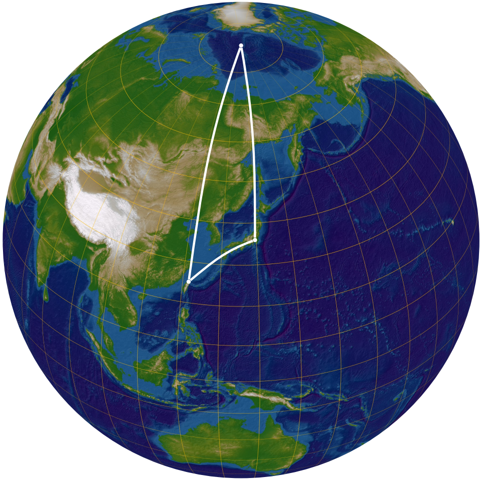
- The scalar curvature controls volume corrections:

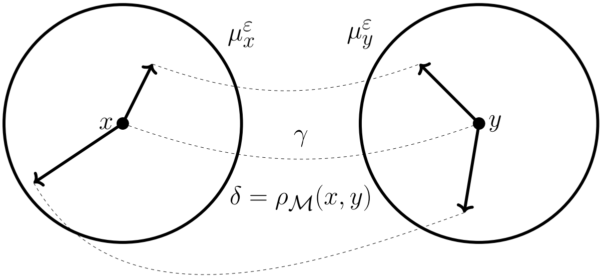
$$\mathrm{vol}(B_\delta^{\mathcal{M}}(p)) = \mathrm{vol}(B_\delta^{\mathbb{R}^D}(0))\left(1 - \frac{R}{6(D+2)}\delta^2 + \mathcal{O}(\delta^4)\right) \quad (21)$$

What is Curvature?

Triangle Comparisons







$$W_{\mathcal{M}}(\mu_x^\varepsilon, \mu_y^\varepsilon) = \delta\left(1 - \frac{\varepsilon^2}{2(D+2)} Ric\left(\frac{\dot\gamma}{|\dot\gamma|}, \frac{\dot\gamma}{|\dot\gamma|}\right) + \mathcal{O}(\varepsilon^2(\varepsilon+\delta))\right)$$

- μ_p^ε is the normalised volume in $\mathbb{B}_\varepsilon(p)$.
- $W_{\mathcal{M}}$ is the *Wasserstein distance*.
- It is obtained from parallel transport in $\mathcal{M}$, but is defined for any MM-space.
- Thus in any MM-space X we define the *Ollivier-Ricci curvature*

$$\kappa_{\delta,\varepsilon}(p,q) = 1 - \frac{W_X(\mu_p^\varepsilon, \mu_q^\varepsilon)}{\rho_X(p,q)}. \quad (22)$$

- Positive ORC says that the *average* distance between two balls is greater than the distance between their centres.

The Brownian Continuum Random Tree is Infinitely Hyperbolic

- This perspective is immediately obvious if we consider the triangle comparison notion of curvature: since triangles in the Brownian continuum random tree are 'infinitely thin' the spaces are infinitely hyperbolic.
- More precisely every continuum tree is a so-called $CAT(K)$ space for all $K \in \mathbb{R}$.
- Intuitively this supports our idea that we may suppress baby universe nucleation using gravitational dynamics, but the statement is qualitative and is hard to turn into a formalism where we can in principle consider a discrete Einstein-Hilbert action and an associated path-integral measure.
- In arXiv:2312.01894 [math.PR] it is shown that

$$-\frac{a}{\delta\ell} \leq \kappa_x(\delta,\ell) \leq -\frac{b}{\delta\ell} \tag{23}$$

for positive constants $a,\ b \in (0,\infty)$ where $\kappa_x(\delta,\ell)$ is a properly normalised Ollivier-Ricci curvature and x is a random point of the continuum random tree.

- Thus *if* we can define a EQG model using the Ollivier-Ricci curvature as a regularisation of the action it will not suffer from the branching singularity pathology.
- In K., Trugenberger, Biancalana (arXiv:2102.02356 [gr-qc]) it was shown how to write down a discrete Einstein-Hilbert action $\mathcal{A}_{DEH}$ such that

$$\mathcal{A}_{DEH}^{\delta,\ell}(\omega_n) \to \mathcal{A}_{EH}(\omega) \tag{24}$$

where ω_n are discrete metric-measure spaces (graphs) that converge appropriately to the closed Riemannian manifold ω.

- There is a regularisation of the curvature and the Euclidean Einstein-Hilbert action using the notion of Ollivier-Ricci curvature.
- Any scaling limit of the Euclidean quantum gravity theory defined using this type of regularisation will not be concentrated in the branched polymer phase.
- Indeed, *if such a scaling limit exists*, then we have a *non-entropic* theory which successfully uses gravitational dynamics to suppress baby universe nucleation in higher dimensions.
- This regularisation of the Einstein-Hilbert action is associated to a topology on the space of triangulations where distances and volumes are coupled locally (see appendix).
- This ensures that symmetries of quantum spacetime cannot be too different from classical diffeomorphisms.

The Problem with the Regge Action?

- The Ollivier-Ricci curvature is by definition concerned with open regions about points.
- The Regge action, however, adopts a notion of curvature where all the information is encoded in monodromy defects localised at *corners*.
- The Regge action cannot see divergences that spread out over open regions!
- Branching singularities associated to the branched polymer phase are manifestly such singularities: with probability 1 a randomly chosen point in the BCRT is not a branching point, but with the same probability every open set contains an infinite number of such branching points.

The above ... suggests that settling the convergence problem in general, might just amount to calculating the *pointwise* limit of an appropriate angle defect. But this turns out to be somewhat misleading. In the general case, the pointwise calculation of $\lim_{\eta \to 0} R^j_\eta$ results in an expression which merely resembles, but does *not* coincide with R^j. ... Recall, however, our actual assertion that $R_\eta \to R$ in the sense of *measures*. This means the following. We fix an open set $U \subseteq M^n$, and then count all contributions corresponding to points lying in U. Thus, when the approximation becomes fine, we are counting a large number of small contributions. So the possibility exists that these might give the correct answer *on the average*, even though they fail to do so individually. Remarkably, this averaging effect does indeed take place, and in this sense, the convergence, $R_\eta \to R$, is *not* a purely local phenomenon.

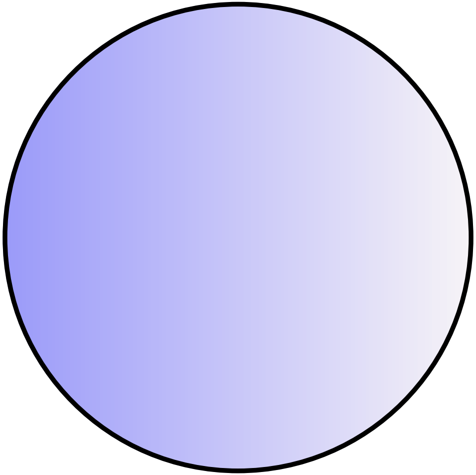
Cheeger, Müller and Schrader, 'On the Curvature of Piecewise Flat Spaces', Comm. Math. Phys, 92, 1984.

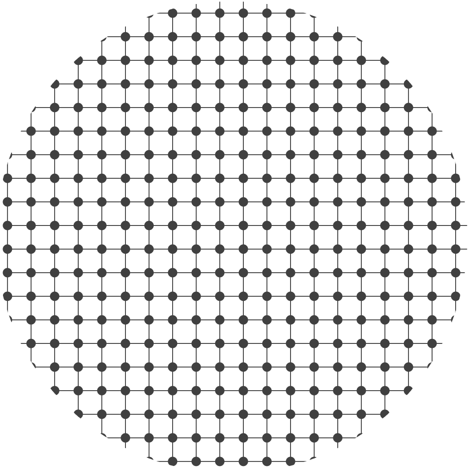
Thank You!

Section 5

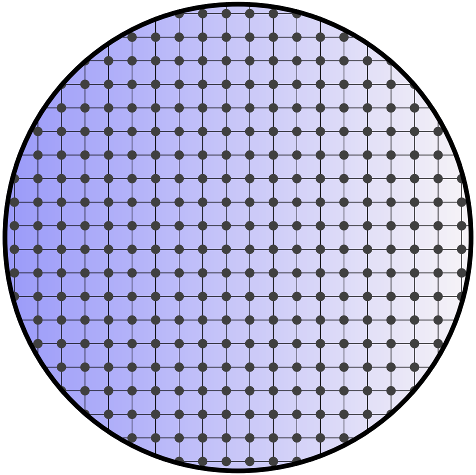
Appendix

Topologies on Spaces of Metric Spaces and the Stability of Ollivier-Ricci Curvature



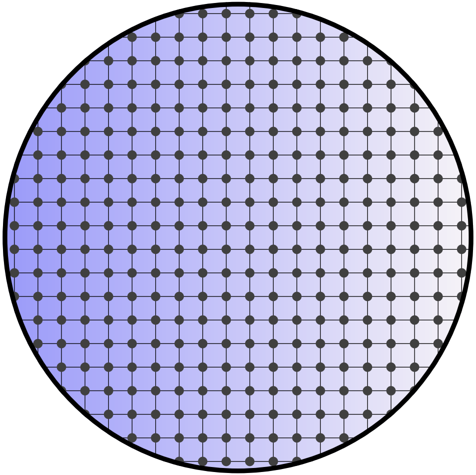


Topologies on Spaces of Metric Spaces and the Stability of Ollivier-Ricci Curvature

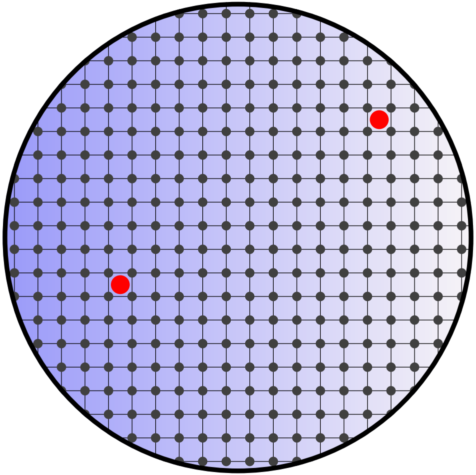


- The two spaces are *Gromov-Hausdorff* similar (i.e. similar as metric spaces) iff there is a nearly isometric imbedding of one into the other.

Topologies on Spaces of Metric Spaces and the Stability of Ollivier-Ricci Curvature



- The two spaces are *Gromov-Hausdorff* similar (i.e. similar as metric spaces) iff there is a nearly isometric imbedding of one into the other.
- This essentially means that we can approximate distances in the blue space by distances in the discrete space.



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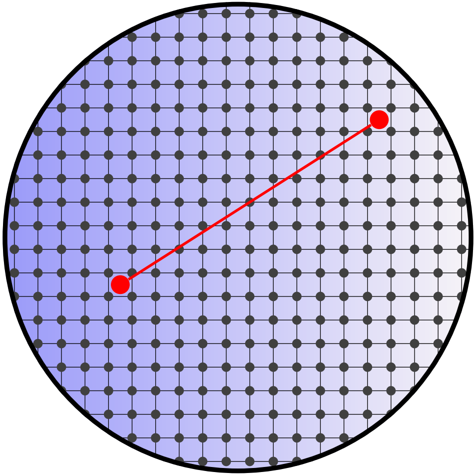

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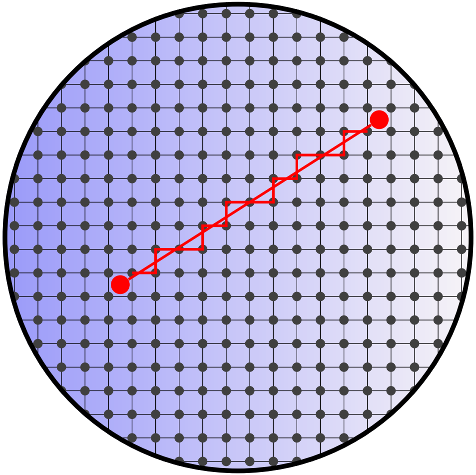

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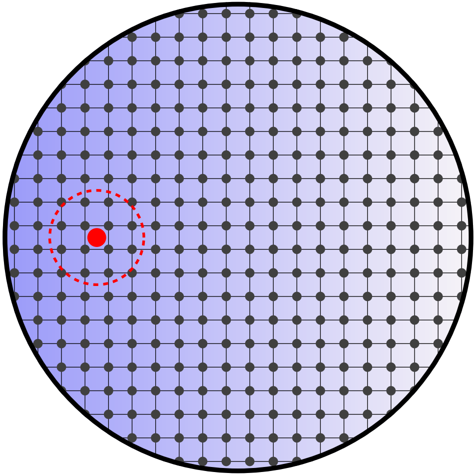
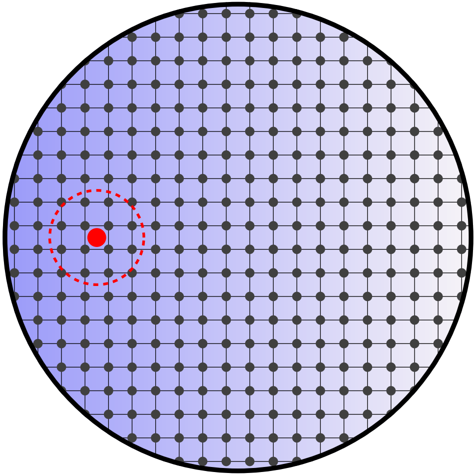
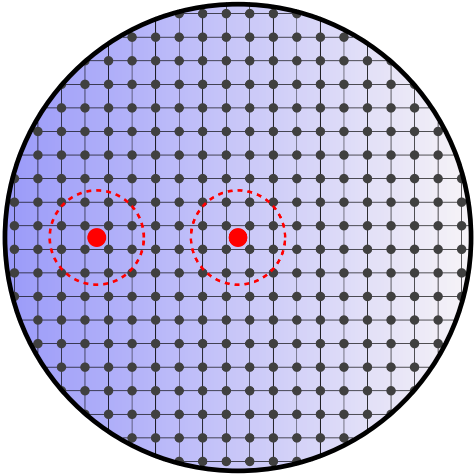


- The two spaces are *Gromov-Hausdorff* similar (i.e. similar as metric spaces) iff there is a nearly isometric imbedding of one into the other.
- This essentially means that we can approximate distances in the blue space by distances in the discrete space.
- Here the spaces are additionally *Prokhorov similar* (i.e. similar as measure spaces).

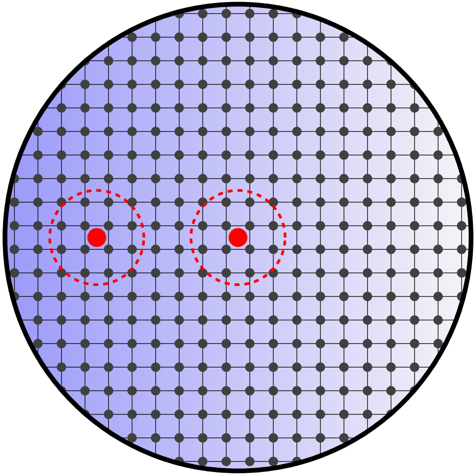
Topologies on Spaces of Metric Spaces and the Stability of Ollivier-Ricci Curvature



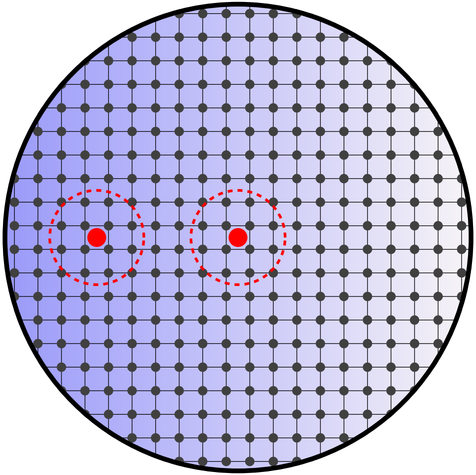
- The two spaces are *Gromov-Hausdorff* similar (i.e. similar as metric spaces) iff there is a nearly isometric imbedding of one into the other.
- This essentially means that we can approximate distances in the blue space by distances in the discrete space.
- Here the spaces are additionally *Prokhorov similar* (i.e. similar as measure spaces).
- In particular this means that the proportion of points in the red ball around any point is approximately the proportion of the volume of the ball to the total volume.



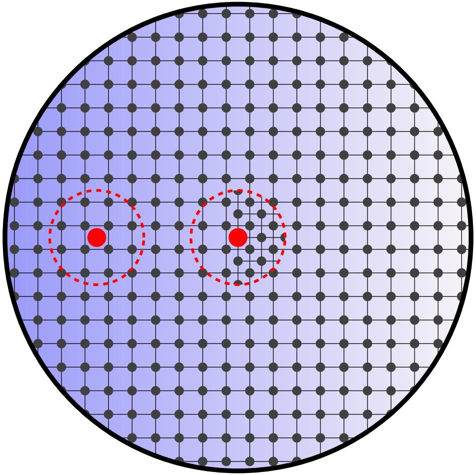
- We may compute the Ollivier curvature by computing the average distance between two small balls in the spaces in question.



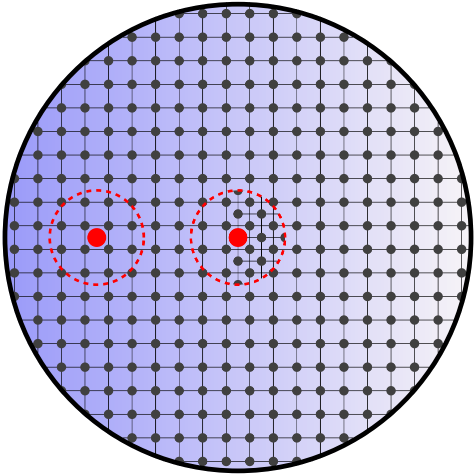
- We may compute the Ollivier curvature by computing the average distance between two small balls in the spaces in question.
- Since the blue space is sampled evenly everywhere by the discrete space, the Ollivier curvature of the discrete space well-approximates the Ollivier curvature of the blue space.



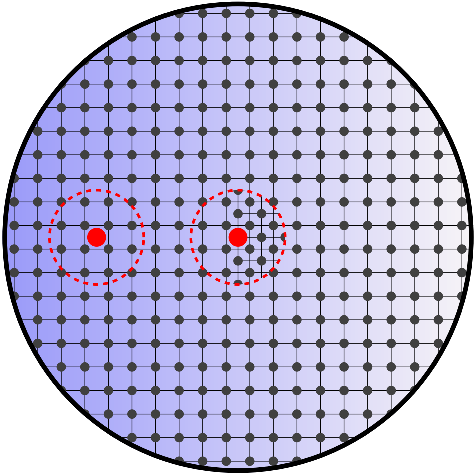
- We may compute the Ollivier curvature by computing the average distance between two small balls in the spaces in question.
- Since the blue space is sampled evenly everywhere by the discrete space, the Ollivier curvature of the discrete space well-approximates the Ollivier curvature of the blue space.
- Hence we can compute the Ricci curvature of the blue space in terms of the Ollivier curvature of the discrete space space.



- However consider a *new* DS that no longer samples the blue space evenly.

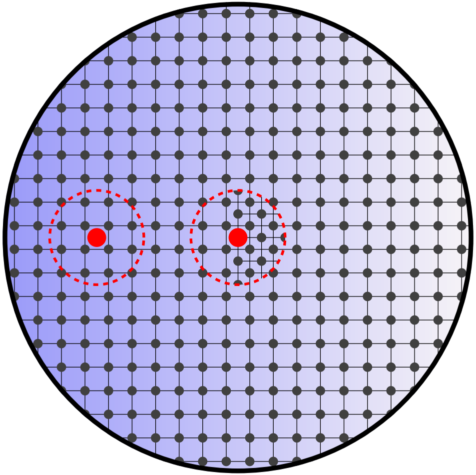


- However consider a *new* DS that no longer samples the blue space evenly.
- This DS is a *better* GH approximation of the blue space.



- However consider a *new* DS that no longer samples the blue space evenly.
- This DS is a *better* GH approximation of the blue space.
- The DS is also a reasonable Prokhorov approximation of the blue space as long as the red balls are small.

Topologies on Spaces of Metric Spaces and the Stability of Ollivier-Ricci Curvature



- However consider a *new* DS that no longer samples the blue space evenly.
- This DS is a *better* GH approximation of the blue space.
- The DS is also a reasonable Prokhorov approximation of the blue space as long as the red balls are small.
- However the average distance between the two red balls in the DS longer approximates the average distance in the blue space and the Ollivier curvature fails to converge.

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