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Comment on "Aging Population, Retirement, and Risk Taking"

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Abstract. Levy [Levy H (2016) Aging population, retirement, and risk taking. *Management Sci.* 62(5):1415–1430.] proposes asymptotic first-degree stochastic dominance (AFSD) as a distribution-ranking criterion for all nonsatiable decision makers with infinite investment horizons. By assuming that the terminal wealth follows a log-normal distribution and that the marginal utility is bounded, he offers the necessary and sufficient distributional condition for AFSD. Given Levy's setting, we provide a counterexample to show that his condition is not necessary and offer the correct equivalent distributional condition for AFSD.

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Keywords: asymptotic stochastic dominance • maximum geometric mean strategy • long-run investment

1. Introduction

Longer life expectancy has become a major challenge in life-cycle planning, and the optimal investment in the long run has become an increasingly important concern. To solve the optimal investment problem with an indefinitely long horizon, Levy (2016) incorporates the impacts of an investment horizon into the concept of stochastic dominance to support the maximum geometric mean (MGM) strategy as a solution for an aging population. He establishes a new notion of stochastic dominance, referred to as "asymptotic first-degree stochastic dominance" (AFSD).

AFSD is a distribution-ranking criterion that defines a distribution as dominating another one in terms of first-degree stochastic dominance (FSD) when the investment period goes to infinity. Assuming that the return in each period is independent and identically log-normally distributed, Levy (2016) claims in his theorem 1 that a strategy dominates others in terms of AFSD if and only if (iff) it generates a higher geometric mean and a higher volatility.

Although Levy's notion of AFSD sheds light on the solution of the investment decision in the long run, the distributional condition for AFSD given in his theorem 1 is inaccurate. In this comment, we first demonstrate that higher geometric means and higher volatilities are not necessary for AFSD. We then show that the correct condition for AFSD is higher geometric means together with the same volatilities. Proofs are relegated to the appendix.

2. A Counterexample to Levy's Condition for AFSD

Consider the buy-and-hold strategy: An investor neither withdraws nor injects new cash into her portfolio, but keeps reinvesting her portfolio in each period. Let x_t denote the rate of gross return at time t and W_T denote the terminal wealth of a uni-dollar investment after T periods. The x_t's are independently and identically distributed. Let U denote the von Neumann–Morgenstern utility function and F_T and G_T denote two cumulative distribution functions of W_T. Then, $E_F U(W_T)$ and $E_G U(W_T)$ represent the expected utility functions of W_T under F_T and G_T, respectively.

Definition 1 (Levy 2016, p. 1416). F_T dominates G_T by AFSD if and only if

$$\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] \geq 0 \text{ for all } U \text{ with } U' \geq 0,$$

and for some nondecreasing U there is a strict inequality.

To gain tractability, Levy (2016) further assumes that $\log x_t$ follows $N(\mu_F, \sigma_F^2)$ and $N(\mu_G, \sigma_G^2)$ under F and G, respectively. The "necessary and sufficient condition" for AFSD provided by Levy (2016, theorem 1) is as follows:

Alleged Theorem 1 (Levy 2016, p. 1417). *Assume an investment horizon of T periods and that F_T and G_T are log-normal distributions. Assume that $\mu_F > \mu_G$; namely, F has a higher geometric mean than G. Then, for $T \to \infty$, F_T and G_T are log-normal distributions, and in this case F_T dominates G_T by asymptotic-FSD iff $\mu_F > \mu_G$ and $\sigma_F \geq \sigma_G$, provided that the marginal utility is bounded.*

The following counterexample shows that Levy's condition is not necessary.

Counterexample 1. *Assume that F_T and G_T are log-normal distributions and that the marginal utility is bounded. F_T dominates G_T by AFSD if*

$$\mu_F - \mu_G \geq (\sigma_G - \sigma_F)\left(\sigma_G + \sqrt{\sigma_G^2 + 2\mu_G}\right)$$

and $\sigma_F < \sigma_G$.

The above case is empirically relevant. For example, let F and G correspond to the MSCI World index and the Standard & Poor's (S&P) 500 index, respectively. From 1926 to 2010, the annualized means and volatilities of the rates of return were 10.81% and 18.06% for the MSCI and 9.19% and 19.96% for the S&P 500 (Bodie et al. 2013, figure 5.3). Assuming that the gross annual returns of both assets follow log-normal distributions, we have $\mu_F = 0.0895$, $\sigma_F = 0.1619$, $\mu_G = 0.0715$, and $\sigma_G = 0.1813$, which satisfy

$$\mu_F - \mu_G = 0.0180 > 0.0117 = (\sigma_G - \sigma_F)\left(\sigma_G + \sqrt{\sigma_G^2 + 2\mu_G}\right)$$

and $\sigma_F < \sigma_G$. Thus, if the underlying dynamics of the two assets persist in the future, then the MSCI dominates the S&P 500 by AFSD, provided that the marginal utility is bounded.

3. A New Characterization of AFSD

In Definition 1, there is no additional restriction on the utility function other than $U' \geq 0$. This feature shares the same spirit as the classic concept of FSD in minimizing specific assumptions about decision-makers' risk attitudes. However, in the characterization of the distributional condition, Levy (2016) imposes the restriction of bounded marginal utility. We now remove this restriction and offer the true necessary and sufficient condition for AFSD.

Theorem 1. *Assume that F_T and G_T are log-normal distributions. F_T dominates G_T by AFSD, if and only if $\mu_F > \mu_G$ and $\sigma_F = \sigma_G$.*

Theorem 1 resolves the debate on the optimality of the MGM strategy in the long run for log-normal distributions. The MGM strategy—choosing F if $\mu_F > \mu_G$—is optimal from the perspective of utility maximization if and only if $\sigma_F = \sigma_G$. Theorem 1 confirms that AFSD yields exactly the same condition as FSD with a finite horizon as found by Levy (1973, theorem 4). Conceptually, AFSD appears to be less demanding than FSD because it only requires that $E_F U(W_T) \geq E_G U(W_T)$ in the limit of $T \to \infty$ and allows for any violation of $E_F U(W_T) \geq E_G U(W_T)$ at finite horizons. However, the potential tolerance of the violation is removed by the power utility class, with which $E_F U(W_T) < E_G U(W_T)$ at some finite horizon T can lead to $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] = -\infty$. Accordingly, the condition in Levy (2016) of bounded marginal utility is not innocuous: It excludes the power utility class, limits the utility loss, and relaxes the important requirement $\sigma_F = \sigma_G$.

Although Theorem 1 spells out the simple distribution condition ensuring that F is preferable to G in the long run for all nonsatiated utility maximizers, the requirement that $\sigma_F = \sigma_G$ is too restrictive to implement it in practice. Thus, how to make the notion of asymptotic stochastic dominance more applicable through excluding the utility functions that are less or not at all interesting from a practical or theoretical perspective is an open question for future research.

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Appendix. Proofs

Proof of Counterexample 1. Levy (2016) ignores the possibility that AFSD can also occur in the case where $\sigma_F < \sigma_G$. To see it, let

$$w_0 = e^{T\left(\frac{\mu_F}{\sigma_F} - \frac{\mu_G}{\sigma_G}\right)/\left(\frac{1}{\sigma_F} - \frac{1}{\sigma_G}\right)}$$

be the unique intersection point of F_T and G_T and write his equation (4) as

$$E_F U(W_T) - E_G U(W_T) = I + II,$$

where $I = \int_0^{w_0} [G_T(w) - F_T(w)]U'(w)dw$ and $II = \int_{w_0}^{\infty} [G_T(w) - F_T(w)]U'(w)dw$. With $\sigma_F < \sigma_G$, F_T intersects G_T from below, and thus $I \geq 0$ while $II \leq 0$. As assumed by Levy (2016), let M denote an upper bound on the marginal utility. The utility loss II can be controlled by

$$|II| \leq M\left|\int_{w_0}^{\infty} [G_T(w) - F_T(w)]dw\right|.$$

In this case, we claim that

$$\lim_{T\to\infty} \int_{w_0}^{\infty} [G_T(w) - F_T(w)]dw = \begin{cases} 0, & \text{if } \mu_F - \mu_G \geq (\sigma_G - \sigma_F)\left(\sigma_G + \sqrt{\sigma_G^2 + 2\mu_G}\right), \\ -\infty, & \text{otherwise,} \end{cases} \tag{A.1}$$

for which a line-by-line derivation is provided at the end of the proof. Therefore, if $\sigma_F < \sigma_G$ and

$$\mu_F - \mu_G \geq (\sigma_G - \sigma_F)\left(\sigma_G + \sqrt{\sigma_G^2 + 2\mu_G}\right),$$

then $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] = \lim_{T\to\infty} I \geq 0$. In addition, under the proposed condition, we also have $\mu_F + \frac{\sigma_F^2}{2} > \mu_G + \frac{\sigma_G^2}{2}$, yielding

$$\lim_{T\to\infty}(E_F W_T - E_G W_T) = \lim_{T\to\infty}\left[e^{(\mu_F+\sigma_F^2/2)T} - e^{(\mu_G+\sigma_G^2/2)T}\right] = \infty,$$

a strict inequality in Definition 1. Taken together, the above verifies that F_T dominates G_T by AFSD, provided that the marginal utility is bounded.

To derive Equation (A.1), let $\Phi(x)$ denote the cumulative distribution function of the standard normal distribution and $\Psi(x) = 1 - \Phi(x)$. We insert

$$G_T(w) - F_T(w) = \Phi\left(\frac{\ln w - T\mu_G}{\sqrt{T}\sigma_G}\right) - \Phi\left(\frac{\ln w - T\mu_F}{\sqrt{T}\sigma_F}\right) = \frac{1}{\sqrt{2\pi}}\int_{\frac{\ln w - T\mu_F}{\sqrt{T}\sigma_F}}^{\frac{\ln w - T\mu_G}{\sqrt{T}\sigma_G}} e^{-y^2/2}dy$$

into $\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw$ and exchange the order of integration to obtain

$$\begin{aligned}
&\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw \\
&= \frac{1}{\sqrt{2\pi}}\int_{e^{T\left(\frac{\mu_F}{\sigma_F} - \frac{\mu_G}{\sigma_G}\right)/\left(\frac{1}{\sigma_F} - \frac{1}{\sigma_G}\right)}}^{\infty}\left(\int_{\frac{\ln w - T\mu_F}{\sqrt{T}\sigma_F}}^{\frac{\ln w - T\mu_G}{\sqrt{T}\sigma_G}} e^{-y^2/2}dy\right)dw \\
&= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{T}\left(\frac{\mu_F - \mu_G}{\sigma_G - \sigma_F}\right)}^{\infty}\left(e^{T\mu_F + \sqrt{T}\sigma_F y} - e^{T\mu_G + \sqrt{T}\sigma_G y}\right)e^{-y^2/2}dy \\
&= e^{(\mu_G+\sigma_G^2/2)T}\left[e^{m(\sigma_G-\sigma_F)T}\Psi\left(\sqrt{T}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)\right)\right. \\
&\quad \left. - \Psi\left(\sqrt{T}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)\right)\right], \qquad \text{(A.2)}
\end{aligned}$$

where $m = \frac{\mu_F - \mu_G}{\sigma_G - \sigma_F} - \frac{1}{2}(\sigma_F + \sigma_G)$. Notice that "$F_T$ dominates G_T by AFSD" can occur only when $\mu_F + \sigma_F^2/2 \geq \mu_G + \sigma_G^2/2$, as otherwise taking $U(W) = W$ yields

$$E_F U(W_T) - E_G U(W_T) = e^{(\mu_F+\sigma_F^2/2)T} - e^{(\mu_G+\sigma_G^2/2)T} \to -\infty,$$

a contradiction to Definition 1. We thus only need to focus on the case where $m \geq 0$. Equation (A.1) is equivalent to

$$\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = \begin{cases} -\infty, & \text{if } 0 \leq m \leq \frac{\sigma_G - \sigma_F}{2}, \\ -\infty, & \text{if } m > \frac{\sigma_G - \sigma_F}{2} \text{ and } \mu_G + \frac{1}{2}\sigma_G^2 - \frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 > 0, \\ 0, & \text{if } m > \frac{\sigma_G - \sigma_F}{2} \text{ and } \mu_G + \frac{1}{2}\sigma_G^2 - \frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 \leq 0. \end{cases}$$

To prove the limits case by case, notice that for any given $a > 0$ and $b > 0$, the L'Hôpital rule implies

$$\lim_{T\to+\infty} e^{aT}\Psi\left(b\sqrt{T}\right) = \lim_{T\to+\infty}\frac{1}{\sqrt{2\pi}}\frac{\int_{b\sqrt{T}}^{+\infty} e^{-y^2/2}dy}{e^{-aT}} = \lim_{T\to+\infty}\left(\frac{b}{2a\sqrt{2\pi T}}\right)e^{(a-b^2/2)T},$$

yielding

$$\lim_{T\to+\infty} e^{aT}\Psi\left(b\sqrt{T}\right) = \begin{cases} +\infty, & \text{if } a > b^2/2 \\ 0, & \text{if } a \leq b^2/2 \end{cases} \text{ and } \lim_{T\to+\infty} e^{b^2T/2}\sqrt{T}\Psi\left(b\sqrt{T}\right) = \frac{1}{b\sqrt{2\pi}}. \qquad \text{(A.3)}$$

If $m = 0$, it is obvious that $\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = -\infty$. If $0 < m \leq \frac{\sigma_G - \sigma_F}{2}$, thanks to $m(\sigma_G - \sigma_F) - \frac{1}{2}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)^2 = -\frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 \leq 0$, we apply (A.3) to obtain

$$\lim_{T\to+\infty} e^{m(\sigma_G - \sigma_F)T}\Psi\left(\sqrt{T}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)\right) = 0,$$

which in turn implies $\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = -\infty$. If $m > \frac{\sigma_G - \sigma_F}{2}$ and $\mu_G + \frac{1}{2}\sigma_G^2 - \frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 > 0$, we formally write the right-hand side of (A.2) as $K(T)/e^{-(\mu_G+\sigma_G^2/2)T}$, where $K(T) \to 0$ as $T \to \infty$ and

$$K'(T) = (\sigma_G - \sigma_F)e^{-\frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 T}X(T)/\sqrt{T}$$

with

$$X(T) = me^{\frac{1}{2}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)^2 T}\sqrt{T}\Psi\left(\sqrt{T}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)\right) - \frac{1}{2\sqrt{2\pi}}.$$

By virtue of the L'Hôpital rule, we obtain

$$\begin{aligned}
&\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw \\
&= \lim_{T\to\infty}\frac{-K'(T)}{(\mu_G + \sigma_G^2/2)e^{-(\mu_G+\sigma_G^2/2)T}} \\
&= \lim_{T\to\infty}\frac{-X(T)}{\sqrt{T}}\left(\frac{\sigma_G - \sigma_F}{\mu_G + \sigma_G^2/2}\right)e^{\left[\mu_G + \frac{1}{2}\sigma_G^2 - \frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2\right]T}.
\end{aligned}$$

According to (A.3) and $m > \frac{\sigma_G - \sigma_F}{2}$, we further have

$$\lim_{T\to\infty} X(T) = \frac{1}{\sqrt{2\pi}}\left[\frac{m}{m + (\sigma_G - \sigma_F)/2} - \frac{1}{2}\right] > 0,$$

which in turn implies $\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = -\infty$. If $m > \frac{\sigma_G - \sigma_F}{2}$ and $\mu_G + \frac{1}{2}\sigma_G^2 - \frac{1}{2}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)^2 \leq 0$, we apply (A.3) to the two terms in the right-hand side of (A.2) independently and obtain that

$$\lim_{T\to\infty} e^{(\mu_G+\sigma_G^2/2)T + m(\sigma_G - \sigma_F)T}\Psi\left(\sqrt{T}\left(m + \frac{\sigma_G - \sigma_F}{2}\right)\right) = 0$$

$$\text{and } \lim_{T\to\infty} e^{(\mu_G+\sigma_G^2/2)T}\Psi\left(\sqrt{T}\left(m - \frac{\sigma_G - \sigma_F}{2}\right)\right) = 0,$$

which in turn implies $\lim_{T\to\infty}\int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = 0$. This completes the proof of Equation (A.1). □

Remark. Another deficiency in the proof of Levy (2016) is that Levy's condition cannot effectively guarantee $\lim_{T\to\infty}|I| = 0$ in the case where $\sigma_F > \sigma_G$ ($I \leq 0$ while $II \geq 0$). To ensure $\lim_{T\to\infty} I = 0$, Levy (2016) controls I by

$$|I| \leq M\left|\int_0^{w_0}[G_T(w) - F_T(w)]dw\right|.$$

However, Levy's claim that "$\lim_{T\to\infty} w_0 = 0$ if $\mu_F > \sigma_F$ and $\sigma_F > \sigma_G$" is incorrect. Indeed, $\lim_{T\to\infty} w_0 = 0$ holds true only if $\mu_F/\sigma_F > \mu_G/\sigma_G$. Otherwise, $w_0 \equiv 1$ if $\mu_F/\sigma_F = \mu_G/\sigma_G$ and $\lim_{T\to\infty} w_0 = \infty$ if $\mu_F/\sigma_F < \mu_G/\sigma_G$. Derivations in the same way as the Proof of Counterexample 1 show that if $\sigma_F > \sigma_G$ and $\mu_F - \mu_G \geq \frac{2(\sigma_F - \sigma_G)\mu_G}{\sigma_G + \sqrt{\sigma_G^2 + 2\mu_G}}$ (a condition weaker than $\mu_F/\sigma_F > \mu_G/\sigma_G$), then F_T dominates G_T by AFSD, provided that the marginal utility is bounded.

Proof of Theorem 1. The "if" part is apparent, because the distributional condition indeed implies that F_T dominates G_T by FSD. To prove the "only if" part, the essential step is to show that $\sigma_F = \sigma_G$ must hold true.

If, by contradiction, $\sigma_F > \sigma_G$, F_T intersects G_T from above and the violation of FSD appears when the wealth is smaller than the intersection point w_0. We can choose a concave power utility function $U(w) = w^\gamma/\gamma$ with a sufficiently large marginal utility in the range of small wealth such that the utility loss from choosing F grows to infinity as $T \rightarrow \infty$. Specifically, let us choose

$$\gamma < 2\min\left\{\frac{\mu_G - \mu_F}{\sigma_F^2 - \sigma_G^2}, 0, -\frac{\mu_G}{\sigma_G^2}\right\}$$

such that $\mu_F + \frac{\gamma}{2}\sigma_F^2 < \mu_G + \frac{\gamma}{2}\sigma_G^2 < 0$. Because

$$E\left(\frac{1}{\gamma}\right)W_T^\gamma = \frac{1}{\gamma}e^{\gamma\left(\mu + \frac{\gamma}{2}\sigma^2\right)T}$$

for $\gamma \neq 0$, we further have

$$E_F\left(\frac{1}{\gamma}\right)W_T^\gamma < E_G\left(\frac{1}{\gamma}\right)W_T^\gamma$$

for any T and, moreover,

$$\lim_{T\rightarrow\infty}\left[E_F\left(\frac{1}{\gamma}\right)W_T^\gamma - E_G\left(\frac{1}{\gamma}\right)W_T^\gamma\right] = -\infty,$$

which is a contradiction of Definition 1.

Similarly, if $\sigma_F < \sigma_G$, F_T intersects G_T from below and the violation of FSD appears when the wealth is larger than w_0. We can instead choose a convex power utility function with a large enough marginal utility in the range of large wealth such that the utility loss from choosing F grows to infinity as $T \rightarrow \infty$. Specifically, we have $0 < \mu_F + \frac{\gamma}{2}\sigma_F^2 < \mu_G + \frac{\gamma}{2}\sigma_G^2$ for any

$$\gamma > 2\max\left\{\frac{\mu_F - \mu_G}{\sigma_G^2 - \sigma_F^2}, 0\right\}.$$

With the fact that $E\left(\frac{1}{\gamma}\right)W_T^\gamma = \frac{1}{\gamma}e^{\gamma\left(\mu + \frac{\gamma}{2}\sigma^2\right)T}$, we further have

$$E_F\left(\frac{1}{\gamma}\right)W_T^\gamma < E_G\left(\frac{1}{\gamma}\right)W_T^\gamma$$

for any T and

$$\lim_{T\rightarrow\infty}\left[E_F\left(\frac{1}{\gamma}\right)W_T^\gamma - E_G\left(\frac{1}{\gamma}\right)W_T^\gamma\right] = -\infty,$$

another contradiction of Definition 1. Therefore, it must be the case that $\sigma_F = \sigma_G$, following which $\mu_F > \mu_G$ arises naturally. □

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