Structure of Spot Rates and Duration Hedging*

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Abstract

The present study proposes a three-factor model using spot rates as proxies for the state variables of the term structure of interest rates. Empirical analysis is carried out on the in-sample explanatory power and the out-of-sample prediction ability of spot-rate models, and comparison is made between the modified Macaulay duration and spot-rate duration hedging for bond portfolios. The results not only show that the optimal three-spot-rate model outperforms the optimal two-spot-rate model proposed by Elton *et al.* (*Journal of Finance*, 45, 1990, 629–642) with respect to explanation ability of unexpected changes in the term structure of interest rates, but also illustrate the importance of capturing the curvature characteristic of the term structure of interest rates for spot-rate duration hedging methods. Moreover, the impressive performance of three-spot-rate duration hedging implies that it is feasible to reduce the dimensions of state variables to three for the purposes of risk exposure prediction and risk management of bond portfolios.

Keywords Risk management; Spot rates; Duration; Immunization; Term structure of interest rates

JEL Classification: E43, G11

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1. Introduction

Over the past 30 years, numerous investigators have studied the term structure of interest rates. Typically, following a stochastic process, one or more state variables have been used to construct the term structure of interest rate models (e.g. Vasicek, 1977; Richard, 1978; Brennan and Schwartz, 1979; Nelson and Schaefer, 1983; Cox *et al.*, 1985). While such theoretical models are applied for empirical tests, one or more spot rates are typically used as proxies for the state variables. In fact, a small set of spot rates is used to explain bond prices or returns, based on the assumption that the whole term structure could be constructed as a function of this set. Among these spot rates, the 1-year rate is frequently used for one-factor models (see Babbel, 1983; Nelson and Schaefer, 1983). Moreover, Nelson and Schaefer (1983) apply long-term and medium-term rates (the 13-year rate and the difference between the 13- and 5-year rates) as proxies for two of the state variables to describe the behavior of the term structure of interest rates, showing that the two-factor model is a better overall fit than one-factor models in describing term structures.

Although previous studies, such as Babbel (1983), Nelson and Schaefer (1983), and Lekkos (2001), use spot rates as driving factors to describe the dynamics of the term structure of interest rates, these spot rates are selected arbitrarily. Elton *et al.* (1990) propose a methodology to identify optimal spot rates as proxies for state variables that drive term structure movements. In essence, the major task is to extract the unknown factors from the term structure itself, rather than from other sources or through arbitrary choice. Using the monthly term structures of spot rates over the 30-year period from 1957 to 1986, Elton *et al.* (1990) asserts that the 4-year spot rate is the optimal proxy for the one-factor model (the "opt 1" model in Elton *et al.*), whereas the 6-year and 8-month spot rates are the optimal proxies for the two-factor model (the "opt 2" model in Elton *et al.*). Elton *et al.* also declare the superiority of the opt 2 model over a series of benchmark models from previous studies, as well as over their own opt 1 model. Navarro and Nave (2001) obtain similar results applying the same methodology, but with data collected from the Spanish Public Debt Market.

In contrast, empirical studies of principal component analysis, such as Steeley (1990), Litterman and Scheinkman (1991), Knez *et al.* (1994), Willner (1996), Bliss (1997), and Byun and Lee (2009), indicate that term structure movements can be classified into three main categories: level (parallel shift), steepness (slope), and curvature. Changes in these three components can explain up to 97% of total variance of changes in the term structure of interest rates. These studies show that it is possible to identify a small set of factors that account for a large proportion of interest rate changes, and, hence, for bond price movement. In addition, Nelson and Siegel (1987) use long-term, short-term, and medium-term components of forward rates to construct a parsimonious model, in which the three parameters of their model can also be interpreted to reflect level, slope, and curvature changes. Due to its good performance, Fabozzi *et al.* (2005), Diebold and Li (2006), and Choi *et al.* (2010) apply the Nelson and Siegel (1987) model to test the predictability of the shape of the term structure of interest rates. Moreover, Soto (2004) examines a series of duration-matching strategies with different numbers of risk factors and points out that three-factor immunization strategies offer the highest immunization benchmarks. If the number of risk factors is larger than three, the immunization performance might deteriorate.

The literature cited above prompts us to model term structure movements in terms of three specific factors (optimal spot rates). We extend the framework of Elton *et al.* (1990) to a three-spot-rate model.[1] Although it is impossible to test the explanation ability of every technique for all possible uses, if a technique performs well both for individual rates and for the returns on a portfolio of bonds, such a technique can be said to be robust. Therefore, we investigate the explanation ability of the unexpected change in interest rates and test the performance of an immunization strategy for the risk exposure of bond portfolios with respect to term structure movements for our three-spot-rate model. The first part in this paper, for comparative purposes, is to test whether our three-spot-rate model is superior to the Elton *et al.* opt 2 model using the same monthly dataset as Elton *et al.* Moreover, updated monthly data over the period July 1997–June 2007 are collected to test the robustness of these models in explanation ability of unexpected changes in the term structure of interest rates.

In addition to investigating the in-sample explanatory power and the out-of-sample prediction ability of these spot-rate models, the second part of this paper examines the performance of the immunization strategy based on the optimal spot rates for managing the interest rate risk of bond portfolios. It is well-known that duration matching strategy is the most common method for institutional investors to conduct interest rate risk management.[2] The practical bucket-based model aims to match the duration of assets with the duration of liabilities in each bucket having different maturity. Usually, maturities for the buckets are consistent with maturities of the spot rates on the term structure. In this paper, we examine a different model by investigating the performance of duration matching (hedging) strategies based solely on optimal spot rates. More specifically, after identifying optimal spot rates, we estimate the coefficients of sensitivities for the spot-rate models, fit the coefficients and the term structure of interest rates smoothly by cubic splines, and

[1]To avoid confusion with factor models that leave the driving factors unspecified, hereafter, we use ‘‘spot-rate model’’ to refer to our model.

[2]The current applications of duration hedging and immunization on interest rate risk management can be found in the fixed income literature. Nawalkha and Soto (2009) classify these immunization strategies that deal with hedging the risk of large, non-parallel term structure movements into four main categories: *M*-absolute/*M*-square models (see Bierwag *et al.*, 1993; Nawalkha and Chambers, 1996), duration vector/*M*-vector models (see Chambers *et al.*, 1988; Nawalkha and Chambers, 1997; Soto, 2001, 2004), key rate duration models (see Ho, 1992) and principal component duration models (see Bliss, 1997; Nawalkha *et al.*, 2005). Nawalkha *et al.* (2005) provide a detail discussion about the advantages and shortcomings of these models.

apply these continuous curves to calculate the corresponding durations for each individual zero-coupon bond (similar to Reitano, 1990; Ho, 1992).[3] Furthermore, these estimated durations are used to perform duration hedging for three typical kinds of bond portfolio (special portfolio) and 100 random bond portfolios (random portfolio). The impressive performance of the three-spot-rate duration hedging strategy implies the feasibility of this model. Moreover, because the number of optimal spot rates is far smaller than the number of buckets, a duration matching strategy based on optimal spot rates might be easier and less costly to implement relative to bucket-based duration matching.

The structure of the present paper is as follows. In Section 2, we introduce the analytical framework, including the methodology of searching for optimal spot rates and estimating sensitivities, deriving the optimal spot-rate durations, and analysis of risk management with duration hedging. The data, consisting of two parts over a 30-year period and an updated 10-year period of the term structure of interest rates, are described in Section 3. In addition, empirical tests for the in-sample explanatory power and the out-of-sample prediction ability of the spot-rate models and for the modified Macaulay duration versus spot-rate durations hedging of bond portfolios are analyzed in Section 3. Section 4 presents our conclusions.

2. Analytical Framework

In Subsection 2.1, we detail the processes for extracting optimal spot rates as proxies for the state variables of the term structure of interest rates and estimating their corresponding sensitivities and durations. In Subsection 2.3, we discuss the methodology for implementing risk management with duration hedging on special bond portfolios and random bond portfolios to evaluate the efficiency of the spot-rate model.

2.1. Searching for Optimal Spot Rates and Estimating Corresponding Sensitivities

Whereas Elton *et al.* (1990) propose either one or two spot rates as proxies for the unknown factors driving the term structure of interest rates. The present paper extends the framework in Elton *et al.* to consider three spot rates to serve as the unknown factors for modeling the dynamics of the term structure of interest rates.

We assume that the changes in the spot rate for the ith maturity are dependent on three unknown explanatory factors, F_1, F_2, and F_3, as follows:

[3] However, there is a major difference between the spot-rate durations and the Ho (1992) key rate durations. The spot-rate durations are estimated according to optimal spot rates derived from the dynamic behavior of the term structure, rather than to a series of key rates stated by Ho (1992) to analyze risk exposures of bond portfolios.

$$dr_{i,t} = \beta_{i,0} + \beta_{i,1}dF_{1,t} + \beta_{i,2}dF_{2,t} + \beta_{i,3}dF_{3,t} + \varepsilon_{i,t}, \quad (1)$$

where $dr_{i,t}$ denotes the unexpected change in the ith maturity spot rate at time t; $dF_{1,t}$, $dF_{2,t}$, and $dF_{3,t}$ represent synchronic changes in the unknown explanatory factors, F_1, F_2, and F_3, at time t; and $e_{i;t}$ is the error term of the unexpected change in the ith maturity spot rate at time t, which is assumed to follow a normal distribution, $N(0, \sigma^2)$.

By employing three optimal spot rates (denoted as r_l, r_m, and r_s) as proxies for the unknown factors, we propose a three-spot-rate model as follows:

$$dr_{i,t} = a_i dr_{l,t} - b_i(dr_{s,t} - dr_{l,t}) - c_i(2dr_{m,t} - dr_{s,t} - dr_{l,t}) + \varepsilon_{i,t}, \quad (2)$$

which can be rewritten as

$$dr_{i,t} = (a_i + b_i + c_i)dr_{l,t} + (-2c_i)dr_{m,t} + (c_i - b_i)dr_{s,t} + \varepsilon_{i,t}, \quad (3)$$

where $dr_{l,t}$, $dr_{m,t}$, and $dr_{s,t}$ represent synchronic changes in the three optimal spot rates. In reality, during the process of finding optimal spot rates, r_l, r_m, and r_s are identified as the optimal long-term, medium-term, and short-term spot rates, and the coefficients $(a_i + b_i + c_i)$, $(-2c_i)$, and $(c_i - b_i)$ represent sensitivities related to such spot rates along the term structure, respectively.

There are two reasons to adopt the terms $(dr_{s,t} - dr_{l,t})$ and $(2dr_{m,t} - dr_{s,t} - dr_{l,t})$ in equation (2). The first is to preclude the problem of multicollinearity (see Nelson and Schaefer, 1983; Elton *et al.*, 1990). Second, because r_l, r_m, and r_s represent, respectively, optimal long-term, medium-term, and short-term spot rates, the term $(dr_{s,t} - dr_{l,t})$ can be viewed as a proxy for the slope change of the term structure. In addition, Diebold and Li (2006) suggest that the term $(2dr_{m,t} - dr_{s,t} - dr_{l,t})$ can be used to represent the curvature changes of the term structure. Therefore, the essence of the three-spot-rate model in equation (2) is to employ the terms $dr_{l,t}$, $(dr_{s,t} - dr_{l,t})$, and $(2dr_{m,t} - dr_{s,t} - dr_{l,t})$ as proxies to capture the level (parallel shift), slope, and curvature changes in the term structure of interest rates.

Furthermore, equation (2) can be discretely approximated for empirical tests as

$$\Delta r_{i,t} = \hat{a}_i \Delta r_{l,t} - \hat{b}_i(\Delta r_{s,t} - \Delta r_{l,t}) - \hat{c}_i(2\Delta r_{m,t} - \Delta r_{s,t} - \Delta r_{l,t}) + \varepsilon_{i,t}, \quad (4)$$

or rearranged as

$$\Delta r_{i,t} = (\hat{a}_i + \hat{b}_i + \hat{c}_i)\Delta r_{l,t} + (-2\hat{c}_i)\Delta r_{m,t} + (\hat{c}_i - \hat{b}_i)\Delta r_{s,t} + \varepsilon_{i,t}. \quad (5)$$

We then conduct a multivariate regression with monthly term structure of interest rates based on equation (4). Coefficients $\hat{a}_i$, $\hat{b}_i$, and $\hat{c}_i$ are estimated from the sample data to explain unexpected changes in interest rates, and the values $(\hat{a}_i + \hat{b}_i + \hat{c}_i)$, $(-2\hat{c}_i)$, and $(\hat{c}_i - \hat{b}_i)$ are estimated sensitivities for the optimal long-term, medium-term, and short-term spot rates. Finally, the values of $(\hat{a}_i + \hat{b}_i + \hat{c}_i)\Delta r_{l,t} + (-2\hat{c}_i)\Delta r_{m,t} + (\hat{c}_i - \hat{b}_i)\Delta r_{s,t}$ are estimations for the unexpected change in the ith maturity spot rate at time t.

For equation (4), the determination coefficient between the spot rate changes, $\Delta r_{i,t}$, and the changes in the three components of the term structure, $\Delta r_{l,t}$, $(\Delta r_{s,t} - \Delta r_{l,t})$, and $(2\Delta r_{m,t} - \Delta r_{s,t} - \Delta r_{l,t})$, is given by

$$R^2_{i,(l,m,s)} = 1 - \frac{Var(\varepsilon_{i,t})}{Var(\Delta r_{i,t})}, \tag{6}$$

or is alternatively represented as

$$R^2_{i,(l,m,s)} \cdot Var(\Delta r_{i,t}) = Var(\Delta r_{i,t}) - Var(\varepsilon_{i,t}). \tag{7}$$

In estimating the model as specified in equation (7), minimizing the residuals' variance is equivalent to maximizing the left-hand side, $R^2_{i,(l,m,s)} \cdot Var(\Delta r_{i,t})$, of the equation. In order to find optimal proxies for the state variables, Elton *et al.* (1990) propose maximizing the weighted average of $R^2_{i,(l,m,s)} \cdot Var(\Delta r_{i,t})$ over various choices of proxy, r_l, r_m, and r_s, across all maturities. That is, we can identify the optimal spot rates (r_l, r_m, and r_s) by maximizing the objective function, $Max_{(l,m,s)} \sum_i w_i R^2_{i,(l,m,s)} \cdot Var(\Delta r_i)$. The value of w_i is the weight allocated to the ith maturity spot rate. Elton *et al.* suggest two approaches to deal with weight w_i: an equal weights approach and a cash flow weights approach. Because the difference between the two approaches is not significant, we simply choose the equal-weighted scheme for our empirical examinations. Moreover, Elton *et al.* use two alternative models, random walk and pure expectation theory, to estimate unexpected movements in spot rates, $\Delta r_{i,t}$; however, their selection of optimal spot rates is based primarily on the former model, which is also adopted in most of the literature that precedes Elton *et al.* (1990) (see Babbel, 1983; Nelson and Schaefer, 1983). Therefore, the movement of interest rates in the present paper is assumed to follow the random walk assumption, which assumes that the yield curve remains unchanged; therefore, any change in spot rates in any given period is assumed to be unexpected.

2.2. Durations of Optimal Spot Rates

The price of a coupon-bearing bond P can be defined as the discounted value of a series of cash flows, C_i, with respect to its appropriate spot rates, r_{t_i} (for $i = 1, 2, \ldots, n$):

$$P = \sum_{i=1}^{n} \frac{C_i}{(1 + r_{t_i})^{t_i}}. \tag{8}$$

To determine the approximate change in price for a small change in interest rates, the first derivative of equation (8) with respect to the interest rate (r) can be calculated as:

$$\begin{aligned} \frac{dP}{P} &= \frac{1}{P}\sum_{i=1}^{n}\left(\frac{\partial P}{\partial r_{t_i}}\frac{dr_{t_i}}{dr}\right)dr \\ &= \frac{-1}{P}\sum_{i=1}^{n}\left[\frac{t_i C_i}{(1 + r_{t_i})^{t_i}}\frac{1}{(1 + r_{t_i})}\frac{dr_{t_i}}{dr}\right]dr = -MD \times dr. \end{aligned} \tag{9}$$

This latter equation operates under the assumption that the term structure of interest rates carries the same magnitude shift for all interest rates (parallel shift) and, therefore, $dr_{t_i}/dr = 1$, and the MD term represents the modified Macaulay duration (modified duration) of the bond, a measure of the sensitivity of the bond price with respect to the shift change in interest rates. The minus sign in equation (9) shows an inverse relationship between the percentage change in price (dP/P), and the modified duration, MD (usually a positive value), for a given interest rate change (dr). This reflects the fact that bond prices move in the opposite direction of the change in interest rates. Although the modified duration is a popular tool used by bond investors for interest-rate risk management, it is a good measure only for a small shift in all interest rates, and cannot capture the effect from changes in the slope and curvature of the term structure of interest rates. The three-spot-rate model developed in this paper remedies this drawback.

Spot-rate durations are analogous to modified duration; they appraise the sensitivity of portfolio value with respect to each of the spot rates. When the optimal three-spot-rate model is considered, the first derivative of equation (8) with respect to the optimal spot rates (r_j, for $j = l$, m, s) becomes

$$\begin{aligned} \frac{dP}{P} &= \frac{1}{P}\sum_{i=1}^{n}\sum_{j=l,m,s}\left(\frac{\partial P}{\partial r_{t_i}}\frac{\partial r_{t_i}}{\partial r_j}dr_j\right) \\ &= \frac{1}{P}\sum_{i=1}^{n}\left(\frac{\partial P}{\partial r_{t_i}}\frac{\partial r_{t_i}}{\partial r_l}dr_l + \frac{\partial P}{\partial r_{t_i}}\frac{\partial r_{t_i}}{\partial r_m}dr_m + \frac{\partial P}{\partial r_{t_i}}\frac{\partial r_{t_i}}{\partial r_s}dr_s\right), \\ &= -D_l dr_l - D_m dr_m - D_s dr_s \end{aligned} \tag{10}$$

which can be represented in the discretized form:

$$\frac{\Delta P}{P} \approx -D_l\Delta r_l - D_m\Delta r_m - D_s\Delta r_s, \tag{11}$$

where D_l, D_m, and D_s denote bond durations corresponding to the optimal long-term (r_l), medium-term (r_m), and short-term (r_s) spot rates, respectively. Based on equations (5) and (8), and some differential calculation, D_l, D_m, and D_s can be estimated as follows:

$$\begin{aligned} \hat{D}_l &= \frac{1}{P}\sum_{i=1}^{n} t_i\cdot(\hat{a}_i + \hat{b}_i + \hat{c}_i)\cdot C_i\cdot(1 + r_{t_i})^{-t_i-1} \\ \hat{D}_m &= \frac{1}{P}\sum_{i=1}^{n} t_i\cdot(-2\hat{c}_i)\cdot C_i\cdot(1 + r_{t_i})^{-t_i-1} \\ \hat{D}_s &= \frac{1}{P}\sum_{i=1}^{n} t_i\cdot(\hat{c}_i - \hat{b}_i)\cdot C_i\cdot(1 + r_{t_i})^{-t_i-1}. \end{aligned} \tag{12}$$

2.3. Risk Management with Duration Hedging and Portfolio Design

As derived above, spot-rate durations for an individual asset are estimated as in equation (12). Such durations can be used to hedge bond portfolios against interest rate risks. For our three-spot-rate model, denote D_j^k (for $j = l$, m, and s) as the kth individual asset's spot-rate durations relative to the three optimal spot rates, r_l, r_m, and r_s. When a portfolio comprising multiple assets is considered, the durations of the portfolio (D_j^P) are the weighted average of the spot-rate durations of each asset in the portfolio, where the weight for each asset is the proportion of total portfolio value contributed by that asset. For instance, simply assume a portfolio consisting of two bonds (i.e. $k = 1, 2$) with values V_1 and V_2 and associated spot-rate durations D_j^1 and D_j^2. Accordingly, the spot-rate durations of the portfolio are:

$$D_j^P = \frac{V_1}{V_1 + V_2} D_j^1 + \frac{V_2}{V_1 + V_2} D_j^2, \text{for } j = l, m, \text{and } s, \tag{13}$$

More generally, for a portfolio consisting of g bonds (for $k = 1, 2, \ldots, g$), its spot-rate durations are:

$$D_j^P = \sum_{k=1}^{g} q_k D_j^k, \text{for } j = l, m, \text{and } s, \tag{14}$$

where the weights q_k are the ratios of each bond value (V_k) to portfolio value ($\sum_{k=1}^{g} V_k$), and may be individually positive or negative but must sum to 1.

2.3.1. Special Portfolios

The purpose of portfolios designed in the paper is to examine the hedging performance of spot-rate models with or without considering the third spot rate. Two kinds of target portfolios, including three special portfolios (hereafter, *SPi*, for $i = 1$, 2, and 3) adapted from Bliss (1997) and a series of random portfolios, are constructed to examine the hedging performance of spot-rate models by using hedge portfolios with matching different number of duration constraints. Bliss (1997) originally construct three special portfolios to compare the Macaulay duration hedging and factor durations hedging, and these three special portfolios are related to specific characteristics of the level, slope, and curvature of term structure movements, respectively. However, different from our model, Bliss (1997) identifies these factors by the principal component analysis rather than by the Elton *et al.* (1990) method.

The first special portfolio *SP*1 holds a single 30-year coupon bond with 5% semiannual coupons. Due to the heavy loading on the level factor and the existence of a partial slope factor, we expect that the hedging performance of modified duration is acceptable and hedging strategies based on two or three spot rates will have superior performance.

The second special portfolio *SP*2 includes equal values of 1-year and 30-year coupon bonds (both paying 5% semiannual coupons). That is, this portfolio consists of variedly divergent-maturity coupon bonds and seems to be sensitive to both changes in levels and changes in the slope of the term structure. Presumably, modified duration hedging will perform even worse on portfolio *SP*2 than on portfolio *SP*1, but two-spot-rate duration hedging might still show reasonable performance in this case.

Finally, the third special portfolio *SP*3 comprises one unit each of 1-year and 30-year zero-coupon bonds in long positions, and one unit of a 16-year zero-coupon bond in a short position. That is, this portfolio mixes long positions on short-maturity and long-maturity bonds (barbell strategy) and a short position on a medium-maturity bond (bullet strategy). The whole portfolio is particularly sensitive to changes in the curvature of the term structure. Intuitively, three-spot-rate duration hedging would capture the interest rate changes of such a portfolio significantly better than modified duration and even two-spot-rate duration hedging.

2.3.2. Hedge Portfolios

The three special portfolios (i.e. *SPi*, for $i = 1$, 2, and 3) and nine hedge portfolios (i.e. *HPiu*, for $i = 1$, 2, and 3; $u = 1$, 2, and 3) related to them are briefly categorized in Table 1.[4] For each special portfolio (e.g. $i = 1$, *SP*1), three hedge portfolios corresponding to $u = 1$, 2, and 3 (i.e. *HP*11, *HP*12, and *HP*13) are constructed to match the value and durations of the portfolio being hedged. This is to immunize the asset–liability (*HP–SP*) portfolio against any dollar price changes due to interest rate moves. By comparing returns between the special portfolio and its three hedge portfolios (i.e. hedging errors), the superior duration-matching model (with smallest error) can be identified.

Corresponding to $u = 1$, the hedge portfolios (i.e. *HPi*1, for $i = 1$, 2, and 3) are, in essence, the modified duration-matching portfolios consisting of two adjacent zero-coupon bonds with maturities 6 months apart,[5] in amounts chosen to match both the value and the modified duration of the special portfolio being hedged.

The two-spot-rate duration-matching portfolios corresponding to $u = 2$ (i.e. *HPi*2, for $i = 1$, 2, and 3) consist of 1-year, 10-year, and 30-year-maturity zero-coupon bonds, in amounts chosen to match the value and both two-spot-rate durations of the special portfolio being hedged.

Similarly, the three-spot-rate duration-matching portfolios corresponding to $u = 3$ (i.e. *HPi*3, for $i = 1$, 2, and 3) are combinations of 1-year, 5-year, 15-year, and 30-year-maturity zero-coupon bonds whose amounts are chosen to match the value and all three spot-rate durations of the special portfolio being hedged.

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[4]Here, i is the index for the special portfolio and $u = 1$, 2 and 3 represent modified duration-matching, two-spot-rate, and three-spot-rate duration-matching portfolios, respectively.

[5]In this work, two adjacent zero-coupon bonds with maturities 6 months apart are chosen for a modified duration-matching portfolio, and these two adjacent zero-coupon bonds are chosen to cover the modified duration of each special portfolio being hedged. The minimum number of assets required to fulfill the duration constraints can be referred to Soto (2001).

Table 1 Specification of special portfolios and hedge portfolios

This table describes the specifications of special and hedge portfolios. Each special portfolio has three kinds of duration-matching portfolio, as defined in hedge portfolios. [a]$u = 1$ represents modified duration-matching portfolios, *HPi*1, for hedging special portfolios, *SPi* (i = 1, 2, and 3), respectively. [b]$u = 2$ represents two-spot-rate duration-matching portfolios, *HPi*2, for hedging special portfolios, *SPi* (i = 1, 2, and 3), respectively. [c]$u = 3$ represents three-spot-rate duration-matching portfolios, *HPi*3, for hedging special portfolios, *SPi* (i = 1, 2, and 3), respectively. [d]l, m, and s are the indices for optimal long-term, medium-term, and short-term spot-rate durations, respectively.

Special portfolios (*SPi*) = Liabilities		Hedge portfolios (*HPiu*) = Assets: Modified duration-matching ($u = 1$)[a]	Two-spot-rate duration-matching ($u = 2$)[b]	Three-spot-rate duration-matching ($u = 3$)[c]
*SP*1 ($i = 1$)	100% synthetic, 5% coupon, 30 year bond	Two adjacent zeros 6 months apart are chosen to match both the value and the modified duration of the special portfolio being hedged; that is, $V^{HPi1} = V^{SPi} = \$100$, and $MD^{HPi1} = MD^{SPi}$	1, 10, and 30 year zeros are chosen to match the value and both two spot-rate durations of the special portfolio being hedged; that is, $V^{HPi2} = V^{SPi} = \$100$, and $\hat{D}_j^{HPi2} = \hat{D}_j^{SPi}$, for $j = l$ and s	1, 5, 15, and 30 year zeros are chosen to match both the value and all three spot-rate durations of the special portfolio being hedged; that is, $V^{HPi3} = V^{SPi} = \$100$, and $\hat{D}_j^{HPi3} = \hat{D}_j^{SPi}$, for $j = l$, m, and s[d]
*SP*2 ($i = 1$)	50% synthetic, 5% coupon, 1 year bond and 50% synthetic, 5% coupon, 30 year bond			
*SP*3 ($i = 1$)	Long each 100%, 1 year zero and 30 year zero, and short 100%, 16 year zero			

2.3.3. Random Portfolios

Instead of up to only three bonds in a special portfolio, portfolios of larger size are used to reconfirm the performance of the three-spot-rate model. Furthermore, the same method described above is applied to generate duration-matching portfolios (or hedge portfolios) for hedging randomly generated bond portfolios. For these random portfolios, we choose a set of randomly distributed parameters (including quantity of bond, time to maturity, and coupon rate) that reasonably reflect bond portfolios of interest to academics and practitioners. The number of bonds in a random portfolio is discretely distributed from 6 to 30 (i.e. 25 different portfolio sizes, each with equal probability 0.04). The time to maturity for each bond is equal-probability distributed from 1 year to 30 years, with an increment of 1 year (i.e. 30 different maturities). The coupon rate for each bond is distributed with equal probability 0.05 and an increment of 0.5%, from 0.5 to 10% (i.e. 20 different coupon rates). Each parameter is selected independently of the others. We simulate 100 random portfolios and, based on these *RP*, the performance of modified duration, two-spot-rate duration, and three-spot-rate duration hedging strategies is compared.

3. Data and Empirical Analysis

Our empirical analysis is divided into two parts in terms of different time periods. In Part I, we use the same dataset (obtained from McCulloch's website)[6] as those in Elton *et al.* (1990), which includes monthly US Treasury term structures of spot rates covering a 30-year period from 1957 to 1986, to compare the results of the three-spot-rate model developed here with that of the optimal two-spot-rate model (opt 2) proposed by Elton *et al.* (1990). Furthermore, more recent 10-year monthly data over the period July 1997 to June 2007 are collected from the same source to examine the robustness of our models and to compare the modified duration and spot-rate duration hedging of various bond portfolios in Part II.[7]

3.1. Part I: January 1957–December 1986 (360 months)

The first dataset constitutes Part I of the empirical study, and includes monthly term structures of spot rates with a series of different maturities over the 30-year period from January 1957 to December 1986. Specifically, for each term structure of

[6]McCulloch and Kwon (1993) originally provided monthly estimates of continuously compounded zero-coupon yields for a series of maturities over the 45-year period from 1947 to 1991. All monthly term structure data represent observations from the last trading day of the month. They are derived from ranges of US Government bond prices and have become an accepted criterion for empirical tests in immunization (e.g. Elton *et al.*, 1990; Nawalkha and Chambers, 1996). The data are available from McCulloch's website: http://economics.sbs.ohio-state.edu/jhm/ts/mcckwon/mccull.htm.

[7]The updated dataset is also available from McCulloch's website: http://economics.sbs.ohio-state.edu/jhm/ts/ts.html.

the first dataset, we select a series of 31 spot rates with various times to maturity (beginning at 1 month, and increasing by monthly intervals to 18 months, quarterly intervals from 18 to 24 months, and yearly intervals from 2 to 13 years), as in Elton *et al.* (1990). The entire 30-year sample period is divided into six 5-year subperiods, with period 1 spanning January 1957 to December 1961, period 2 spanning January 1962 to December 1966, and so on. Among these subperiods, the data from periods 1 and 4 are defined as the in-sample data, and are used only to identify optimal spot rates for the spot-rate models. The remaining data are defined as the out-of-sample periods, and are used to test the fitness of the estimated models.

Following the methodology of Elton *et al.* (1990), the 4-year spot rate (4yr) is identified as the optimal spot rate for the one-spot-rate (opt 1) model through in-sample data from periods 1 and 4, while the optimal spot rates for the two-spot-rate (opt 2) model are the 8-month and the 6-year spot rates (8mo, 6yr). These results are consistent with Elton *et al.* By extending the Elton *et al.* framework with an equally weighted scheme, and with the assumption that innovation in spot rates follows a random walk, we identify the 6-month, 3-year, and 9-year spot rates (6mo, 3yr, and 9yr) as the three optimal proxies for the state variables of the three-spot-rate model (opt 3) in describing unexpected movements in the term structure of interest rates.

The criteria we use to evaluate the fit and performance of the estimated models include computing the mean square errors (MSE) on unexpected changes in interest rates. We then apply paired sample *t*-statistics to test the significance of the differences in mean between these models for each out-of-sample period (60 months) and the overall period, combining four out-of-sample periods (240 months).[8] Sensitivities of optimal spot rates for all models (see equation 5) are estimated from the previous 5-year period,[9] and each maturity is weighted equally to calculate the mean and variance of the mean square errors for each competing model.

[8]First, for each model (i.e. opt 1, opt 2, and opt 3), square errors are computed monthly for the interest rate of each maturity. More specifically, this procedure is repeated each month in examined periods, resulting in a matrix of 60 (months) by 31 (maturities) for single out-of-sample period (i.e. Periods 2, 3, 5, and 6, as shown in Table 2), and 240 (months) by 31 (maturities) for the overall period (combining the four out-of-sample periods into one, as reported in the attached table of Figure 1). Next, the mean square errors (MSE) are calculated for each maturity through 60 months for each single period and 240 months for the overall period. Finally, for each model, the mean and the standard deviation of the MSE (i.e. $\overline{MSE}$) are computed along the dimension of different maturities. In addition, to compare the performance of two models, the paired sample *t*-test is applied to the series of the difference between correspondent MSE with the same maturity in these two competing models.

[9]That is, coefficients estimated from Periods 1, 2, 4, and 5 are applied to explain unexpected changes in interest rates of the out-of-sample Periods 2, 3, 5, and 6, respectively. Periods 1 and 4 are in-sample data, exclusive of estimation.

Table 2 Performance of explanation ability and paired sample *t*-test of competing models for various out-of-sample periods

This table shows the mean and standard deviation of mean square errors (i.e. $\overline{MSE}$) of three optimal spot-rate models over various out-of-sample periods. Based on the data in periods 1 and 4, the opt 1, opt 2, and opt 3 models denote using (4-year), (8-month, 6-year), and (6-month, 3-year, 9-year) optimal spot rates as state variables, respectively. The mean, variance, and *t*-statistics of paired sample *t*-tests of the mean square errors calculated across varied maturities among these optimal spot-rate models for each out-of-sample time period are provided. The figures 6.94, 1.87, and 8.68 from the last two columns of period 2 denote the *t*-statistics for the paired sample *t*-test under the one-tailed tests of opt 1 versus opt 2 (i.e. $H_0: \overline{MSE}_{opt1} \le \overline{MSE}_{opt2}$ and $H_a: \overline{MSE}_{opt1} > \overline{MSE}_{opt2}$), opt 2 versus opt 3 (i.e. $H_0: \overline{MSE}_{opt2} \le \overline{MSE}_{opt3}$ and $H_a: \overline{MSE}_{opt2} > \overline{MSE}_{opt3}$), and opt 1 versus opt 3 (i.e. $H_0: \overline{MSE}_{opt1} \le \overline{MSE}_{opt3}$ and $H_a: \overline{MSE}_{opt1} > \overline{MSE}_{opt3}$), respectively. SD refers standard deviation. *** or ** asterisks associated with the *t*-statistics indicate differences significant at the 1 or 5%, level, respectively.

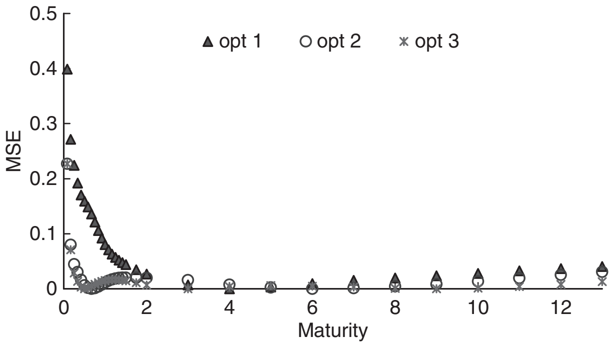
Period	Model	$\overline{MSE}$ (SD)	*t*-statistics	
2 (1962–1966)	opt 1	0.0122 (0.0101)	6.94***	8.68***
	opt 2	0.0041 (0.0071)	1.87**	
	opt 3	0.0033 (0.0073)		
3 (1967–1971)	opt 1	0.0464 (0.0326)	6.11***	7.15***
	opt 2	0.0168 (0.0188)	2.11**	
	opt 3	0.0129 (0.0156)		
5 (1977–1981)	opt 1	0.1856 (0.2177)	4.89***	5.13***
	opt 2	0.0395 (0.0867)	1.79**	
	opt 3	0.0352 (0.0947)		
6 (1982–1986)	opt 1	0.1056 (0.1049)	6.10***	6.67***
	opt 2	0.0307 (0.0540)	4.98***	
	opt 3	0.0201 (0.0476)		

As reported in Table 2, the mean value of MSE (defined as $\overline{MSE}$) cross the whole term structure of the opt 3 model is always significantly lower than that of the opt 2 and opt 1 models over each out-of-sample period. For instance, in period 2, the $\overline{MSE}$ of the opt 1 model (0.0122) is significantly higher than that of the opt 2 and opt 3 models (0.0041 and 0.0033) at the 1% level, with *t*-statistics of 6.94 and 8.68, respectively. Additionally, the $\overline{MSE}$ of the opt 3 model is smaller than that of the opt 2 model, at the 5% level, with a *t*-statistic of 1.87. The results of paired sample *t*-tests are consistent in each subperiod.[10]

Figure 1 plots the MSE results of different models with respect to different maturities for the overall period (i.e. a combination of the four out-of-sample periods). The figures in the last two columns, 5.47, 3.40, and 5.90, denote the *t*-statistics

[10]One might argue that the opt 2 model will have one more exact match explaining the unexpected changes in interest rates than the opt 1 model (as with the opt 3 versus opt 2 models). The authors also modify the MSE computed from the opt 1, opt 2, and opt 3 models by multiplying by 31/30, 31/29, and 31/28, respectively, to eliminate this effect. The conclusion still holds.

Figure 1 Overall out-of-sample performance of competing models.



Period	Model	$\overline{MSE}$ (SD)	*t*-statistics	
Overall	opt 1	0.0875 (0.0905)	5.47***	
	opt 2	0.0228 (0.0410)		5.90***
	opt 3	0.0179 (0.0410)	3.40***	

This figure displays the mean square errors (MSE) of three optimal spot-rate models across various maturities. The attached table shows the mean and standard deviation of mean square errors (i.e. $\overline{MSE}$) and *t*-statistics of paired sample *t*-test results over time among these models for the overall four out-of-sample periods (periods 2, 3, 5, and 6). The figures 5.47, 3.40, and 5.90 denote the *t*-statistics for paired sample *t*-test under the one-tailed tests of: opt 1 versus opt 2 (i.e. $H_0 : \overline{MSE}_{opt1} \leq \overline{MSE}_{opt2}$ and $H_a : \overline{MSE}_{opt1} > \overline{MSE}_{opt2}$), opt 2 versus opt 3 (i.e. $H_0 : \overline{MSE}_{opt2} \leq \overline{MSE}_{opt3}$ and $H_a : \overline{MSE}_{opt2} > \overline{MSE}_{opt3}$), and opt 1 versus opt 3 (i.e. $H_0 : \overline{MSE}_{opt1} \leq \overline{MSE}_{opt3}$ and $H_a : \overline{MSE}_{opt1} > \overline{MSE}_{opt3}$), respectively. *** The three asterisks associated with the *t*-statistics indicate a difference significant at the 1% level during this 240-month sample period.

for paired sample *t*-test under the one-tailed tests of opt 1 versus opt 2 ($H_0 : \overline{MSE}_{opt1} \leq \overline{MSE}_{opt2}$ and $H_a : \overline{MSE}_{opt1} > \overline{MSE}_{opt2}$), opt 2 versus opt 3 ($H_0 : \overline{MSE}_{opt2} \leq \overline{MSE}_{opt3}$ and $H_a : \overline{MSE}_{opt2} > \overline{MSE}_{opt3}$), and opt 1 versus opt 3 ($H_0 : \overline{MSE}_{opt1} \leq \overline{MSE}_{opt3}$ and $H_a : \overline{MSE}_{opt1} > \overline{MSE}_{opt3}$) models, respectively, and they demonstrate that all these *t*-statistics have differences significant at the 1% level during this 240-month sample period. Patterns showing that higher MSE generally occur at shorter maturities are consistent in each subperiod and are in accordance with Navarro and Nave (2001). Overall, the opt 3 model has a lower MSE than the opt 2 and opt 1 models across the entire maturity spectrum. Because of its significantly smaller MSE than the opt 1 and opt 2 models in each subperiod, as well as in the overall period, the opt 3 model shows explanation superiority. As a result, the opt 3 model provides better estimation with respect to unexpected changes in interest rates and is significantly superior to the opt 1 and even the opt 2 models proposed by Elton *et al.* (1990).

3.2. Part II: July 1997–June 2007 (120 months)

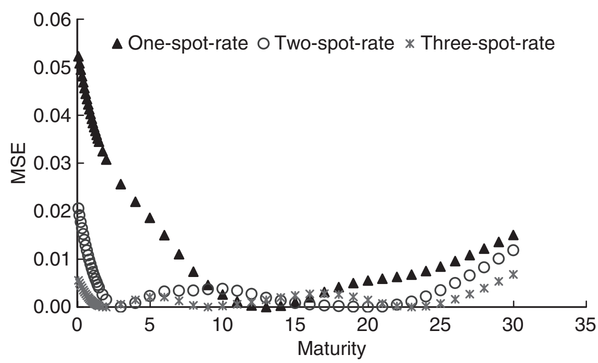
The second part of this empirical study verifies the robustness of the model's prediction ability using updated and extended data; furthermore, it compares the results of portfolio hedging by matching modified duration and spot-rate durations for executing immunization strategies.

3.3. Optimal Spot Rates and Explanation Ability

The updated monthly term structure of spot rates that cover the period July 1997–June 2007 (120 months) is divided into two subsamples. The first 60-month period is defined as the in-sample data used to identify the optimal spot rates for the spot-rate models, and the remaining 60-month period is defined as the out-of-sample data for examining the prediction ability and the explanatory power of the estimated models. To extend the empirical evidence for matching US government bonds' longest time to maturity, which can be up to 30 years, we lengthen the maturity of the term structure data to 30 years for this Part II. As a result, each term structure of spot rates for the second dataset comprises a series of 48 maturities (including the 31 maturities defined in Part I, plus 17 additional annual maturities, which are 14, 15, …, and 30 years).

Using the same methodology as in Part I with the updated in-sample data from July 1997 to June 2002, we identify the 13-year spot rate (13yr) as the optimal spot rate for the one-spot-rate model. For the two-spot-rate model, the optimal spot

Figure 2 Out-of-sample performance of competing models for updated samples.



Period	Model	$\overline{MSE}$ (SD)	*t*-statistics	
July 2002 – June 2007 (out-of-sample of Part II)	One-spot-rate	0.0219 (0.0177)	7.98***	7.98***
	Two-spot-rate	0.0059 (0.0056)	6.03***	
	Three-spot-rate	0.0019 (0.0016)		

This top figure displays mean square errors (MSE) of three optimal spot-rate models across varied maturities with updated data over the 5-year out-of-sample period from July 2002 to June 2007. All monthly term structure data represents observations on the last trading day of each month. The attached table shows the mean and standard deviation of mean square errors (i.e. $\overline{MSE}$) and *t*-statistics of paired sample *t*-test results among these models. The optimal spot rates for one-spot-rate, two-spot-rate, and three-spot-rate models with updated samples are (13 year), (3, 20 year), and (2, 9, and 23 year), respectively. The figures, 7.98, 6.03, and 7.98, denote the *t*-statistics for paired sample *t*-test of one- versus two-spot-rate, two- versus three-spot-rate, and one- versus three-spot-rate models, respectively, and show that all *t*-statistics have differences significant at the 1% level during the out-of-sample period of July 2002–June 2007.

rates are the 3-year and 20-year spot rates (3yr, 20yr) simultaneously. For the three-spot-rate model, the optimal spot rates are the 2-year, 9-year, and 23-year spot rates (2yr, 9yr, and 23yr). Extending the maturity of the term structure of spot rates in Part II up to as long as 30 years results in longer maturities of optimal spot rates relative to those in Part I.

We reconfirm the explanation ability of these models with respect to unexpected changes in interest rates using the out-of-sample data from July 2002 to June 2007. Figure 2 illustrates that the $\overline{MSE}$ for the three-spot-rate model is still smallest (0.0019, compared with the two-spot-rate and one-spot-rate models, 0.0059 and 0.0219, respectively), and the differences in $\overline{MSE}$ between each spot-rate-model pair are significant at the 1% level. These results once again support the superior explanation ability of the three-spot-rate model relative to its counterparts, the one-spot-rate and two-spot-rate models.

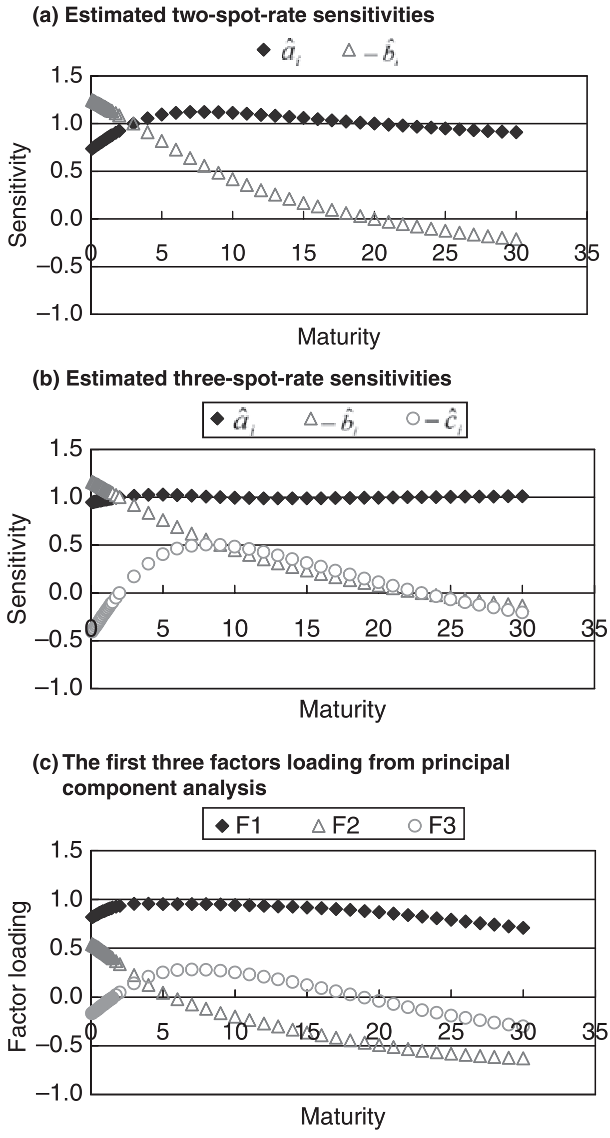
3.4. Estimated Consequent Sensitivities

After optimal spot rates are selected, the consequent sensitivities of the changes in these proxy factors to each spot rate on the term structure can be estimated based on the regression of equations (4) or (5). Figure 3(a) and (b) exhibits the sensitivities of the two-spot-rate ($\hat{a}_i$ and $-\hat{b}_i$) and the three-spot-rate ($\hat{a}_i, -\hat{b}_i,$ and $-\hat{c}_i$) models estimated from July 1997 to June 2002, respectively. The estimated sensitivities are used to calculate their corresponding durations. For further comparison, Figure 3(c) illustrates the first three factors loading ($F1$, $F2$, and $F3$) from principal component analysis, which have been chosen to represent the level, slope and curvature of the term structure of interest rates in many published studies (e.g. Steeley, 1990; Litterman and Scheinkman, 1991; Knez *et al.*, 1994; Willner, 1996; Bliss, 1997; Byun and Lee, 2009).

By comparing Figure 3(a) and (b), we can infer the effect of incorporating the third spot rate. For the two-spot-rate model in Figure 3(a), the humped shape of $\hat{a}_i$ can be viewed as a mixed influence of level and curvature. Although the third spot rate is introduced in Figure 3(b), it slightly affects the slope ($-\hat{b}_i$) of the two-spot-rate model, but indeed extracts curvature influence ($-\hat{c}_i$) from $\hat{a}_i$ in Figure 3(a) of the two-spot-rate model. Therefore, the level coefficient $\hat{a}_i$ of the three-spot-rate model (in Figure 3(b)) becomes an approximately flat line. Moreover, in comparison with Figure 3(c), the pattern of the sensitivity $-\hat{c}_i$ in Figure 3(b) has a similar characteristic to the loading of the third factor ($F3$). From principal component analysis, we can infer that this evidence supports that the third spot rate captures the curvature of the term structure of interest rates.

As described in equations (9) and (12), we also explore the hedging performance of modified duration and spot-rate durations (i.e. $\hat{D}_l, \hat{D}_m,$ and $\hat{D}_s$). Continuous curves of the term structure of interest rates and spot-rate sensitivities are more convenient for estimating spot-rate durations for cash flows with any time to maturity. We simply employ the widely used cubic spline method to perform the curve fitting for these coefficients with respect to different maturities.

Figure 3 Estimated sensitivities of the two-spot-rate and three-spot-rate model versus principal component analysis for updated samples.



(a, b and c) Display estimated sensitivities of the two- and three-spot-rate models from the updated in-sample data (from July 1997 to June 2002). The two- and three-spot-rate models respectively employ 3- and 20-year spot rates and 2-, 9-, and 23-year spot rates as their optimal state variables. For comparative purposes, (c) illustrates the first three factors loading from principal component analysis.

3.5. A Comparison of Modified and Spot-rate Duration Hedging

In this Subsection, we investigate whether the third spot rate added to a spot-rate model enhances the explanation of the term structure of interest rates, especially with respect to curvature, and whether it improves the efficiency of prediction on portfolio returns by comparing the performance of modified duration hedging with two-spot-rate and three-spot-rate duration hedging. The empirical data for execution of duration hedging are the same term structures of interest rates obtained from McCulloch's website over the period July 1997–June 2007, as described in Subsection 3.3.

The objective of duration hedging is the construction of a portfolio that is immunized to changes in interest rates. More specifically, for each special portfolio (*SP*) or random portfolio (*RP*), a hedge portfolio (*HP*) is constructed to match the value and the modified or spot-rate durations of the target portfolio. In our experiments, following Bliss (1997), the face values of the component bonds in the *SP* and *RP* (defined in Subsection 2.3) are adjusted such that the initial value of each component bond and the initial investment amount of the portfolio are \$100 at the beginning of each month. In contrast, the initial investment amount of the *HP* is fixed at \$100, and the values (or the weights) of the component bonds in the *HP* are determined by solving a set of simultaneous equations specifying the duration matching criteria. Taking the three-spot-rate model as an example, we need to solve the weights, q_1, q_2, q_3, and q_4, of the four component bonds. The set of simultaneous equations consists of four equations: three equations for spot-rate duration matching, that is, $\hat{D}_j^{HP} = \hat{D}_j^{SP}$, for $j = l$, m, and s, according to equations (12) and (14), and one equation to assure that the weights sum to 1 ($\sum_{k=1}^{4} q_k = 1$). Once equipped with the solutions of q_k, the corresponding values of the component bonds can be decided through $V_k = 100q_k$. In addition, the face value of each component bond can be derived based on the value V_k and the real term structure at the beginning of the examined period. These component bonds and, therefore, "*SP*"s (or "*RP*"s) and "*HP*"s are revalued on the last trading day of each month according to the real term structure at the end of the examined time period. Finally, the absolute difference between the values of *SP* (or *RP*) and *HP* for each month is computed to analyze hedge performance.

Similar to the methodology in Elton *et al.* (1990), the 120-month data are split up equally into in-sample and out-of-sample periods. The spot-rate durations, as derived in equation (12), are calculated by applying sensitivities $\hat{a}_i$, $\hat{b}_i$, and $\hat{c}_i$ estimated over the first 60-month in-sample period (from July 1997 to June 2002), as reported in Figure 3(b). Because the coefficients $\hat{a}_i$, $\hat{b}_i$, and $\hat{c}_i$ are estimated based on the in-sample data, in the out-of-sample period (from July 2002 to June 2007), we would expect the performance of spot-rate models to worsen over time, especially when sensitivity estimates are not updated. In practice, more frequent recalibrations might yield precise estimation but might also induce frequent trading turnover, thereby increasing transaction costs. Our purpose, however, is only to illustrate the relative performance of these hedging strategies under fair conditions.

3.5.1. Duration Hedging on Special Portfolios

Table 3 presents the performance of portfolio hedging on *SP* using modified duration and spot-rate durations with 10-year monthly term structures from July 1997 to

June 2007. Panels A, B, and C present the results for in-sample, out-of-sample, and overall-sample periods, respectively. From the paired sample *t*-test for these methods, the results strongly support the notion that hedge strategies based on two-spot-rate and three-spot-rate durations outperform hedge strategies based on the modified duration. In addition, modified duration does not provide acceptable performance for hedging a portfolio with widely divergent cash flows, like portfolios *SP*2 and *SP*3, in which nonparallel shifts or intense term structure movements occur. As expected, the mean of the absolute value of monthly return errors of two-spot-rate duration hedging for portfolios *SP*1 and *SP*2 is fairly close to that of three-spot-rate duration hedging (e.g. 0.0002 versus 0.0000 and 0.0001 versus 0.0000 for *SP*1 and *SP*2 for the in-sample period, respectively). However, for portfolio *SP*3, the mean value of hedging errors of the three-spot-rate durations is smaller than that of the other two methods, with *t*-statistics indicating a difference significant at the 1% level for all three periods. For example, the hedging errors of the 10-year overall sample period are 0.0006, 0.0046, and 0.0228 for three-spot-rate, two-spot-rate, and modified durations, respectively. These results show that incorporating the third spot rate into the two-spot-rate model not only enhances understanding of the feature of curvature, but also significantly reduces hedging errors, by approximately 40 basis points on average (for the entire 120 months).

3.5.2. Duration Hedging on Random Portfolios

Duration hedging results for the three *SP* in Table 3 do not represent a sufficiently large sample to draw firm conclusions about our three-spot-rate model. Therefore, larger and more random bond portfolios are needed to evaluate the efficiency of the spot-rate model. These *RP* are simulated by choosing a random distribution of parameters, as described in Subsection 2.3.3. We randomly generate 100 *RP* with different quantities of bonds (6–30 types), times to maturity (1–30 years), and coupon rates (0.5–10%). For purposes of comparison, the same 100 *RP* are applied to each of the 10-year monthly term structures from July 1997 to June 2007.

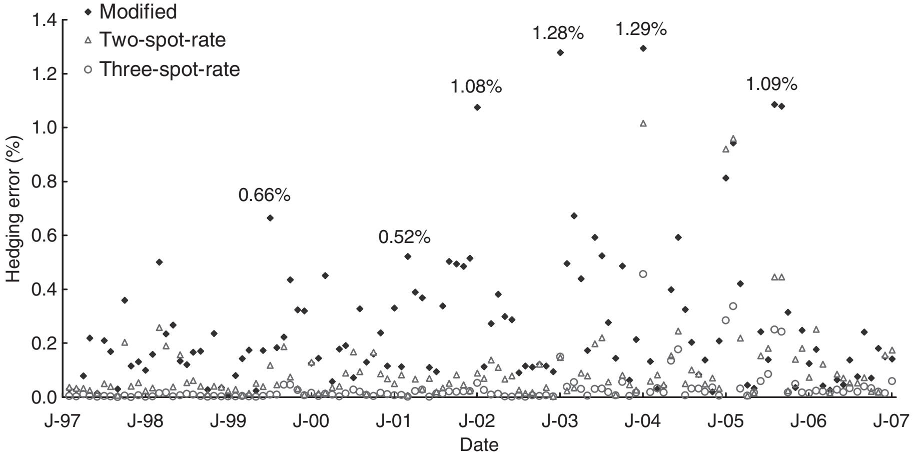
Figure 4 shows the hedging performance of various duration-matching methods on *RP* across the 120-month period. The data consist of the average of absolute values of hedging errors on 100 *RP*. Results from the first 60 months illustrate the domination of the three-spot-rate duration-matching method in the in-sample period. Even in the out-of-sample period, the three-spot-rate method outperforms the two-spot-rate and modified duration-matching methods. Table 4 provides the mean, the standard deviation and the paired sample *t*-test of the absolute value of monthly duration hedging errors between each duration-matching method pair. The results still show that hedge strategies based on two-spot-rate and three-spot-rate durations outperform those based on modified duration. For example, the average hedging errors on 100 *RP* over the overall sample period are 0.03, 0.10, and 0.26% for three-spot-rate, two-spot-rate, and modified durations, respectively. Although the differences in hedging errors among *RP* are smaller than those of *SP*, all *t*-statistics in Table 4 are significant at the 1% level across the entire period. These results reconfirm that incorporating the third spot rate into the two-spot-rate

Table 3 Hedging performance of modified versus spot-rate duration-matching methods on special portfolios

This table presents the performance of portfolio hedging using modified duration and spot-rate durations with 10-year monthly term structures. Panels A, B, and C show results for in-sample, out-of-sample and overall-sample periods, respectively. From the paired sample *t*-test of the absolute value of monthly duration hedging errors, the results strongly support the notion that hedge strategies based on two-spot-rate and three-spot-rate durations outperform hedge strategies based on modified duration. Adding the third spot rate into the two-spot-rate model could significantly reduce hedging errors, by 40 basis points, and enhance understanding of curvature, as shown for hedging special portfolio *SP*3 within an overall sample period. SD refers standard deviation. ***indicate a difference significant at the 1% level.

Special Portfolios (*SP*)	Duration-matching	Mean (SD)	*t*-statistics	
Panel A: In-sample period (July 1997–June 2002)				
*SP*1	Modified	0.0065 (0.0069)	7.04***	7.17***
	Two-spot-rate	0.0002 (0.0002)	5.95***	
	Three-spot-rate	0.0000 (0.0000)		
*SP*2	Modified	0.0082 (0.0074)	8.48***	8.53***
	Two-spot-rate	0.0001 (0.0001)	5.97***	
	Three-spot-rate	0.0000 (0.0000)		
*SP*3	Modified	0.0242 (0.0238)	7.48***	7.85***
	Two-spot-rate	0.0029 (0.0028)	7.67***	
	Three-spot-rate	0.0001 (0.0001)		
Panel B: Out-of-sample period (July 2002–June 2007)				
*SP*1	Modified	0.0062 (0.0082)	5.13***	5.75***
	Two-spot-rate	0.0015 (0.0030)	3.69***	
	Three-spot-rate	0.0002 (0.0004)		
*SP*2	Modified	0.0063 (0.0056)	8.31***	8.71***
	Two-spot-rate	0.0007 (0.0015)	3.62***	
	Three-spot-rate	0.0001 (0.0002)		
*SP*3	Modified	0.0213 (0.0266)	5.50***	6.14***
	Two-spot-rate	0.0064 (0.0106)	4.76***	
	Three-spot-rate	0.0010 (0.0020)		
Panel C: Overall-sample period (July 1997–June 2007)				
*SP*1	Modified	0.0063 (0.0076)	8.57***	9.07***
	Two-spot-rate	0.0008 (0.0022)	3.89***	
	Three-spot-rate	0.0001 (0.0003)		
*SP*2	Modified	0.0072 (0.0066)	11.52***	11.95***
	Two-spot-rate	0.0004 (0.0011)	3.79***	
	Three-spot-rate	0.0001 (0.0001)		
*SP*3	Modified	0.0228 (0.0252)	9.15***	9.85***
	Two-spot-rate	0.0046 (0.0079)	6.73***	
	Three-spot-rate	0.0006 (0.0015)		

Figure 4 Hedging performance of modified versus spot-rate duration-matching methods on random portfolios.



This figure displays the hedging performance of various duration-matching methods on random portfolios across the 120-month period July 1997–June 2007. The data are averaged from the absolute value of hedging errors on 100 random portfolios. The six figures above "◆" are each the first three largest modified-duration hedging errors for in-sample and out-of-sample periods, respectively.

model not only improves the understanding of term structure but also reduces hedging error and variance. Moreover, the results support the notion that the estimated sensitivities of spot rates are sustainable over a wide horizon; that is, hedging performance does not deteriorate significantly even if sensitivities of spot rates are not re-estimated after as much as 5 years. The persistence of the sensitivity of spot rates suggests that the three-spot-rate duration hedging strategy is a more convenient and easier way for those using this technique to manage interest rate risk.

Figures 5(a) and (b) display three vivid examples corresponding to the three largest errors associated with modified duration hedging strategies in Figure 4, for in-sample and out-of-sample periods, to illustrate the significant changes in slope and curvature of the term structures. The monthly dynamics of the term structure of interest rates are estimated by McCulloch and are observed from the last trading day of each month.

The first graph of Figure 5(a) reveals that the 1-month spot rate increases 40 basis points, but the 30-year spot rate decreases 30 basis points; that is, the term structures become flatter from December 1999 to January 2000. In contrast, the short-term spot rate plunges 89 basis points and the long-term spot rate increases 21 basis points in the second graph of Figure 5(a); that is, the term structures become steeper from August 2001 to September 2001. The average hedging errors for modified, two-spot-rate and three-spot-rate duration hedging within these two periods are: 0.66, 0.52; 0.12, 0.01; and 0.01, 0.00%, respectively. The third graph of Figure 5(a) shows that the 1-month spot rate increases 35 basis points, but that the

Table 4 Hedging performance of modified versus spot-rate duration-matching methods on random portfolios

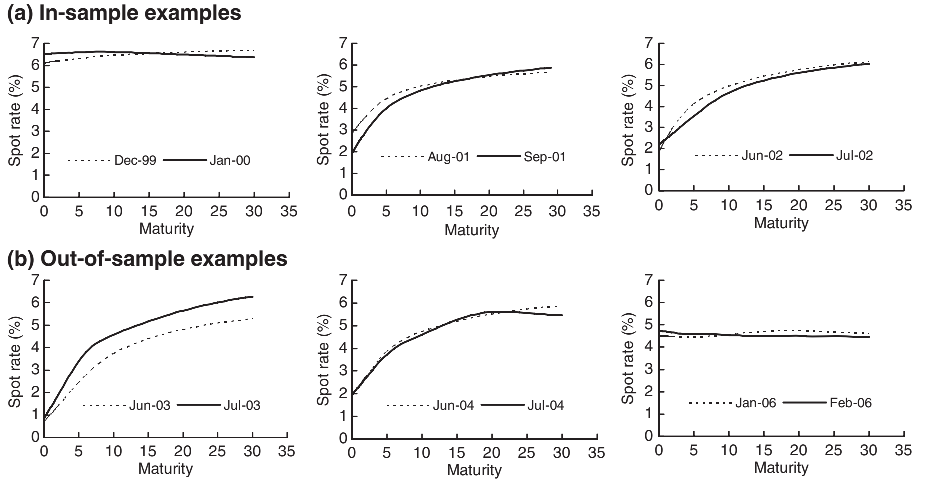
This table demonstrates the performance of hedging on random portfolios using modified duration and spot-rate durations with 10-year monthly term structures. Panels A, B, and C show results for in-sample, out-of-sample, and overall-sample periods, respectively. From the paired sample *t*-test of the absolute value of monthly duration hedging errors, the results strongly support the notion that hedge strategies based on two-spot-rate and three-spot-rate durations outperform hedge strategies based on the modified duration. SD refers standard deviation. *** asterisks on the upper right side of the *t*-statistics indicate a difference significant at the 1% level during this 10-year sample period.

Random Portfolio (*RP*)	Duration-matching	Mean (SD)	*t*-statistics	
Panel A: In-sample period (July 1997–June 2002)				
RP	Modified	0.0023 (0.0019)	6.94***	9.15***
	Two-spot-rate	0.0006 (0.0006)	7.49***	
	Three-spot-rate	0.0001 (0.0001)		
Panel B: Out-of-sample period (July 2002–June 2007)				
RP	Modified	0.0030 (0.0031)	5.42***	7.62***
	Two-spot-rate	0.0014 (0.0021)	5.08***	
	Three-spot-rate	0.0005 (0.0009)		
Panel C: Overall-sample period (July 1997–June 2007)				
RP	Modified	0.0026 (0.0026)	8.66***	11.63***
	Two-spot-rate	0.0010 (0.0016)	7.36***	
	Three-spot-rate	0.0003 (0.0007)		

30-year spot rate drops 12 basis points from June 2002 to July 2002; the corresponding hedging errors for modified, two-spot-rate and three-spot-rate duration hedging over this period are 1.08, 0.09, and 0.05%, respectively.

For out-of-sample examples, the left graph of Figure 5(b) shows that the short-term spot rate increases only 18 basis points, whereas the long-term spot rate dramatically increases by 98 basis points from June 2003 to July 2003. That is, the spread between 1-month and 30-year spot rates widens 80 basis points during the period. In contrast, in the second graph of Figure 5(b), two term structures cross each other more frequently. That is, the curvature of the term structure of interest rates changes profoundly over the period June 2004–July 2004. We expect a significant improvement in hedging performance from incorporating the third optimal spot rate, because of its superior ability to explain the curvature of the term structure of interest rates for the three-spot-rate model. The final graph of Figure 5(b) shows that the 1-month spot rate increases 21 basis points, but that the 30-year spot rate decreases 15 basis points from January 2006 to February 2006. Similarly, for these three out-of-sample examples, the average hedging errors of modified, two-spot-rate and three-spot-rate duration-matching methods are: 1.28, 0.15, and 0.15%; 1.29, 1.02, and 0.46%; and 1.09, 0.45, and 0.25%, respectively.

Figure 5 Examples of changes in slope and curvature of the term structure of interest rates.



(a,b) Display each three monthly dynamics of the term structure of interest rates (observed from the last trading day of the month) corresponding to the first three largest hedging errors in Figure 4, for in-sample and out-of-sample periods, respectively. The term structures of interest rates are estimated by McCulloch and reported in his website (http://economics.sbs.ohio-state.edu/jhm/ts/ts.html). This figure demonstrates that large hedging errors for the modified duration-matching correspond to significant changes in slope and curvature of term structures.

According to the above analysis, it can be concluded that these three examples illustrate the significant changes in slope and curvature of the term structures, and the more severe the changes in the term structure of interest rates, the larger the difference in hedging error between modified and spot-rate duration-matching methods. In addition, the three-spot-rate model always shows its superiority to the two-spot-rate model for reducing hedging errors, especially when there is a significant change in the curvature of the term structure of interest rates.

In order to explore the effect of investment horizon longer than 1 month, the six term structures of interest rates corresponding to the three largest hedging errors for in-sample and out-of-sample periods in Figure 5 are chosen to investigate the hedging performance of different models on the 100 *RP* for longer investment horizons. Table 5 reports the hedging performance of modified duration and spot-rate durations for these 6-month term structures given the investment horizon of 2, 3 and 4 months. The paired sample *t*-test of the absolute value of monthly duration hedging errors is performed and the two and three asterisks on the upper right side of the *t*-statistics indicate a difference significant at the 5 and 1% level, respectively. The results show that the hedging performance between the two-spot-rate and the modified duration matching are no longer significantly different while the investment horizon is longer than 1 month. However, the hedging errors of the three-spot-rate model are still within an acceptable level, even for a longer investment horizon (e.g. the mean

Table 5 Effect of longer investment horizons on hedging performance

This table shows results of using modified duration and spot-rate durations with six monthly term structures for holding periods as long as 2 months, 3 months, and 4 months, respectively. The six monthly term structures of interest rates are associated to the first three largest hedging errors in Figures 4 and 5, for in-sample and out-of-sample periods. From the paired sample *t*-test of the absolute value of monthly duration hedging errors, the results strongly support the notion that even the longer investment horizon hedge strategies based on three-spot-rate durations still outperform hedge strategies based on two-spot-rate and the modified durations. SD refers standard deviation. ** and *** asterisks on the upper right side of the *t*-statistics indicate a difference significant at the 5% and 1% level, respectively.

Random Portfolio (*RP*)	Duration-matching	Mean (SD)	*t*-statistics	
Panel A: 2-month holding period				
RP	Modified	0.0088 (0.0068)	−0.81	3.25**
	Two-spot-rate	0.0108 (0.0109)	2.54**	
	Three-spot-rate	0.0013 (0.0018)		
Panel B: 3-month holding period				
RP	Modified	0.0096 (0.0052)	0.17	4.59***
	Two-spot-rate	0.0090 (0.0107)	2.08**	
	Three-spot-rate	0.0013 (0.0016)		
Panel C: 4-month holding period				
RP	Modified	0.0126 (0.0056)	0.03	4.04***
	Two-spot-rate	0.0125 (0.0098)	2.77**	
	Three-spot-rate	0.0023 (0.0022)		

of 0.23% for 4-month holding period). The results strongly support that even with a longer investment horizon, hedge strategies based on three-spot-rate durations still outperform hedge strategies based on two-spot-rate and modified durations.

The deterioration of the performance of the two-spot-rate durations for longer horizons further consolidates the contribution of introducing the third spot rate, which not only enhances the understanding of the curvature feature of the term structure of interest rates, but can also reduce the rebalancing frequency and, therefore, effectively save transaction costs of the risk management.

In summary, empirical tests of duration hedging on *SP* and *RP* (both are target portfolios) show that the second spot rate contributes significantly toward lessening hedging error. However, three constraints related to the level, slope, and curvature of term structure movements are necessary to assure a return of *HP* close to that of target portfolios; that is, incorporating the third spot rate not only enriches our understanding of the term structure of interest rates, especially the feature of curvature, but also improves the explanation ability for the dynamics of the term structure and reduces hedging errors for bond portfolios to a sufficiently small magnitude. The excellent explanatory power and the fairly small hedging errors of our three-spot-rate model suggest that adding a fourth or more spot rates into a spot-rate model is unnecessary.[11] These empirical results are consistent with Litterman and Scheinkman (1991), Bliss (1997), and Soto (2004).[12] Furthermore, in contrast to previous models, the spot-rate model simply extracts the optimal spot rates (as driving factors) from the term structure itself, rather than from other sources, which is an important feature of the hedging strategy in this paper.

4. Conclusion

Because the framework in Elton *et al.* (1990) is reliable and performs well in selecting optimal spot rates as proxies for unknown driving state variables of the term structure of interest rates, we propose a three-spot-rate model and identify state variables using the methodology of Elton *et al.* (1990). The concise and elegant three-spot-rate model determined from the information content of the term structure of interest rates itself provides better description for the parallel, slope, and curvature characteristics of the term structure of interest rates than two-spot-rate model.

[11]Higher-degree spot-rate models might enhance both the performance of estimating the unexpected changes of the term structure of interest rates and the performance of hedging bond portfolios. However, the improvement might be marginal or even insignificant. In addition to the results of one-spot-rate, two-spot-rate, and three-spot-rate models reported in Figure 1, we further provide empirical evidence for the same examined period that the optimal four-spot-rate-model (opt 4 model with 2-month, 13-month, 3-year, and 9-year spot rates) does not significantly outperform the optimal three-spot-rate model (opt 3 model with 6-month, 3-year, and 9-year spot rates) in prediction ability of unexpected changes of the term structure. The $\overline{MSE}$ (standard deviation) for opt 4 is 0.0102 (0.0168), smaller than that of opt 3, 0.0179 (0.0410), but the t-statistic for the paired sample t-test of the difference between opt 3 and opt 4 is insignificant (1.46). Furthermore, our results are consistent with the findings in Soto (2004, *Journal of Banking and Finance* 28, p. 1089), in which Soto argues that: "the number of risk factors considered has a greater influence on the result than the particular model chosen and three-factor immunization strategies offer the highest immunization benchmark."

[12]There are several differences between Soto (2004) and the present paper. First, Soto (2004) introduces a three-factor model based on the principal component analysis to implement immunization strategies, but the three-spot-rate model (optimal keyrate model in Soto's term) proposed in this paper is not considered in Soto (2004). Second, Soto (2004) adopts the maximum diversification criterion to construct hedging portfolios, but our method is to minimize the estimation error for individual rates and to diminish the difference between the returns of target and hedge portfolios. Third, we focus on the US Treasury market, which is the largest in the world, whereas the Spanish Treasury market is examined in Soto (2004). Finally, to reflect the practical problems on risk management for financial institutions, a unique feature of the present paper is the examination of different duration-matching methods for randomly generated bond portfolios.

Empirical results show that the model introduced in the present study is consistently superior over a variety of horizons. The optimal three-spot-rate model performs very well in estimating the unexpected change in the dynamics of the term structure and in executing spot-rate duration hedging of various bond portfolios. In addition, the results indicate that incorporating the third spot rate not only enhances the understanding of the curvature feature of the term structure of interest rates, but also implies that it is feasible to reduce the dimensions of state variables, extracted from the term structure itself, to three for the risk management of bond portfolios. This model is more convenient and easier to implement, and likely carries lower transaction costs, in modeling interest rate risk management for investors in bond portfolios.

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