Variance Reduction for Multivariate Monte Carlo Simulation

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In practice, the rate of convergence for Monte Carlo simulation is often unsatisfactory when a large number of underlying variables are involved. The reason behind this deficiency is the mismatch between the prespecified and the sample variance-covariance matrices of the underlying multivariate random samples, upon which the value of the option of interest is highly dependent. In this article, a new method, termed the inverse Cholesky decomposition transformation, is proposed to rectify this problem, in which crude independent standard normal distributed random samples are viewed as weakly correlated, normally distributed random samples transformed from truly independent standard normal distributed random samples via the Cholesky decomposition transformation. In simulation of European calls on the maximum of 10 assets, the proposed method achieves a root mean squared error (RMSE) about one-third of that of the standard Monte Carlo simulation under the same number of simulations. Furthermore, the analysis of the RMSE and the computational time demonstrates this new method's superior efficiency compared with the traditional variance-reduction techniques. Accordingly, the proposed method is suggested as one of the standard procedures in multivariate Monte Carlo simulation.

Since the advent of the pricing formula of European options by Black and Scholes [1973], derivatives markets have become highly evolved. However, not all options currently in financial markets have analytic pricing formulae. For those that lack analytic solutions, various numerical methods, including numerical integration, lattice models, finite difference methods, and Monte Carlo simulation, are employed to find theoretical option values.

Among these methods, Monte Carlo simulation, introduced by Boyle [1977], has been generally regarded as the most flexible and practical. For example, Monte Carlo simulation has shown great success when the payoffs are path dependent or when multiple underlying variables are considered. Still, some deficiencies exist with the Monte Carlo approach. One drawback is that, in many cases, an exceedingly large number of simulations are necessary to obtain satisfactory precision, as the standard error of a Monte Carlo estimate decreases with the square root of the number of simulations. A better alternative to this brute-force method is to incorporate variance-reduction techniques, which improve the rate of convergence and hence generate a more accurate estimate with fewer simulations.

Boyle, Broadie, and Glasserman [1997] surveyed different variance-reduction techniques for Monte Carlo simulation, including the antithetic variate approach, control variate approach, moment matching method, stratification sampling, Latin hypercube sampling, importance sampling, and the conditional Monte Carlo method. In addition to these techniques, Duan and Simonato [1998] introduced another variance-reduction technique known as empirical martingale simulation, in which the martingale property is imposed on the present value of the cross-sectional average of stock prices at every future point in time. This technique is generally viewed as an extension of the moment matching method in the space of the stock price, and significant convergence improvement over standard Monte Carlo simulation is verified.

Nearly all aforementioned variance-reduction techniques are originally designed for the case of one stochastic variable only. To price options on multiple underlying processes, Monte Carlo simulation is more appealing than other numerical methods. Nevertheless, the requisite brute-force simulation to obtain the desired precision highlights the importance of designing variance-reduction techniques tailored for multivariate Monte Carlo simulation. This article seeks to present a new method focusing on variance reduction for multivariate Monte Carlo simulation.

Barraquand [1995] was the first to develop a variance-reduction technique, termed quadratic resampling, for multivariate Monte Carlo simulation. This technique is similar to the multidimensional control variate method, in which the gain matrix replaces the variance-minimizing scalar. Some success has been achieved over standard Monte Carlo simulation on pricing multivariate European rainbow options, but comparisons with traditional variance-reduction techniques are absent in his numerical results. Pellizzari [2001] introduced another type of control variate method to reduce the variance of simulated option prices with multiple underlying assets. It is constructed based on replacing partial underlying values in the payoff function with their unconditional expectations. The numerical results show that this method can effectively reduce the variance.

In addition to the previously mentioned variance-reduction techniques, Galanti and Jung [1997] used low-discrepancy sequences (also termed quasi-Monte Carlo random variables) to achieve high-level accuracy for multivariate Monte Carlo simulation. The simulation's success depends on the evenness with which these low-discrepancy sequences are dispersed throughout the domain of the uniform distribution. In general, the more uniformly these deterministic points are dispersed, the smaller the discrepancy and the higher the level of accuracy. However, many existing studies, for example, Brately, Fox, and Niederreiter [1992], have shown that the performance of low-discrepancy sequences deteriorates with an increase in the number of underlying assets.

In this article, a new variance-reduction technique designed for multivariate Monte Carlo simulation is developed, which does not rely on the concepts of the existing control variate techniques and low-discrepancy sequences. The motivation follows from the fact that the values of options on multiple underlying assets, in general, depend to a great extent on the correlation structure. However, a mismatch occurs between the prespecified and sample correlation structures in multivariate random samples, which significantly erodes the precision of Monte Carlo simulation. Based on the these observations, a novel variance-reduction technique is derived by rectifying the sampling errors in the correlations with independent standard normal distributed random samples. In addition to matching the means and variances, the random samples are also adjusted to ensure that sampling errors of the correlation coefficients vanish.

The remainder of the article is organized as follows. The basic concept of the Monte Carlo simulation, some traditional variance-reduction techniques, and the Cholesky decomposition transformation are described in the next section. In the third section, a new variance-reduction technique termed the inverse Cholesky decomposition transformation designed for multivariate Monte Carlo simulation is proposed. The results of the numerical experiments are presented in the fourth section. The last section concludes.

MONTE CARLO SIMULATION ON OPTION PRICING

According to Black and Scholes [1973], the value of a European call option is the expectation of the present value of its payoff under the risk-neutral probability measure, that is,

$$c = E^{Q}[e^{-rT}\max(S(T)-K,0)]$$

where $S(T)$ is the stock price at maturity T, K is the strike price, r is the risk-free interest rate, and Q represents the risk-neutral probability measure. Under the risk-neutral probability measure, the lognormal random variable $S(T)$ can be simulated as

$$S_i(T) = S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z_i} \text{ for } i = 1, \cdots, M$$

where $S_i(T)$ is the i-th simulation of the terminal stock price, $S(0)$ is the current stock price, σ is the standard deviation of the stock return, $\{z_i\}$ are independent random samples from a standard normal distribution, and M is the number of simulations. In the Monte Carlo simulation, the arithmetic average of the present values of the terminal cash flows is used as an estimate of the option price. Specifically, the estimate can be calculated by

$$\hat{c}(M) = \frac{1}{M}\sum_{i=1}^{M} e^{-rT}\max(S_i(T) - K, 0)$$

If one wants to estimate the price of a European call option on the maximum of multiple underlying assets, $S_i(T)$ can be replaced with $\max(S_{i1}(T), S_{i2}(T), \ldots, S_{iN}(T))$ in the preceding equation, where N is the number of underlying assets and $S_{ij}(T)$ is the i-th simulated price of the j-th underlying asset at maturity T in the risk-neutral world. The estimate of the option value then becomes

$$\hat{c}(M) = \frac{1}{M}\sum_{i=1}^{M} e^{-rT}\max(\max(S_{i1}(T), S_{i2}(T), \cdots, S_{iN}(T)) - K, 0)$$

Instead of increasing the number of simulations to generate more accurate estimates, it is common to improve the convergence rate by variance-reduction techniques. One of the simplest and most widely used variance-reduction techniques is the antithetic variate approach. Following the symmetry of a normal distribution, for every simulated sample z_i, the corresponding antithetic sample $-z_i$ is also used. This method can halve the number of simulations and also reduce the variance of the random samples, thus improving the accuracy of the Monte Carlo simulation method.

The moment matching method is also a commonly used variance-reduction technique, which post-processes the generated simulation samples and forces the generated samples to satisfy certain moment conditions of the prespecified distribution. In this article's experiments, the first and second moments of the underlying normal distribution are matched whenever the moment matching method is employed.

The control variate method is another classic variance-reduction technique. Suppose one is interested in estimating the expected option value, $E[X]$, and has information about another random variable Y, which may be correlated with X in some respects and with the expectation $\mu_Y = E[Y]$. The basic idea of the control variate method is to use the additional information about Y by estimating $E[P] = E[X + \beta(Y - \mu_Y)]$, rather than estimating $E[X]$ directly. In practice, the variance-minimizing parameter β is often estimated directly via the regression of Y on X from the historical data. If ρ_{XY}, the correlation between X and Y, is near −1 or 1, working on $E[P]$ has a strict efficiency improvement over working directly on $E[X]$.

The Cholesky Decomposition Transformation

The Cholesky decomposition transformation method transforms independent standard normal random variables into correlated normally distributed random variables within a given variance-covariance structure. Suppose the prespecified variance-covariance matrix of a N-variate normal distribution is a symmetric $N \times N$ matrix C, in which $\rho_{jk}\sigma_j\sigma_k$ is the covariance between the returns of the j-th and k-th underlying assets. The essence of the Cholesky decomposition is to construct an upper triangular $N \times N$ matrix A satisfying $C = A^T A$. Once the upper triangular matrix A is derived, the independent standard normal distributed random samples $[z_1\ z_2 \cdots z_N]$ can be transformed into correlated normally distributed random samples $[x_1\ x_2 \cdots x_N]$ via the following equation:

$$[x_1\ x_2 \cdots x_N] = z_1\ z_2 \cdots z_N] \times A \qquad (1)$$

Because the random samples $[z_1\ z_2 \cdots z_N]$ are independently standard normal distributed, the variance-covariance matrix of $[x_1\ x_2 \cdots x_N]$ is then

$$E\left[\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}\begin{bmatrix} x_1 x_2 \cdots x_N \end{bmatrix}\right] = E\left[A^T \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}\begin{bmatrix} z_1 z_2 \cdots z_N \end{bmatrix} A\right]$$

$$= A^T E\left[\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}\begin{bmatrix} z_1 z_2 \cdots z_N \end{bmatrix}\right] A = A^T I A = A^T A = C$$

where I is the identity matrix.

The upper triangular matrix A of the Cholesky decomposition exists if C is positive semidefinite, which holds generally except for the cases in which all variables are perfectly positively correlated or when there are theoretical inconsistencies in the correlation structure. When Cholesky decomposition fails, eigenvalue decomposition is often used as an alternative to achieve the same effect. However, eigenvalue decomposition has much higher computational complexity relative to the Cholesky decomposition. Therefore, in the following paragraphs, only the Cholesky decomposition transformation is discussed. All arguments can apply to the eigenvalue decomposition transformation as well.

THE NEW METHOD

In a multivariate Monte Carlo simulation, random variables with a prespecified variance-covariance structure are generated by simulating vectors of independent standard normal random variables and multiplying them by the A matrix obtained from a Cholesky factorization. But, in practice, the sample variance-covariance matrix of the simulated vectors will not turn out to be the identity matrix. In option pricing, due to these variance-covariance errors, the variance-covariance matrix of simulated underlying random samples will not equal exactly the prespecified variance-covariance matrix. Moreover, during the Cholesky decomposition transformation, these variance-covariance errors propagate with the increase of the number of underlying variables, and this error propagation problem further affects the variance-covariance structure of simulated underlying random samples. Therefore, the accuracy of multivariate Monte Carlo simulation is undermined, and a large number of simulations are needed to average out the deviation caused by these problems. In this article, a new variance-reduction technique, termed the inverse Cholesky decomposition transformation, is proposed to control the variance-covariance structure among multivariate normally distributed random samples.

The central idea of the inverse Cholesky decomposition transformation is that because the correlations among independent standard normal distributed random samples are not exactly zero, they can be viewed as weakly correlated, normally distributed random samples transformed from truly independent, standard normally distributed random samples via the Cholesky decomposition transformation. By reversing the procedure of the Cholesky decomposition transformation, it is possible to reconstruct truly independent standard normal random samples.

Suppose there are N underlying assets and one desires to generate M random samples for each underlying asset via multivariate Monte Carlo simulation. Step-by-step details of the inverse Cholesky decomposition transformation are as follows.

Step 1: Generate independent standard normal distributed random samples for each underlying asset and obtain a matrix of random samples,

$$Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1N} \\ z_{21} & z_{22} & \cdots & z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_{M1} & z_{M2} & \cdots & z_{MN} \end{bmatrix} = \left[z_1 z_2 \cdots z_N \right]$$

Step 2: Calculate the variance-covariance matrix $\tilde{C}$ between $\tilde{z}_j = z_j - \hat{\mu}_j 1$ and $\tilde{z}_k = z_k - \hat{\mu}_k 1$, where $\hat{\mu}_j$ and $\hat{\mu}_k$ are the sample means of z_j and z_k, and 1 is a $M \times 1$ vector of ones.[1]

$$\tilde{C} = \begin{bmatrix} Var(\tilde{z}_1) & Cov(\tilde{z}_1, \tilde{z}_2) & \dots & Cov(\tilde{z}_1, \tilde{z}_N) \\ Cov(\tilde{z}_2, \tilde{z}_1) & Var(\tilde{z}_2) & \dots & Cov(\tilde{z}_2, \tilde{z}_N) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(\tilde{z}_N, \tilde{z}_1) & Cov(\tilde{z}_N, \tilde{z}_2) & \dots & Var(\tilde{z}_N) \end{bmatrix} \quad (2)$$

Step 3: Based on the covariance matrix $\tilde{C}$, perform the Cholesky decomposition $\tilde{C} = \tilde{A}^T\tilde{A}$ to obtain the corresponding linear transformation $\tilde{A}$. Find the inverse matrix $\tilde{A}^{-1}$ such that $I = \tilde{A}\tilde{A}^{-1}$, and $\tilde{A}^{-1}$ is referred to as the inverse Cholesky decomposition matrix. If the eigenvalue decomposition is employed as an alternative of the Cholesky decomposition to derive the linear transformation $\tilde{A}$, the corresponding $\tilde{A}^{-1}$ is termed as the inverse eigenvalue decomposition matrix in the following paragraphs.[2]

Step 4: Since the variance-covariance matrix of $[\tilde{z}_1 \tilde{z}_2 \dots \tilde{z}_N]$ is not exactly the identity matrix, it can be viewed as a group of correlated normally distributed random samples obtained from $[z_1' z_2' \dots z_N'] \times \tilde{A}$, where $[z_1' z_2' \dots z_N']$ are truly independent, standard normally distributed random samples. By applying the inverse Cholesky decomposition matrix $\tilde{A}^{-1}$, the matrix of truly independent standard normal distributed random samples Z' can be obtained by

$$Z' = [z_1' z_2' \cdots z_N'] = [\tilde{z}_1 \tilde{z}_2 \cdots \tilde{z}_N] \times \tilde{A}^{-1} = \tilde{Z} \times \tilde{A}^{-1}$$

After completing these four steps, the correlations between each new random vector z_j' are exactly zero, that is, $Cov(z_j', z_k') = 0$. In addition, the mean and the variance of each new random vector z_j' are also standardized to be zero and one simultaneously.

In practice, the inverse Cholesky decomposition transformation (for generating truly independent standard normally distributed random samples) is followed by another application of the Cholesky decomposition transformation (for converting the truly independent standard normally distributed random samples into correlated normally distributed random samples to satisfy the prespecified variance-covariance matrix). For the prespecified variance-covariance matrix C, define the corresponding Cholesky decomposition matrix by A. Then, correlated normally distributed random samples $[x_1 x_2 \cdots x_N]$ with zero means and the variance-covariance matrix C can be derived from $[x_1 x_2 \cdots x_N] = [\tilde{z}_1 \tilde{z}_2 \cdots \tilde{N}] \times A^*$, where $A^* = \tilde{A}^{-1} \times A$. Since $\tilde{A}^{-1}$ is close to an identity matrix I, A^* is, in general, not far from A. It is easily found that this new method changes nothing beyond replacing the traditional Cholesky decomposition matrix A with a slightly different matrix A^*.

In theory, both the Cholesky and eigenvalue decomposition transformations can deal with multivariate normal transformation, and both the inverse Cholesky and eigenvalue decomposition transformations have the same effect of correcting the errors of correlations between random samples. In a preliminary study, different combinations of the inverse Cholesky or eigenvalue transformation and the Cholesky or eigenvalue decomposition transformation were found to do exactly the same thing and end up with very similar performance. Since there is little difference in performance and the Cholesky decomposition transformation admits less computational complexity, this article focuses mainly on the Cholesky transformation.

NUMERICAL RESULTS

The experimental results illustrating the efficiency improvement of the inverse Cholesky decomposition transformation are presented in this section. European calls on the maximum of multiple assets are taken as an example, of which the payoff at the expiration date T can be expressed as

$$\max(\max(S_1(T), S_2(T), \cdots, S_N(T)) - K, 0)$$

In this article, the set of basic parameters for the maximum call option includes the current stock price for the j-th underlying asset $S_j(0) = 40$, $j = 1,\ldots,N$, the risk-free interest rate $r = 0.1$, the volatility of the rate of return of the j-th underlying asset $\sigma_j = 0.2$, the strike price $K = 40$, and the time to maturity $T = 0.25$. The cases of $N = 5$ and $N = 10$ are examined.

To verify the efficiency improvement of the inverse Cholesky decomposition transformation, the same crude random samples are used in comparing the following four methods: standard Monte Carlo simulation without any variance-reduction technique (SMC), moment matching method (SMC+MM), inverse Cholesky decomposition transformation (SMC+IC), and inverse eigenvalue decomposition transformation (SMC+IE). Using the same crude samples helps to isolate the benefits of individual variance-reduction techniques from the effects of different realizations of simulated samples. Similarly, the same crude random samples will be used in the four methods with antithetic variates: (SMC+Anti), (SMC+Anti+MM), (SMC+Anti+IC), and (SMC+Anti+IE).

The root mean squared errors (RMSEs) and the required CPU times corresponding to different variance-reduction techniques will be reported. The RMSEs are computed by replicating each simulation method several times. To be more explicit, suppose each Monte Carlo simulation uses M runs to generate one estimate. Then, the Monte Carlo simulation is repeated several times and the results are used to derive the corresponding RMSE. In this article, the number of simulations M ranges from 1,280 to 12,800, and the number of replications is 10. The RMSE for a variance-reduction technique with M simulations is defined by RMSE $= \sqrt{\frac{1}{10}\sum_{l=1}^{10} e_l^2}$, where $e_l = \hat{c}_l(M) - c$ is the pricing error of the l-th replication, $\hat{c}_l(M)$ is the estimated option value of the l-th replication, and c is the "true" option value derived based on 50,000 replications of 1,000,000 simulated and 1,000,000 antithetic independent standard normally distributed random samples. In addition, since the actual CPU times are machine dependent, only the ratios of CPU times relative to the standard Monte Carlo simulation with 1,280 simulations are reported.

There are two main parts to the numerical results. In the first part, the comparisons between the inverse Cholesky and eigenvalue decomposition transformations and some traditional variance-reduction techniques are provided for pricing maximum call options involving 5 and 10 underlying assets. In addition to the examples with equal volatility for all assets and equal value for all correlations, the performance of different variance-reduction techniques given randomly generated variance-covariance matrices is examined. In the second part, whereas the inverse Cholesky decomposition transformation effectively improves the convergence rate for the Monte Carlo estimates of option prices, it would be interesting to know if this new method also works with low discrepancy sequences. In this article, the effect of applying the inverse Cholesky decomposition transformation method to the simplest low discrepancy sequences, Halton sequences, is also investigated.

European Calls on the Maximum of 5 Assets

This subsection reports on the simulation results of European calls on the maximum of 5 assets with different variance-reduction techniques. After applying different variance-reduction techniques, including the proposed inverse Cholesky and eigenvalue decomposition transformations, to adjust the crude independent standard normally distributed random samples, the Cholesky decomposition transformation is employed to further transform the adjusted samples into correlated normally distributed random samples with the desired variance-covariance matrix. Exhibits 1 and 3 compare the RMSEs and the relative CPU times of different variance-reduction techniques for $\rho_{jk} = 0.1$ and $\rho_{jk} = 0.5$, respectively. Meanwhile, the rates of convergence, represented by the values of log(RMSE) in relation to the number of simulations, are plotted in Exhibits 2 and 4 as well.

In the upper diagram of Exhibit 2, the inverse Cholesky and eigenvalue decomposition transformations are compared with the classical moment matching method. In general, all three techniques can reduce the RMSEs of the estimates, and the inverse Cholesky and eigenvalue decomposition transformations perform more effectively than the method of moment matching. In the lower diagram of Exhibit 2, the antithetic variate approach is also considered. Compared to the upper diagram, the results show that all the antithetic variate–based approaches have smaller RMSEs than their nonantithetic counterparts. The advantages of the inverse Cholesky and the eigenvalue decomposition transformations over the moment matching method remain, even with the presence of the antithetic variate technique. For example, when the number of simulations is 12,800, the RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.00818 and 0.00526, respectively, which indicates the RMSE improvement of the inverse Cholesky decomposition transformation over the moment matching method is about 36%.

For the results in the case of $\rho_{jk} = 0.5$ in Exhibits 3 and 4, the advantages of the inverse Cholesky and eigenvalue decomposition transformations over the moment matching method become slight. To explain the performance of the inverse Cholesky decomposition transformation method with highly correlated underlying random variables—for example, $\rho_{jk} = 0.5$—consider the following two-variate example with the variance-covariance matrix C:

$$C = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

After performing the Cholesky decomposition transformation to derive the upper triangular matrix A, according to Equation (1), the correlated normally distributed random variables $[x_1 x_2]$ are represented as $x_1 = \sigma_1 z_1$ and $x_2 = \sigma_2(\rho_{12} z_1 + \sqrt{1-\rho_{12}^2} z_2)$.

Suppose the traditional moment matching method was employed, which corrects the first and second moments of z_j; that is, $E[z_j] = 0$ and $Var(z_j)$, for $j = 1$ and 2, but the $Cov(z_1, z_2)$ is not exactly equal to zero. Let e_{12} denote the value of $Cov(z_1, z_2)$. The variance-covariance matrix for x_1 and x_2 is as follows:

$$Var\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2(\rho_{12} + \sqrt{1-\rho_{12}^2} e_{12}) \\ \sigma_1\sigma_2(\rho_{12} + \sqrt{1-\rho_{12}^2} e_{12}) & \sigma_2^2(1 + 2\rho_{12}\sqrt{1-\rho_{12}^2} e_{12}) \end{bmatrix} \quad (3)$$

By analyzing the preceding variance-covariance structure, it is clear that when the absolute value of ρ_{12} is relatively large, the effect of the term $\sqrt{1-\rho_{12}^2} e_{12}$ relative to ρ_{12} in $Cov(x_1, x_2)$ becomes comparatively small.[3] Therefore, the inverse Cholesky decomposition transformation performs only slightly better than the moment matching method in the case of $\rho_{jk} = 0.5$, since the benefit of enforcing $e_{jk} = 0$ is less significant. In the real world, however, it is rare that the correlations among assets underlying a rainbow option are as high as 0.5. Instead, small positive or negative correlations are the most common scenarios; thus, the inverse Cholesky decomposition transformation method should offer significant improvement in the real world.

Exhibits 1 and 3 also contain the CPU time comparisons of different variance-reduction techniques, and several implications can be made. First, as expected, the antithetic variate technique costs less CPU time due to the fact that only half the simulated samples are drawn. Second, for the inverse Cholesky and eigenvalue decomposition transformations, about 20% more computational time is necessary to generate RMSEs of only one-third to one-quarter of the magnitude when compared with the CPU times and the RMSEs of the standard Monte Carlo simulation. Third, since the inverse Cholesky decomposition transformation and the Cholesky decomposition transformation are combined in a single step by multiplying the crude independent standard normal distributed random samples with the matrix A^*, the CPU times of the inverse Cholesky decomposition transformation method (SMC+IC+C or SMC+Anti+IC+C) are less than those of the moment matching method (SMC+MM+C or SMC+Anti+MM+C). Finally, when the tradeoff between the CPU times and the RMSEs is considered, the antithetic variate approach, together with the inverse Cholesky decomposition transformation, is the most efficient variance-reduction technique.

European Calls on the Maximum of 10 Assets

When the number of underlying assets rises to 10, the results of RMSEs and relative CPU times are reported in Exhibits 5 and 7, which include the cases of different correlation coefficients, $\rho_{jk} = 0.1$ and $\rho_{jk} = 0.5$, respectively. Similar to the previous subsection, a two-stage process is employed. Different variance-reduction techniques are first applied to crude independent standard normal distributed random samples, and the Cholesky decomposition transformation is used to further convert the random samples into correlated normally distributed random samples. The values of log(RMSE) in relation to the number of simulations are plotted in Exhibits 6 and 8 to illustrate the convergence rates of different variance-reduction techniques.

In the upper diagram of Exhibit 6, the inverse Cholesky and eigenvalue decomposition transformations are compared with the classical moment matching method. All three techniques can reduce the RMSE of the estimation, and the inverse Cholesky decomposition transformation is superior to the method of moment matching. In the lower diagram of Exhibit 6, the antithetic variate technique is incorporated into each respective variance-reduction technique. It is easily seen that the moment matching method accounts for a larger improvement than the antithetic variate approach. In addition, the inverse Cholesky and eigenvalue decomposition transformations again significantly outperform the traditional moment matching method with the presence of the antithetic variate technique. For instance, when the number of simulations is 12,800, the RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.01045 and 0.00927, respectively, which shows that the RMSE improvement of the inverse Cholesky decomposition transformation over the moment matching method is about 11%. Compared to the results in the case of 5 assets, the advantage of the inverse Cholesky and eigenvalue decomposition transformations over the traditional variance-reduction techniques remains, with the increase of the number of underlying assets.

From Exhibit 8, it is shown that when $\rho_{jk} = 0.5$, the advantage of the inverse Cholesky decomposition transformation relative to the moment matching method still exists, but diminishes. Similar to the discussion in the previous subsection, this phenomenon can be explained via the analysis of Equation (3). The effect of errors of correlations among simulated samples becomes less important when the absolute values of correlations ρ_{jk} are relatively large.

As for the relative CPU times in Exhibits 5 and 7, the results follow the same trend of those in the 5 asset case. For the inverse Cholesky and eigenvalue decomposition transformations, significantly smaller RMSEs can be achieved compared with the standard Monte Carlo simulation method within a little marginal CPU time. Taking the methods of SMC+C and SMC+IC+C for comparison, SMC+IC+C costs about 20% more CPU time than SMC+C, but the RMSEs of SMC+IC+C are only one-half to one-third of the magnitude, compared with the RMSEs of SMC+C. In addition, due to the one-step transformation through the matrix A^*, the inverse Cholesky or eigenvalue decomposition transformations cost less time than the traditional moment matching method. Results in Exhibits 5 and 7 suggest using the inverse Cholesky decomposition transformation with the antithetic variate approach as the most efficient variance-reduction technique when both the CPU times and the RMSEs are taken into account.

EXHIBIT 1

RMSEs and Relative CPU Times of Variance-Reduction Techniques for a Call Option on the Maximum of 5 Assets when $\rho_{jk} = 0.1$

This exhibit compares the RMSEs and relative CPU times of different variance-reduction techniques for pricing a European call on the maximum of 5 assets under the assumption of $\rho_{jk} = 0.1$ for $j \neq k$. The true value for comparison is 5.567073, which is estimated with 50,000 replications of 1,000,000 simulated and 1,000,000 antithetic independent standard normally distributed random samples. The reported RMSEs are based on 10 replications of *M* simulations, and the results of RMSEs are also illustrated in Exhibit 2. The relative CPU time is ratio of CPU time of each method relative to the CPU time of standard Monte Carlo simulation plus the Cholesky decomposition transformation (SMC+C) with 1,280 simulations. From the results, it is clear that the proposed inverse Cholesky decomposition transformation provides the highest rate of convergence with limited marginal calculation time, compared with standard Monte Carlo simulation.

RMSEs

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	0.11351	0.10223	0.05361	0.03848	0.04407	0.01850	0.02886	0.01745
2,560	0.06020	0.06933	0.03891	0.02143	0.03324	0.00762	0.02260	0.01028
3,840	0.04697	0.03931	0.02823	0.01990	0.02214	0.00860	0.01967	0.01238
5,120	0.03760	0.03188	0.02099	0.01554	0.01549	0.00774	0.01915	0.00995
6,400	0.04028	0.03122	0.01673	0.01460	0.01248	0.00928	0.01100	0.00980
7,680	0.03964	0.03148	0.01499	0.01369	0.01139	0.00918	0.01156	0.00938
8,960	0.03605	0.02695	0.01273	0.01058	0.00902	0.00730	0.00955	0.00839
10,240	0.03268	0.02933	0.01356	0.01101	0.01022	0.00669	0.00643	0.00806
11,520	0.03312	0.02531	0.01207	0.00894	0.00883	0.00603	0.01081	0.00883
12,800	0.02548	0.02235	0.01151	0.00818	0.00893	0.00526	0.01026	0.00719

Relative CPU times

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	1.000	0.765	1.399	1.132	1.166	0.934	1.267	0.932
2,560	1.966	1.565	2.665	2.198	2.299	1.900	2.365	1.932
3,840	2.932	2.267	3.832	3.198	3.465	2.797	3.465	2.800
5,120	3.998	3.064	5.365	4.399	4.697	3.697	4.765	3.731
6,400	4.998	3.866	6.729	5.497	5.932	4.731	5.930	4.797
7,680	5.996	4.631	8.062	6.595	7.062	5.565	7.162	5.631
8,960	6.996	5.362	9.362	7.631	8.162	6.495	8.230	6.529
10,240	7.864	6.130	10.727	8.761	9.394	7.463	9.429	7.497
11,520	8.996	6.930	12.060	9.861	10.563	8.330	10.593	8.362
12,800	9.962	7.663	13.294	10.595	11.759	9.094	11.795	9.160

SMC: standard Monte Carlo simulation; Anti: antithetic variate approach; MM: moment matching method; IC: inverse Cholesky decomposition transformation; IE: inverse eigenvalue decomposition transformation; C: Cholesky decomposition transformation.

EXHIBIT 2

Comparison of Rates of Convergence of Variance-Reduction Techniques in the 5 Asset Case when ρ_{jk} = 0.1

In order to obtain a better understanding of the rates of convergence, the values of log(RMSE) in relation to the number of simulations are graphed. The exhibit shows that the proposed inverse Cholesky and eigenvalue decomposition transformations are significantly more efficient than the traditional moment matching method, especially when the antithetic variate approach is employed to derive option values. For example, when the number of simulations is 12,800, the RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.00818 and 0.00526, respectively, which indicate the RMSE improvement of the inverse Cholesky decomposition transformation over the moment matching method is about 36%.

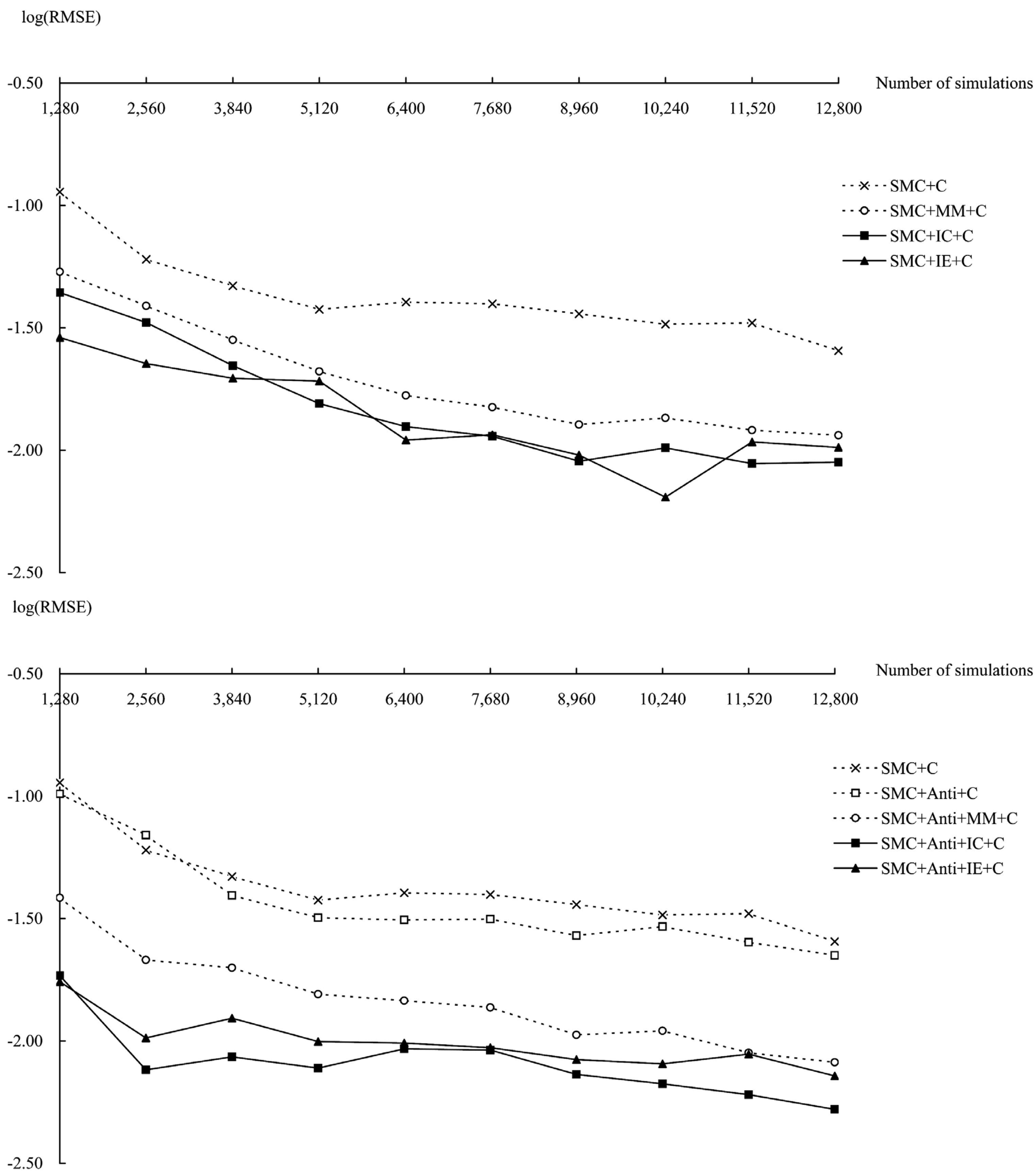


Exhibit 3

RMSEs and Relative CPU Times of Variance-Reduction Techniques for a Call Option on the Maximum of 5 Assets when $\rho_{jk} = 0.5$

This exhibit compares the RMSEs and relative CPU times of different variance-reduction techniques for pricing a European call on the maximum of 5 assets under the assumption of $\rho_{jk} = 0.5$ for $j \neq k$. The true option value is estimated to be 4.529253 with 50,000 replications of 1,000,000 simulated and 1,000,000 antithetic independent standard normally distributed random samples. The reported RMSEs are based on 10 replications of M simulations, and the results of RMSEs are illustrated in Exhibit 4 as well. The relative CPU time is the ratio of CPU time of each method relative to the CPU time of standard Monte Carlo simulation plus the Cholesky decomposition transformation (SMC+C) with 1,280 simulations. Both the inverse Cholesky and eigenvalue decomposition transformations still perform well, but due to the highly correlated characteristic between underlying assets, the advantage over the traditional moment matching method is not as significant as the results in Exhibit 1.

RMSEs

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	0.10737	0.05118	0.02746	0.01803	0.02811	0.01679	0.02401	0.00888
2,560	0.06954	0.03456	0.01807	0.00904	0.01774	0.00856	0.01209	0.01038
3,840	0.05118	0.02524	0.01464	0.00900	0.01406	0.00867	0.01260	0.00605
5,120	0.04042	0.02126	0.01241	0.00624	0.01201	0.00579	0.00771	0.00737
6,400	0.04048	0.01902	0.01276	0.00714	0.01231	0.00687	0.01265	0.00745
7,680	0.03200	0.01863	0.01247	0.00788	0.01203	0.00767	0.01004	0.00410
8,960	0.03726	0.01452	0.01029	0.00608	0.00990	0.00588	0.01185	0.00689
10,240	0.03005	0.01413	0.00893	0.00615	0.00853	0.00603	0.01056	0.00482
11,520	0.03420	0.01541	0.00792	0.00525	0.00764	0.00504	0.00668	0.00346
12,800	0.03458	0.01609	0.00938	0.00517	0.00908	0.00519	0.00723	0.00634

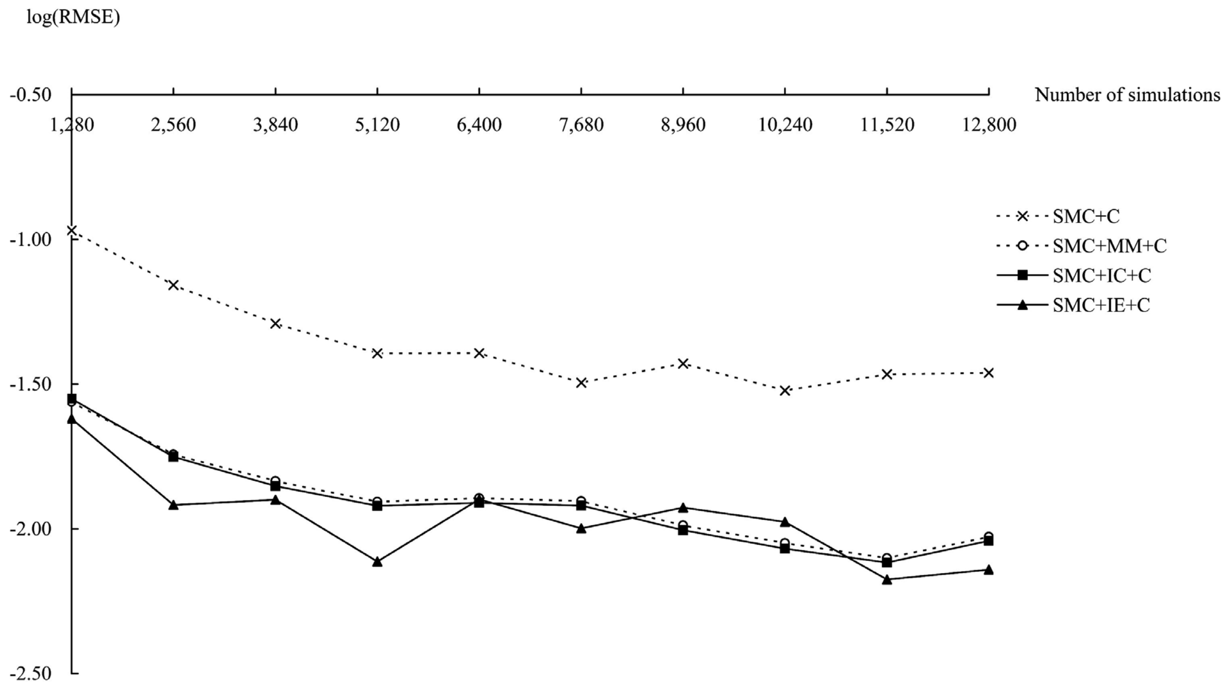
Relative CPU times

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	1.000	0.742	1.388	1.066	1.227	0.872	1.225	0.969
2,560	1.905	1.421	2.519	2.130	2.227	1.808	2.260	1.903
3,840	2.808	2.194	3.744	3.099	3.324	2.680	3.390	2.746
5,120	3.907	2.874	5.165	4.229	4.552	3.616	4.583	3.616
6,400	4.843	3.744	6.554	5.360	5.715	4.519	5.715	4.616
7,680	5.713	4.457	7.814	6.360	6.812	5.424	6.909	5.457
8,960	6.781	5.229	9.039	7.360	8.006	6.262	8.006	6.293
10,240	7.651	5.843	10.426	8.362	9.136	7.167	9.136	7.231
11,520	8.717	6.715	11.688	9.556	10.233	8.103	10.264	8.136
12,800	9.620	7.329	13.041	10.461	11.169	8.975	11.430	9.039

SMC: standard Monte Carlo simulation; Anti: antithetic variate approach; MM: the moment matching method; IC: inverse Cholesky decomposition transformation; IE: inverse eigenvalue decomposition transformation; C: Cholesky decomposition transformation.

EXHIBIT 4
Comparison of Rates of Convergence of Variance-Reduction Techniques in the 5 Asset Case when ρ_{jk} = 0.5

European calls on the maximum of 5 assets are taken as examples. In order to obtain a better understanding of the rates of convergence, the values of log(RMSE) in relation to the number of simulations are plotted. In general, the inverse Cholesky or eigenvalue decomposition transformation generates the results with less RMSEs, which demonstrates that these two proposed methods have higher convergence rates than those traditional variance-reduction techniques.



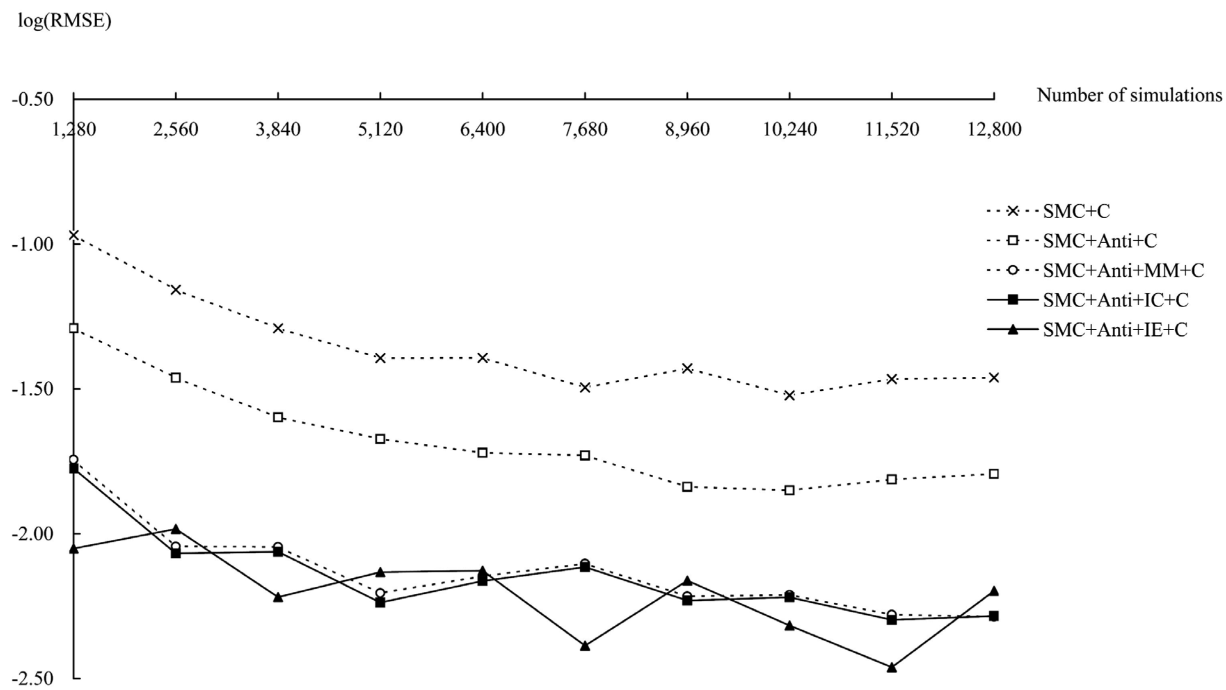


Exhibit 5

RMSEs and Relative CPU times of Variance-Reduction Techniques for Call Option on Maximum of 10 Assets when $\rho_{jk} = 0.1$

This exhibit compares the RMSEs and relative CPU times of different variance-reduction techniques for pricing a European call on the maximum of 10 assets under the assumption of $\rho_{jk} = 0.1$ for $j \neq k$. The true option value is 7.139944, which is calculated from 50,000 replications of 1,000,000 simulated and 1,000,000 antithetic independent standard normally distributed random samples. The reported RMSEs are based on 10 replications of M simulations, and the results of RMSEs are also graphed in Exhibit 6. The relative CPU time is the ratio of CPU time of each method relative to the CPU time of standard Monte Carlo simulation plus the Cholesky decomposition transformation (SMC+C) with 1,280 simulations. The performance of the inverse Cholesky and eigenvalue decomposition transformations are better than that of traditional variance-reduction techniques, and the combination of the antithetic variate approach and the inverse Cholesky decomposition transformation generates the results with the highest rate of convergence.

RMSEs

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	0.07625	0.08616	0.05988	0.02208	0.05517	0.02187	0.04047	0.02555
2,560	0.04841	0.04633	0.03377	0.02897	0.03040	0.02736	0.02563	0.02447
3,840	0.05445	0.04425	0.02331	0.02657	0.02098	0.02409	0.01735	0.01577
5,120	0.04242	0.03924	0.01860	0.01955	0.01712	0.01696	0.02400	0.01662
6,400	0.03718	0.03438	0.01816	0.02114	0.01764	0.01762	0.01893	0.01286
7,680	0.03726	0.02836	0.01627	0.01751	0.01551	0.01445	0.02384	0.01173
8,960	0.03167	0.02754	0.01564	0.01642	0.01485	0.01397	0.01864	0.01103
10,240	0.02846	0.02430	0.01341	0.01283	0.01340	0.01125	0.01476	0.01228
11,520	0.02363	0.02027	0.01065	0.01061	0.01092	0.00930	0.01007	0.00846
12,800	0.01958	0.02024	0.00949	0.01045	0.00941	0.00927	0.01514	0.01025

Relative CPU times

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	1.000	0.768	1.286	1.089	1.197	1.018	1.197	0.983
2,560	1.911	1.482	2.661	2.125	2.339	1.946	2.446	2.017
3,840	2.874	2.267	3.893	3.107	3.482	2.893	3.518	2.875
5,120	3.857	3.089	5.215	4.447	4.696	4.018	4.733	3.965
6,400	4.875	3.875	6.678	5.375	5.805	4.803	5.857	4.714
7,680	5.929	4.642	8.197	6.785	7.160	5.857	7.214	5.839
8,960	6.821	5.322	9.339	7.643	8.143	6.839	8.179	6.571
10,240	7.911	6.339	10.946	9.178	9.571	8.339	9.625	7.893
11,520	8.786	7.000	12.018	10.054	10.536	8.983	10.571	8.643
12,800	9.857	7.857	13.696	11.499	11.965	9.874	12.035	9.874

SMC: standard Monte Carlo simulation; Anti: antithetic variate approach; MM: moment matching method; IC: inverse Cholesky decomposition transformation; IE: inverse eigenvalue decomposition transformation; C: Cholesky decomposition transformation.

Exhibit 6

Comparison of Rates of Convergence of Variance-Reduction Techniques in the 10 Asset Case when ρ_{jk} = 0.1

The RMSEs are adapted as the proxy of the rates of convergence of different variance-reduction techniques. The values of log(RMSE) in relation to the number of simulations are plotted for the sake of better understanding the rates of convergence. The inverse Cholesky or eigenvalue decomposition transformation outperforms the traditional variance-reduction techniques, including the antithetic variate approach or the moment matching method, and accelerates the rate of convergence. For instance, when the number of simulations is 12,800, the RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.01045 and 0.00927, respectively, which indicate the RMSE improvement of the inverse Cholesky decomposition transformation over the moment matching method is about 11%.

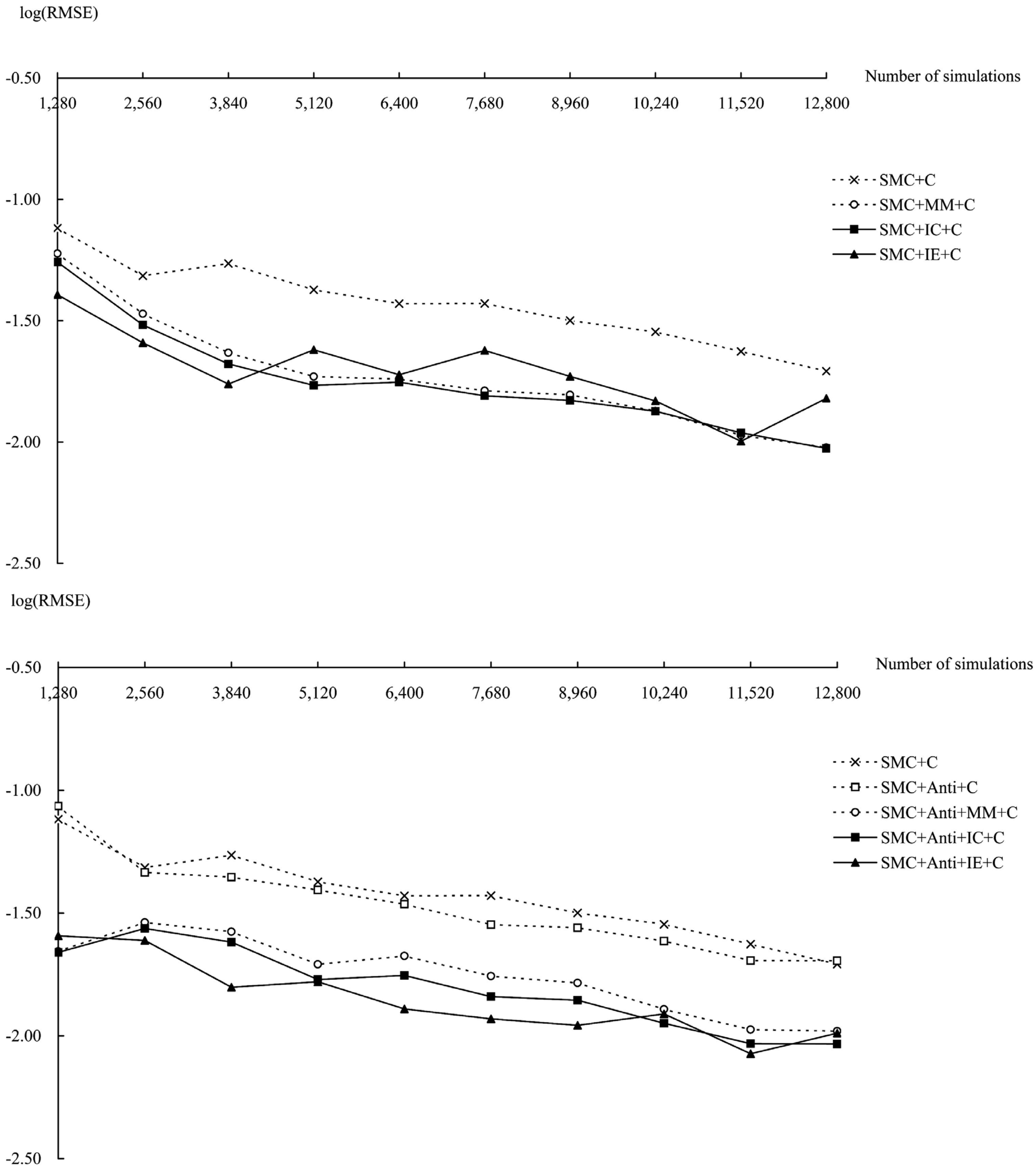


EXHIBIT 7

RMSEs and Relative CPU Times of Variance-Reduction Techniques for a Call Option on Maximum of 10 Assets when $\rho_{jk} = 0.5$

This exhibit compares the RMSEs and relative CPU times of different variance-reduction techniques for pricing a European call on the maximum of 10 assets under the assumption of $\rho_{jk} = 0.5$ for $j \neq k$. The true option value is 5.585270, which is derived from 50,000 replications of 1,000,000 simulated and 1,000,000 antithetic independent standard normally distributed random samples. The reported RMSEs are based on 10 replications of M simulations, and the results of RMSEs are plotted in Exhibit 8 as well. The relative CPU time is the ratio of CPU time of each method relative to the CPU time of standard Monte Carlo simulation plus the Cholesky decomposition transformation (SMC+C) with 1,280 simulations. This exhibit shows that the inverse Cholesky decomposition transformation performs better than the moment matching method or the antithetic variate approach. Furthermore, if both the RMSE and the calculation time are taken into account, combining the antithetic variate approach with the inverse Cholesky decomposition transformation is the most efficient variance-reduction technique.

RMSEs

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	0.07644	0.03908	0.03140	0.01921	0.03244	0.01824	0.03774	0.02161
2,560	0.05873	0.03066	0.01949	0.01088	0.01914	0.01072	0.02371	0.01546
3,840	0.05913	0.03115	0.01473	0.01063	0.01408	0.01118	0.01848	0.01108
5,120	0.03475	0.02793	0.01134	0.00788	0.01091	0.00821	0.01203	0.00981
6,400	0.02298	0.02419	0.01066	0.00635	0.01040	0.00670	0.01546	0.01037
7,680	0.02840	0.02290	0.01200	0.00601	0.01162	0.00588	0.00945	0.00829
8,960	0.03224	0.02176	0.00946	0.00615	0.00909	0.00589	0.01550	0.01171
10,240	0.03125	0.01928	0.00989	0.00628	0.00965	0.00593	0.01126	0.00691
11,520	0.02294	0.01779	0.01002	0.00528	0.00990	0.00495	0.01002	0.00753
12,800	0.02634	0.01667	0.01022	0.00509	0.01013	0.00486	0.00868	0.00769

Relative CPU times

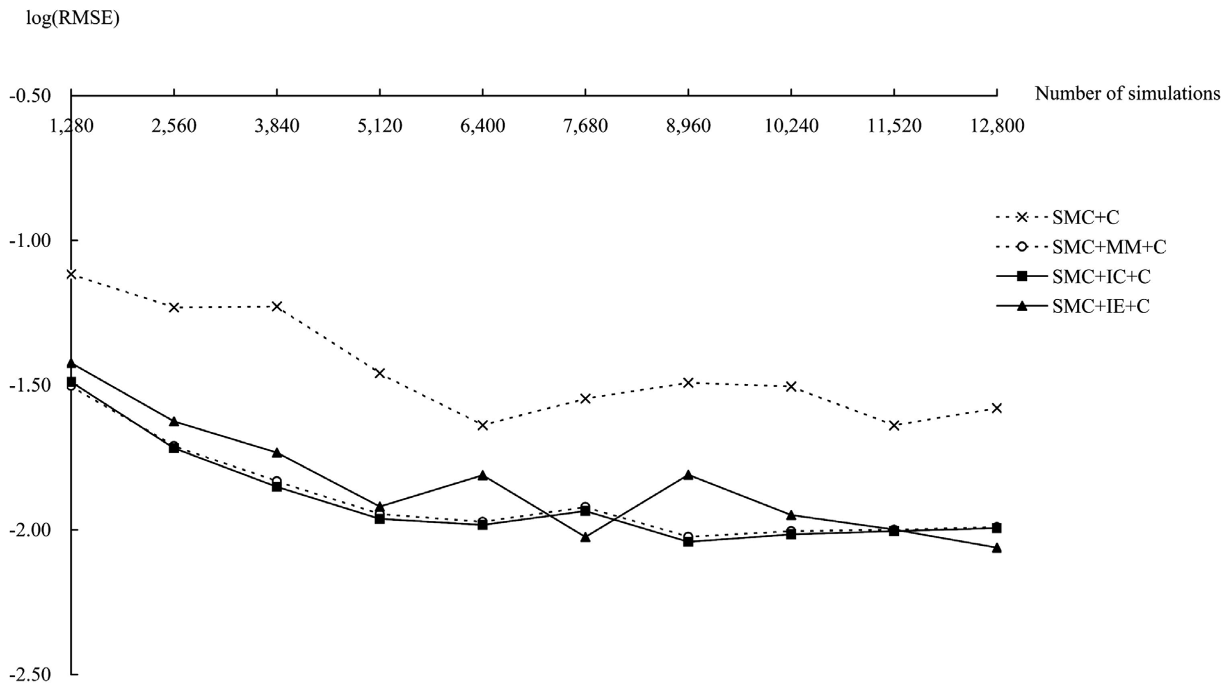
M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	1.000	0.816	1.390	1.205	1.224	1.038	1.241	1.019
2,560	2.021	1.631	2.688	2.353	2.483	2.020	2.521	2.039
3,840	3.020	2.428	4.133	3.374	3.614	2.929	3.652	2.929
5,120	4.042	3.262	5.523	4.633	4.949	4.023	4.986	4.077
6,400	4.967	3.948	6.821	5.783	6.024	4.837	6.079	4.912
7,680	6.154	4.819	8.508	7.006	7.433	6.005	7.470	6.043
8,960	7.079	5.542	9.675	7.951	8.452	6.784	8.507	6.821
10,240	8.174	6.580	11.363	9.528	9.935	8.155	9.991	8.192
11,520	9.120	7.302	12.456	10.435	10.936	8.916	10.992	8.951
12,800	10.231	8.211	14.216	11.880	12.418	10.212	12.456	10.269

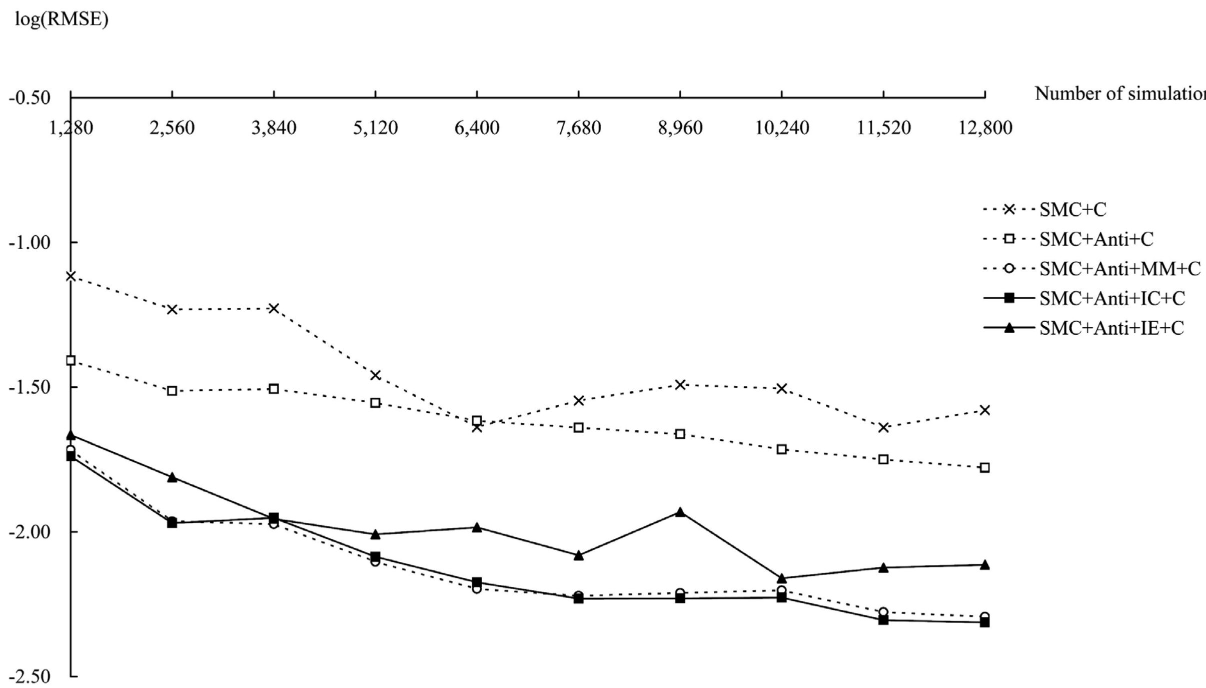
SMC: standard Monte Carlo simulation; Anti: antithetic variate approach; MM: moment matching method; IC: inverse Cholesky decomposition transformation; IE: inverse eigenvalue decomposition transformation; C: Cholesky decomposition transformation.

EXHIBIT 8

Comparison of Rates of Convergence of Variance-Reduction Techniques in the 10 Asset Case when ρ_{jk} = 0.5

European calls on the maximum of 10 assets are considered here. In order to obtain a better understanding of the rates of convergence, the values of log(RMSE) in relation to the number of simulations are graphed. The exhibit shows that when the antithetic variate approach, the moment matching method, and the proposed inverse Cholesky or eigenvalue decomposition transformation are used, the RMSEs can be reduced. The inverse Cholesky decomposition transformation still shows the strongest convergence rate. Nevertheless, due to the large ρ_{jk} value, there is little performance discrepancy between the moment matching method and the inverse Cholesky decomposition transformation.





Results for Randomly Generated Variance-Covariance Matrices

In the preceding two subsections, only examples with equal volatility for all assets and constant correlations of $\rho_{jk} = 0.1$ and $\rho_{jk} = 0.5$ are considered. They are not enough to draw any firm conclusions about the proposed new method. In this subsection, several random scenarios of the variance-covariance structures are generated to reflect the real world condition, and the performance of different variance-reduction techniques is examined for these scenarios. European calls on the maximum of 10 assets are taken as examples. All parameter values are the same as before, except that the volatility of each asset is assumed to follow the uniform distribution from 10% to 50%, and the correlations between assets are uniformly distributed between 0 and 0.5. Ten scenarios of variance-covariance structures are generated randomly, and the reported RMSEs and relative CPU times in Exhibit 9 are the averages across all scenarios. Meanwhile, the comparison of the average rates of convergence of different variance-reduction techniques is shown in Exhibit 10.

In Exhibits 9 and 10, it is apparent that the inverse Cholesky and eigenvalue decomposition transformations show the strongest performance in variance reduction. For instance, SMC+IC+C generates average RMSEs of about one-third of the magnitude compared with that for the standard Monte Carlo simulation (SMC+C). In addition, when the number of simulations is 12,800, the average RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.02451 and 0.01484, respectively, an improvement over the moment matching method of about 39%. Comparing Exhibit 10 with previous results, because of the absence of the homogeneous variance-covariance structures, the average RMSEs are a little larger than the results in Exhibits 6 and 8. Meanwhile, due to the average effect, the superiority of the inverse Cholesky and eigenvalue decomposition transformations becomes more stable and evident, which further supports that the proposed new method can perform well in the real world. In addition, for the relative CPU times in Exhibits 9, the results are very similar to those in Exhibits 1, 3, 5, and 7, which all indicate that employing the inverse Cholesky decomposition transformation with the antithetic variate approach is the most efficient variance-reduction technique when both the CPU times and the RMSEs are taken into account.

The Inverse Cholesky Decomposition Transformation Applied to Low-Discrepancy Sequences

From the previous discussion, the combination of the inverse Cholesky decomposition transformation and the Cholesky decomposition transformation effectively reduces the variance for multivariate Monte Carlo simulation. In this subsection, these sequential methods are applied to low-discrepancy Halton sequences.[4] European calls on the maximum of 5 assets are taken as examples, with correlations $\rho_{jk} = 0.1$ and $\rho_{jk} = 0.5$.

It should be noted that the Halton sequences simulate uniform distributions. After deriving the Halton sequences, it is necessary to transform them into independent standard normal distributed sequences. In this article, a function in Excel named "NORMSINV" is used to accomplish the transformation. Hereafter, the word "Halton" represents a series of independent standard normal distributed samples transformed from the uniform distributed Halton sequences.

The effects of applying the inverse Cholesky decomposition transformation on the Halton sequences are examined in Exhibits 11 and 12. It is clear that the inverse Cholesky decomposition transformation further improves the convergence rate of the Halton sequences in the case of $\rho_{jk} = 0.1$. When $\rho_{jk} = 0.5$, the inverse Cholesky decomposition transformation still demonstrates its efficiency, but the advantage is less than for $\rho_{jk} = 0.1$. Similar to the discussion in the previous subsection, the phenomenon can be attributed to the effect of errors of correlations e_{jk}s becoming comparatively small when the absolute value of ρ_{jk} is large. As a result, correcting errors of correlations by the inverse Cholesky decomposition transformation brings slight improvement to the results of option values.

By analyzing the correlations between the independent standard normal distributed sequences generated from the uniform-distributed Halton sequences in Exhibit 13, it is found that the values of correlations among different vectors of generated samples are almost the same and positive, and they decrease to zero with the increase in the number of simulations. Therefore, in Exhibit 12, the option value increases with the increase of the number of simulations, because of the well-known phenomenon that with the decrease of the correlations among assets, the calls on the maximum of multiple assets become more valuable. Furthermore, since the correlations are corrected to zero, the

Exhibit 9

Average RMSEs and Relative CPU Times of Variance-Reduction Techniques for Call Options on the Maximum of 10 Assets Given Randomly Generated Variance-Covariance Matrices

This exhibit reports the average RMSEs and relative CPU times of different variance-reduction techniques for pricing European maximum call options given randomly generated variance-covariance matrices. The volatility for each asset is assumed to be distributed uniformly between 0.1 and 0.5, and the correlations are uniformly distributed between 0 and 0.5. Ten scenarios of variance-covariance structures are generated randomly, and the reported RMSEs and relative CPU times are the average of the RMSEs and relative CPU times across all scenarios. The results of average RMSEs are also plotted in Exhibit 10. This exhibit shows that, on average, the performance of the proposed inverse Cholesky and eigenvalue decomposition transformations is superior than that of traditional variance-reduction techniques. For instance, when the number of simulations is 12,800, the average RMSEs of SMC+Anti+MM+C and SMC+Anti+IC+C are 0.02451 and 0.01484, respectively, which indicates the RMSE improvement of the inverse Cholesky decomposition transformation over the moment matching method is about 39%.

Average RMSEs

M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	0.19809	0.17643	0.09791	0.07755	0.09248	0.04332	0.06586	0.04660
2,560	0.12102	0.10287	0.06401	0.05795	0.05551	0.03315	0.05000	0.03132
3,840	0.14062	0.08720	0.04869	0.04604	0.03923	0.02904	0.03590	0.02629
5,120	0.10949	0.07885	0.04142	0.03787	0.03525	0.02367	0.03501	0.02383
6,400	0.09830	0.06968	0.03952	0.03415	0.03351	0.02145	0.02963	0.01902
7,680	0.08602	0.05863	0.03469	0.02997	0.02890	0.01778	0.03564	0.02148
8,960	0.07283	0.05832	0.03175	0.02886	0.02730	0.01836	0.02643	0.01797
10,240	0.06617	0.05435	0.03017	0.02685	0.02597	0.01642	0.02295	0.01719
11,520	0.06249	0.04891	0.02711	0.02553	0.02450	0.01525	0.02196	0.01480
12,800	0.05275	0.04661	0.02338	0.02451	0.02110	0.01484	0.02394	0.01427

Average relative CPU times

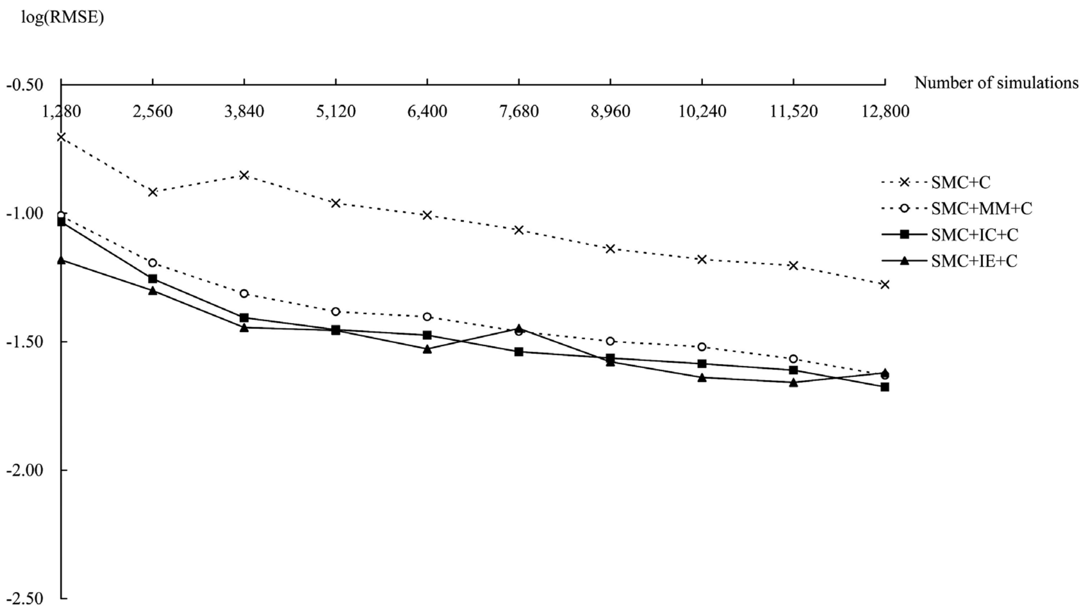
M simulations	SMC+C	SMC+Anti+C	SMC+MM+C	SMC+Anti+MM+C	SMC+IC+C	SMC+Anti+IC+C	SMC+IE+C	SMC+Anti+IE+C
1,280	1.00000	0.78410	1.34844	1.10928	1.20039	0.92731	1.19966	0.97674
2,560	2.00242	1.54446	2.71020	2.24933	2.39981	1.90841	2.40804	1.93458
3,840	2.95299	2.31718	3.95299	3.19893	3.39617	2.73734	3.46063	2.72983
5,120	3.98667	3.11219	5.37243	4.44899	4.75939	3.80906	4.79259	3.86213
6,400	4.93676	3.81585	6.61449	5.47056	5.79598	4.63436	5.81948	4.64139
7,680	6.30724	4.90695	8.72232	7.16307	7.61352	6.08796	7.64332	6.11437
8,960	7.26945	5.62661	9.94621	8.14417	8.66634	6.91325	8.71214	6.92852
10,240	8.42767	6.54907	11.67967	9.63121	10.17325	8.15508	10.21856	8.20790
11,520	9.35498	7.28810	12.82748	10.50666	11.18028	8.87497	11.22147	8.94257
12,800	10.51393	8.23068	14.58735	12.02350	12.74364	10.19918	12.78919	10.29077

SMC: standard Monte Carlo simulation; Anti: antithetic variate approach; MM: moment matching method; IC: inverse Cholesky decomposition transformation; IE: inverse eigenvalue decomposition transformation; C: Cholesky decomposition transformation.

EXHIBIT 10

Average Rates of Convergence of Variance-Reduction Techniques in the 10 Asset Case Given Randomly Generated Variance-Covariance Matrices

This exhibit compares the average rates of convergence of different variance-reduction techniques for 10 randomly generated variance-covariance structures. The volatility for each asset is assumed to be distributed uniformly between 0.1 and 0.5. The correlations are uniformly distributed between 0 and 0.5. This exhibit shows that the inverse Cholesky or eigenvalue decomposition transformation, on average, outperforms the traditional variance-reduction techniques and accelerates the rate of convergence, especially under the cases in which the antithetic variate approach is combined with the proposed new method. Since generating variance-covariance matrices randomly is a reasonable reflection of various possible scenarios in the real world, the results in this exhibit demonstrate that the proposed inverse Cholesky and eigenvalue decomposition transformations should perform well in the real world.



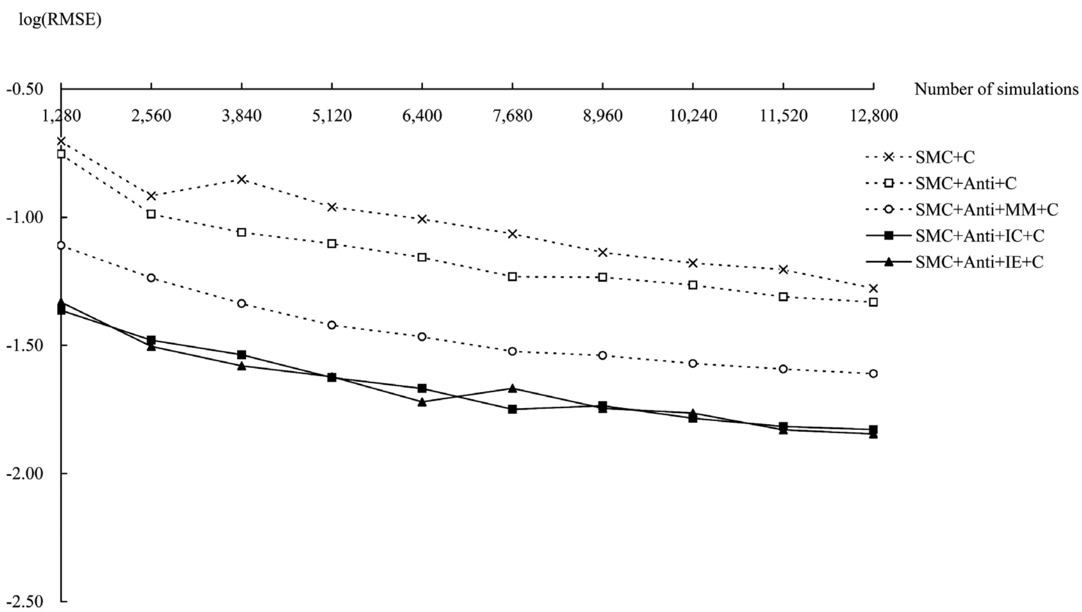


Exhibit 11

Inverse Cholesky Decomposition Transformation Applied to Halton Sequences

This exhibit contains the results of the estimates obtained from applying the inverse Cholesky decomposition transformation to independent standard normal distributed samples generated from the uniformly distributed Halton sequences. European calls on the maximum of 5 assets with ρ_{jk} = 0.1 and ρ_{jk} = 0.5 for $j \neq k$ are taken as examples. According to the previous results, the true values of these options are 5.567073 and 4.529253, respectively. For the case of ρ_{jk} = 0.1, it is obvious that when the inverse Cholesky decomposition transformation is applied, the derived option value converges to the true value more quickly. In the case of ρ_{jk} = 0.5, the inverse Cholesky decomposition does help to improve the convergence rate, but the improvement is not as large as that in the case of ρ_{jk} near zero.

	$\rho_{jk} = 0.1$		$\rho_{jk} = 0.5$	
Number of Simulations	Halton+C	Halton+IC+C	Halton+C	Halton+IC+C
1,280	5.48968	5.53928	4.46108	4.46185
2,560	5.53195	5.55658	4.49800	4.49561
3,840	5.53412	5.55367	4.50234	4.50313
5,120	5.54332	5.55804	4.51107	4.51112
6,400	5.54482	5.55759	4.51353	4.51452
7,680	5.54713	5.55835	4.51354	4.51537
8,960	5.55015	5.55999	4.51676	4.51840
10,240	5.55091	5.56019	4.51727	4.51884
11,520	5.55343	5.56167	4.51834	4.52023
12,800	5.55452	5.56172	4.52043	4.52156

option prices estimated by the inverse Cholesky decomposition transformation converge to their true values more quickly, even when the number of simulations is small. This observation shows that the inverse Cholesky decomposition transformation helps to improve the convergence rate of the estimates even when the low-discrepancy sequences are used.

CONCLUSION

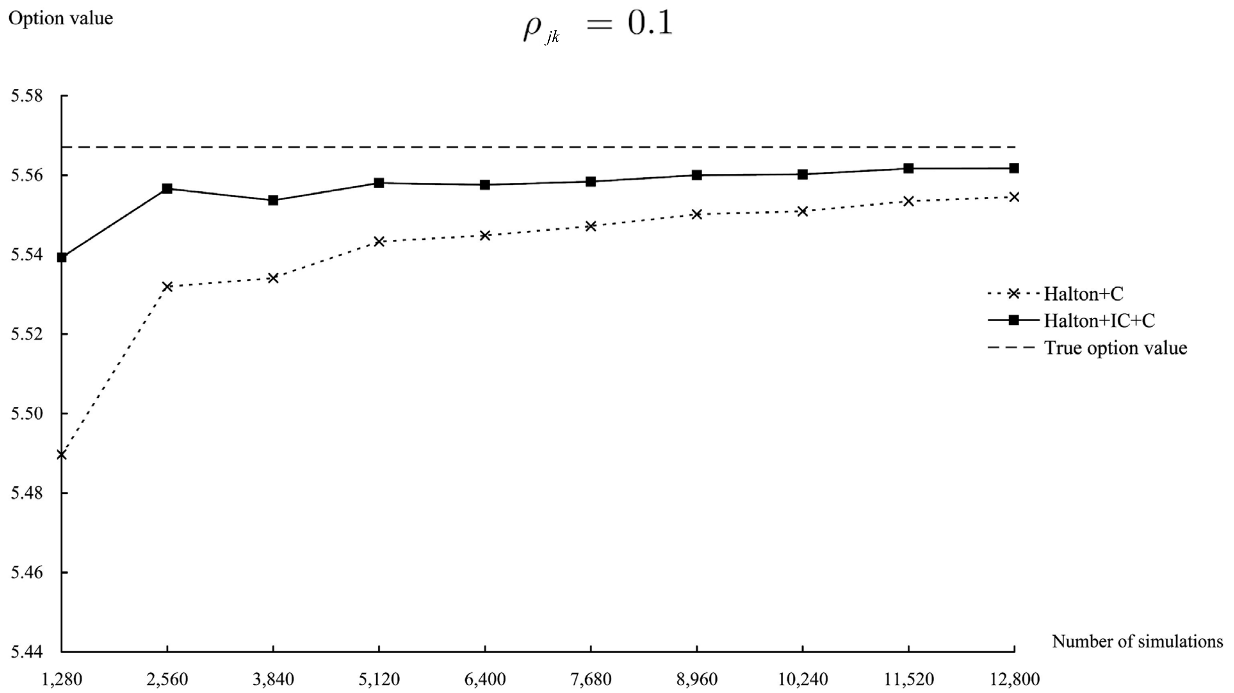
This article introduces a new variance-reduction technique, the inverse Cholesky decomposition transformation, which diminishes the sample errors of correlations among multivariate independent standard normal distributed random samples. The result is a set of random samples that is a better representative for a multivariate independent standard normal distribution with an identity variance-covariance matrix. Since the errors of the sample variances and covariances among underlying assets can be eliminated, this new method, invented for multivariate Monte Carlo simulation, is superior to the traditional variance-reduction techniques for pricing rainbow options.

Numerical results conclude that the combination of the inverse Cholesky decomposition transformation with the antithetic variate approach results in the most efficient estimation of European calls on the maximum of multiple assets. Because the eigenvalue decomposition is computationally more expensive, it is suggested that a simple combination of the inverse Cholesky decomposition transformation and the Cholesky decomposition transformation achieves an excellent balance between the computational effort and the accuracy of the estimates. Furthermore, the inverse Cholesky decomposition transformation also helps to improve the convergence rate of the estimates generated based on the low-discrepancy sequences. Verified by these extensive simulation results, the substantial convergence-rate improvement and the feature of easy implementation for the inverse Cholesky decomposition transformation encourages the use of this new method as a standard variance-reduction procedure in multivariate Monte Carlo and quasi-Monte Carlo simulations.

EXHIBIT 12

Option Values from Halton Sequences with and without Inverse Cholesky Decomposition Transformation

European calls on the maximum of 5 assets with ρ_{jk} = 0.1 and ρ_{jk} =0.5 are taken as examples. The true values of these options are 5.567073 and 4.529253, respectively. For ρ_{jk} = 0.1, the upper diagram demonstrates that even for the low-discrepancy sequence, incorporating the inverse Cholesky decomposition transformation helps to derive a more accurate option value with fewer simulations. However, similar to the previous discussion, the improvement of the inverse Cholesky decomposition is less significant when the absolute value of ρ_{jk} is relatively large.



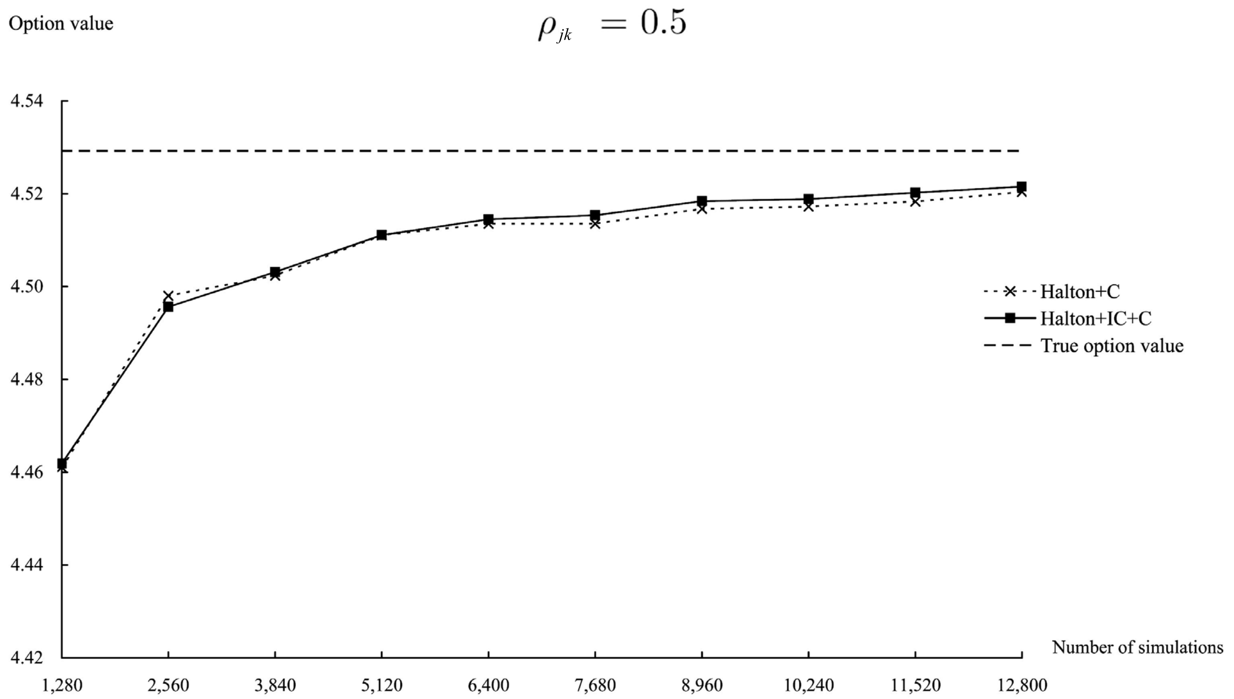


EXHIBIT 13

Correlations between Independent Standard Normal Distributed Samples Generated from Uniformly Distributed Halton Sequences

This exhibit contains the correlations between the independent standard normal distributed samples generated based on the uniformly distributed Halton sequences. As the number of simulations increases, the correlations draw closer to zero. Because the option value increases with respect to the decrease of the values of correlations, option values increase with the increase of the number of simulations as demonstrated in Exhibit 12.

Number of Simulations is 1,280

	z_1	z_2	z_3	z_4	z_5
z_1	1	0.0440	0.0449	0.0452	0.0464
z_2		1	0.0453	0.0435	0.0455
z_3			1	0.0484	0.0490
z_4				1	0.0463
z_5					1

Number of Simulations is 2,560

	z_1	z_2	z_3	z_4	z_5
z_1	1	0.0221	0.0233	0.0223	0.0238
z_2		1	0.0222	0.0224	0.0228
z_3			1	0.0237	0.0236
z_4				1	0.0239
z_5					1

Number of Simulations is 3,840

	z_1	z_2	z_3	z_4	z_5
z_1	1	0.0150	0.0151	0.0160	0.0138
z_2		1	0.0164	0.0150	0.0152
z_3			1	0.0148	0.0140
z_4				1	0.0158
z_5					1

ENDNOTES

[1]If the antithetic variate approach is combined with the inverse Cholesky decomposition transformation method, this detrend process is not necessary and $\tilde{C}$ is the variance-covariance matrix of $[z_1 z_2 \cdots z_N]$.

[2]In fact, the only time to use the inverse eigenvalue decomposition transformation is when the Cholesky decomposition fails, which occurs when the covariance matrix $\tilde{C}$ is not positive semidefinite (PSD). The only possibility a variance-covariance matrix constructed from data becomes non-PSD is when there are missing observations, but that does not happen in a simulated series. Therefore, it is almost impossible for the Cholesky decomposition to fail here.

[3]The error term e_{12} affects $Var(x_2)$ as well. However, the influence of $2\rho_{12}\sqrt{1-\rho_{12}^2}e_{12}$ relative to 1 in $Var(x_2)$ is generally much less than the influence of $\sqrt{1-\rho_{12}^2}e_{12}$ relative to ρ_{12} in $Cov(x_1, x_2)$. Therefore, the analysis here focuses only on the effect of e_{12} on $Cov(x_1, x_2)$.

[4]Refer to Galanti and Jung [1997] for information about how to generate the Halton sequences.

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