Optimal Incentives for Sequential Production

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The Model

- Players act sequentially in the order $1, 2, \ldots, n$.
- $d_i \in \{0, 1\}$ the investment decision of player i.
  - $d_i = 1$ – investment.
  - $d_i = 0$ – non-investment.
- c is the cost of investment.
  - Task succeeds with probability $0 < \alpha < 1$ if no investment and with probability 1 with investment.
- Project succeeds iff all tasks succeed.

The Model

- The principal pays v_i to agent i if the project succeeds,
  - and 0 if fails,
- Payoffs:
  - $f_i(d) = v_i\, \alpha^{d(\text{-}i)} - c$ if $d_i = 1$
  - $f_i(d) = v_i\, \alpha^{d(\text{-}i)+1}$ if $d_i = 0$,
  - where $d(\text{-}i) = |\{j \neq i \mid d_j = 0\}|$ is the number of shirking agents
- $G(v)$ the extensive form game induced by the vector of rewards $v = (v_1, v_2, \ldots, v_n)$.

The Model

- We say that the mechanism v is investment-inducing (*INI*) when the unique SPE of the game $G(v)$ entails investment by all players, i.e., $d = (1, ..., 1)$.
- We will say that an *INI* mechanism v is optimal if $\sum_{i\in N} v_i' \geq \sum_{i\in N} v_i$ for every other *INI* mechanism v'
- Proposition 1: A mechanism v is an optimal investment-inducing mechanism iff $$v_j = \frac{c}{1-\alpha^{n-j+1}}$$

Agents' Competence

- Let c_i be i's level of competence and $c_1 > c_2 > \ldots > c_n$
- a mechanism is now a pair $m = (w, v)$ where
  - w is a permutation of $N = \{1, 2, \ldots, n\}$, and
  - v is a vector of rewards.
- For game $G_{m,c}$, m is an optimal *INI* mechanism if
  - Every SPE of $G_{m,c}$ leads to a probability 1 success and no other mechanism $m' = (w', v')$ exist such that:

1. all SPEs of $G_{m',c}$ lead to a probability 1 success, and
2. m' is cheaper than m: $\sum_{i \in N} v_i > \sum_{i \in N} v'_i$

Proposition 2

- In the differential costs model, $m = (w, v)$ is an optimal investment-inducing mechanism iff
- w is the identity permutation.
  - (i.e., lower-cost individuals are assigned to move later)
- Furthermore, $v_j = \dfrac{c_j}{1 - \alpha^{n-j+1}}$

Tasks' Importance

- Let $\alpha_i > \alpha_{i+1}$ be the order of importance of tasks
- The mechanism in this framework is a pair $m = (\theta, v)$ such that
- θ is a permutation on N specifying the allocation of tasks to different slots of the process and
- v is a vector of rewards.

Proposition 3

- In the model with differential probabilities of success, $m = (\theta, v)$ is an optimal investment-inducing mechanism if and only if
- θ is the identity permutation
  - (i.e., tasks with lower α are assigned to be conducted by agents whose effort exertion is less observable)
- and $v_j = \dfrac{c}{1 - \prod_{k=j}^{n} \alpha_k}$

Reducing Rewards at the Cost of Lower Success

- For a probability $0 < p \leq 1$ we say that a mechanism v is p-investment-inducing if
  - Every SPE of the game G (which depends on α and c) leads to the project's success with probability of at least p.
  - More precisely, $\alpha^{s(d)} \geq p$
- Proposition 4: If α is small enough, then for any $\alpha^n < p < 1$ the mechanism v is optimal p-investment-inducing iff it is optimal investment-inducing.
  - (i.e., for $p = 1$)

Proposition 5

- Let α be close enough to 1 and $1 < k < n$.
- In the optimal $p(k)$-*INI* mechanisms, the principal selects a group of k players $K = \{i_1, i_2, \ldots, i_k\}$ with
- $i_1 < i_2 < \ldots < i_k$ (which will be induced to invest) and pays them $v_{ij} = \dfrac{c}{\alpha^{n-k} - \alpha_{n-j+1}}$
  - while $v_r = 0$ for r in $N \backslash K$.
- Furthermore, the total reward in an optimal $p(k)$-investment-inducing mechanism is increasing with k.

General Technologies

- p is a function from $\{0,1\}^N$ to $[0,1]$
- Monotonicity: $T \subset S$ implies $p(T) < p(S)$
- Convexity: $p(S \cup \{i\}) - p(S) > p(T \cup \{i\}) - p(T)$, whenever $T \subset S$.
- Proposition 1*: A mechanism v is an optimal investment-inducing mechanism if and only if $v_j = \dfrac{c}{p(N) - p(S_j)}$ where S_j are the set of players who act before player j.

Proposition 3*

- Definition: i's position is more important than j's position iff $p(S \cup \{i\}) > p(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.
- Proposition 3*: Assume that the order of importance of positions is complete so that player i's position is more important than player $i+1$'s ($i = 1, 2, \ldots, n$-1).
- Then (θ, v) is an optimal investment-inducing mechanism iff θ is the identity permutation and

(θ assigns higher positions to late movers)

$$v_j = \frac{c}{p(N) - p(S_j)}$$

Decreasing Returns to Scale

- Incentives should not be affected by the order of moves:
  - EX: $p(0) = 0$, $p(1) = 1/3$, $p(2) = 1/2$, and $p(3) = 7/12$
  - Thus, the first investor increases the success probability by $1/3$, the second by $1/6$, and the third by $1/12$.
- Because of decreasing returns players' incentives to invest decline as more players contribute.
  - For example, if 1 and 2 invest, then player 3 will invest only if his reward is at least $12c$, while he can be induced to contribute at a reward of $3c$ if 1 and 2 shirk.

Decreasing Returns to Scale

- Consequently, the optimal mechanism imposes a uniform reward of $12c$ on all players.
  - Under this mechanism each agent chooses to invest even if he is the only one investing.
- Definition: We say that the technology p possesses decreasing returns to scale if
- $p(S \cup \{i\}) - p(S) < p(T \cup \{i\}) - p(T)$,
- whenever $T \subset S$.

Decreasing Returns to Scale

- Proposition 6: If p has decreasing returns to scale then the optimal *INI* mechanism pays player i the amount

$$v_j = \frac{c}{p(N) - p(N \setminus \{i\})}$$

- In particular, agents' rewards are independent of their information about peer investment.