Attitudes Toward Risk

Joseph Tao-yi Wang
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(Lecture 11, Micro Theory I)

Dealing with Uncertainty

- Preferences over risky choices (Section 7.1)
- One simple model: Expected Utility
  $$U(x_1, x_2, x_3) = \pi_1 v(x_1) + \pi_2 v(x_2) + \pi_3 v(x_3)$$
- How can old tools be applied to analyze this?
- How is "risk aversion" measured? (ARA, RRA)
- What about differences in risk aversion?
- How does a risk averse person trade state claims? (Wealth effects? Individual diff.?)

Risk Neutrality

- Consequence x_s happens in state $s = 1, \cdots, S$
- Assign (subjective) probability π_s to state s
- A prospect $(\pi; x) = ((\pi_1, \cdots, \pi_S); (x_1, \cdots, x_S))$
  - People have preferences for these prospects
- Fix and Relabel states so that $x_1 \succ x_2 \succ x_3$
  - First focus on probabilities (like 7.1)
- If one's Expected Utility is $U^0(x) = \sum_{s=1}^{S} \pi_s \underline{\underline{x_s}}$
- This person is Risk Neutral!

Risk Neutrality

- Consider two prospects $\hat{\pi}, \pi = \hat{\pi} + \Delta\pi$
  - Changing the first three probabilities
- Change in EU (=EV!) is: $\Delta U^0 = \sum_{s=1}^{3} \Delta\pi_s x_s$
- Probabilities change only in the first 3 states:

$$\sum_{s=1}^{3} \Delta\pi_s = 0 \;\; \Rightarrow \Delta\pi_2 = -\Delta\pi_3 - \Delta\pi_1$$

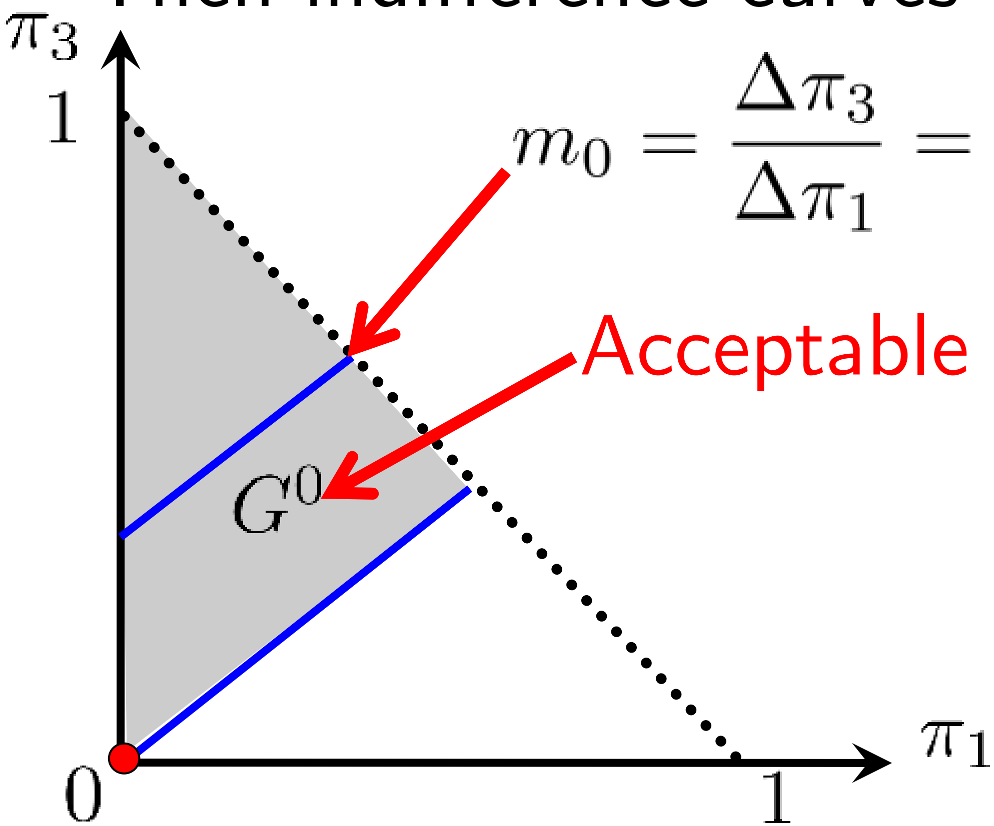
- So, $\Delta U^0 = \Delta\pi_3(x_3 - x_2) - \Delta\pi_1(x_2 - x_1)$

Risk Neutrality

- If $\Delta U^0 = \Delta\pi_3(x_3 - x_2) - \Delta\pi_1(x_2 - x_1)$
- Then indifference curves have slope

$$m_0 = \frac{\Delta\pi_3}{\Delta\pi_1} = \frac{x_2 - x_1}{x_3 - x_2}$$

Acceptable Gambles: EU=EV>0



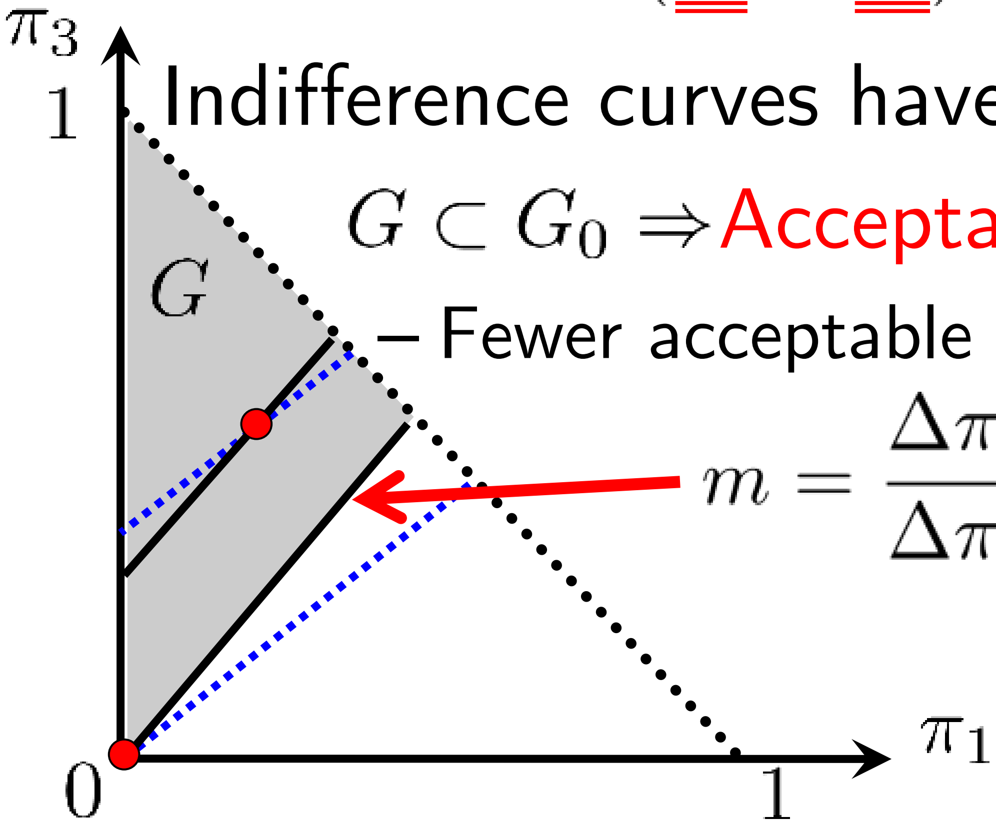
- Risk averse VNM utility $u_s = u(x_s)$

$$\Delta U = \Delta\pi_3(\underline{\underline{u_3}} - \underline{\underline{u_2}}) - \Delta\pi_1(\underline{\underline{u_2}} - \underline{\underline{u_1}})$$

- Indifference curves have slope $m = \frac{u_2 - u_1}{u_3 - u_2} > m_0$

$G \subset G_0 \Rightarrow$ Acceptable Gambles: Smaller!

  – Fewer acceptable gambles = Risk Aversion



$$m = \frac{\Delta\pi_3}{\Delta\pi_1} = \frac{u_2 - u_1}{u_3 - u_2}$$

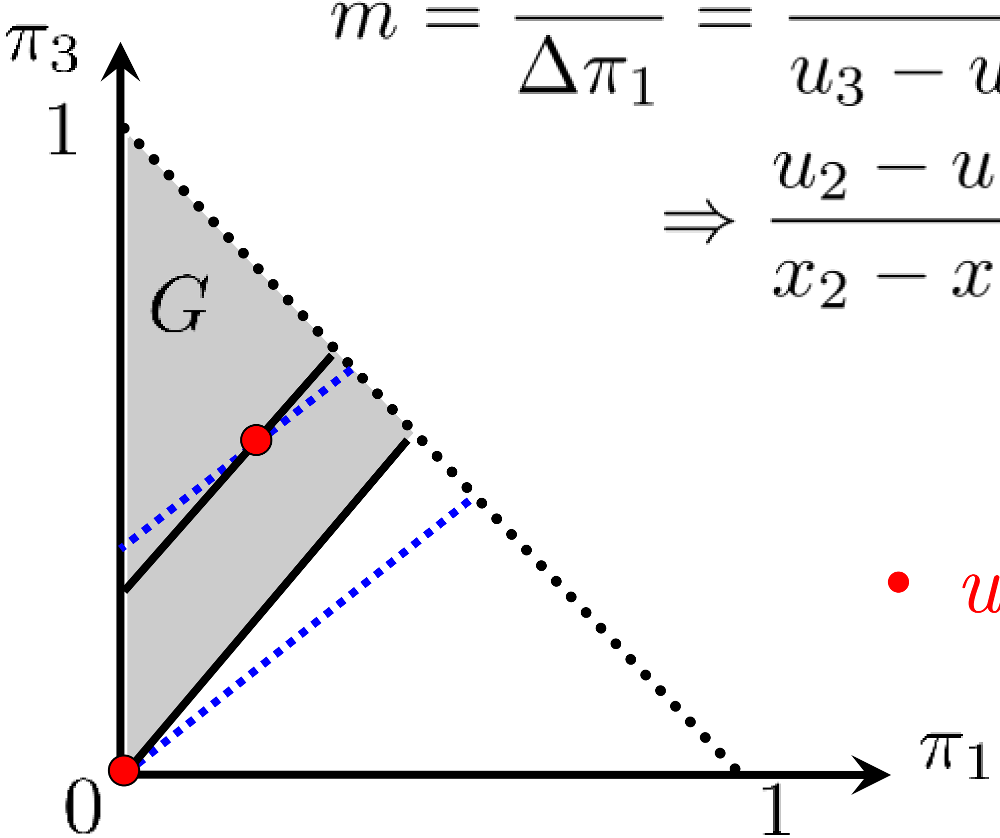
$$> m_0 = \frac{x_2 - x_1}{x_3 - x_2}$$

Risk Aversion vs. Risk Neutrality

- $G \subset G_0$ and $G \neq G_0$ (Risk Averse) if and only if

$$m = \frac{\Delta\pi_3}{\Delta\pi_1} = \frac{u_2 - u_1}{u_3 - u_2} > \frac{x_2 - x_1}{x_3 - x_2} = m_0$$

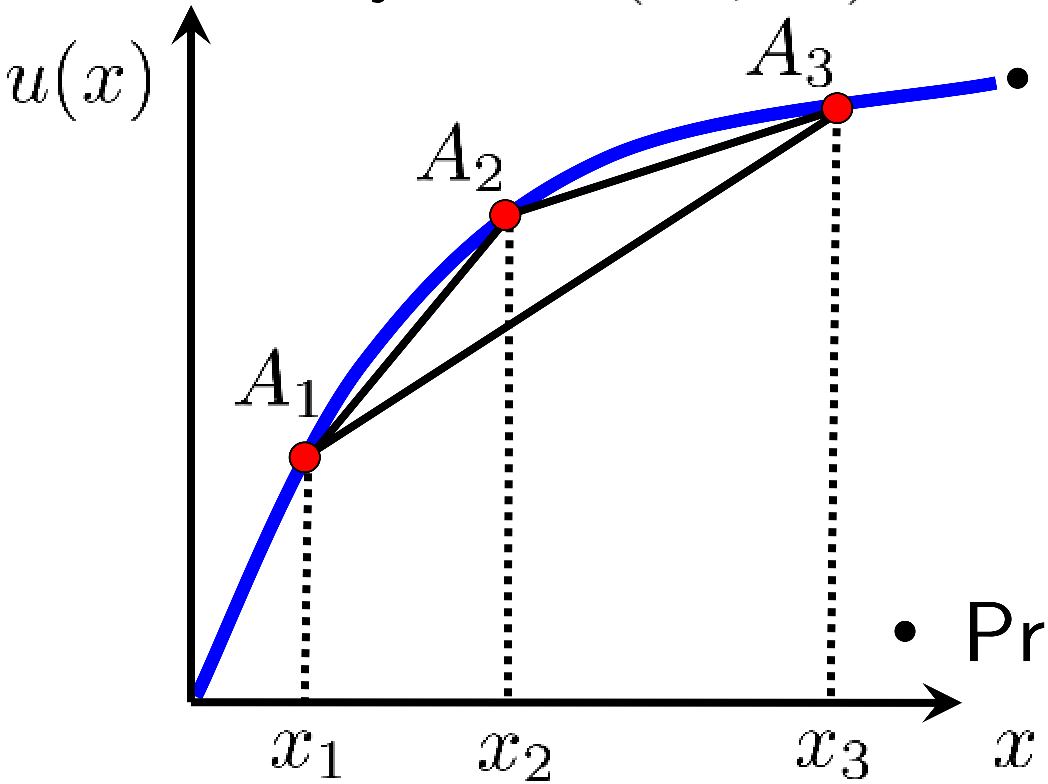
$$\Rightarrow \frac{u_2 - u_1}{x_2 - x_1} > \frac{u_3 - u_2}{x_3 - x_2}$$



- $u(.)$ is strictly concave!
  - In fact, we have...

Lemma 7.2-1: Strictly Concave Function

- $u(x), x \in \mathbb{R}$ is strictly concave if and only if for any $x_2 \in (x_1, x_3)$



- i.e. A_2 is above $\overline{A_1 A_3}$

$$\frac{u(x_2) - u(x_1)}{x_2 - x_1} > \frac{u(x_3) - u(x_2)}{x_3 - x_2}$$

- Proof: (Exercise 7.2-6)

Victor and Ursula: Set of Acceptable Gambles

- Victor and Ursula have utility functions $v(\cdot), u(\cdot)$
- If $v = g(u)$ where g increasing strictly concave
- Then, Victor has a smaller set of acceptable gambles. (I.e. Victor more risk averse than Ursula)
- Proof: Lemma 7.2-1 means g strictly concave if and only if for all $u_2 \in (u_1, u_3)$

$$m^v = \frac{v_2 - v_1}{v_3 - v_2} = \frac{g(u_2) - g(u_1)}{g(u_3) - g(u_2)} > \frac{u_2 - u_1}{u_3 - u_2} = m^u$$

Absolute Risk Aversion (ARA)

- Victor and Ursula have utility functions $v(\cdot), u(\cdot)$
- If $v = g(u)$ (g increasing strictly concave)
- Then, $v'(x) = g'(u(x))u'(x)$
- Thus, $\ln v'(x) = \ln g'(u(x)) + \ln u'(x)$

$$\Rightarrow \frac{\partial}{\partial x} \ln v'(x) = \frac{v''(x)}{v'(x)} = \frac{g''(u)}{g'(u)} + \frac{u''(x)}{u'(x)}$$

< 0

> 0

- Absolute Risk Aversion (ARA):

$$A^v(x) = -\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} = A^u(x)$$

Small Risk and Absolute Risk Aversion (ARA)

- Consider $(x_1, x_2, x_3) = (w, w+z, w+2z)$
- Choose extreme lottery $(\pi_1, 0, \pi_3) \sim (0, 1, 0)$
- Indifferent between earning z for sure and winning $2z$ with prob. π_3 (otherwise 0)

$$m(z) = \frac{u_2 - u_1}{u_3 - u_2} = \frac{\pi_3}{\pi_1} = \frac{u(w+z) - u(w)}{u(w+2z) - u(w+z)}$$

$$\boxed{\text{Claim: } m(0) = 1, m'(0) = -\frac{u''(w)}{u'(w)}}$$

$$\Rightarrow m(z) \approx m(0) + m'(0)z = 1 + A(w)z$$

- ARA = Measure of "small risk"

Small Risk and Absolute Risk Aversion (ARA)

- Consider $(x_1, x_2, x_3) = (w, w+z, w+2z)$
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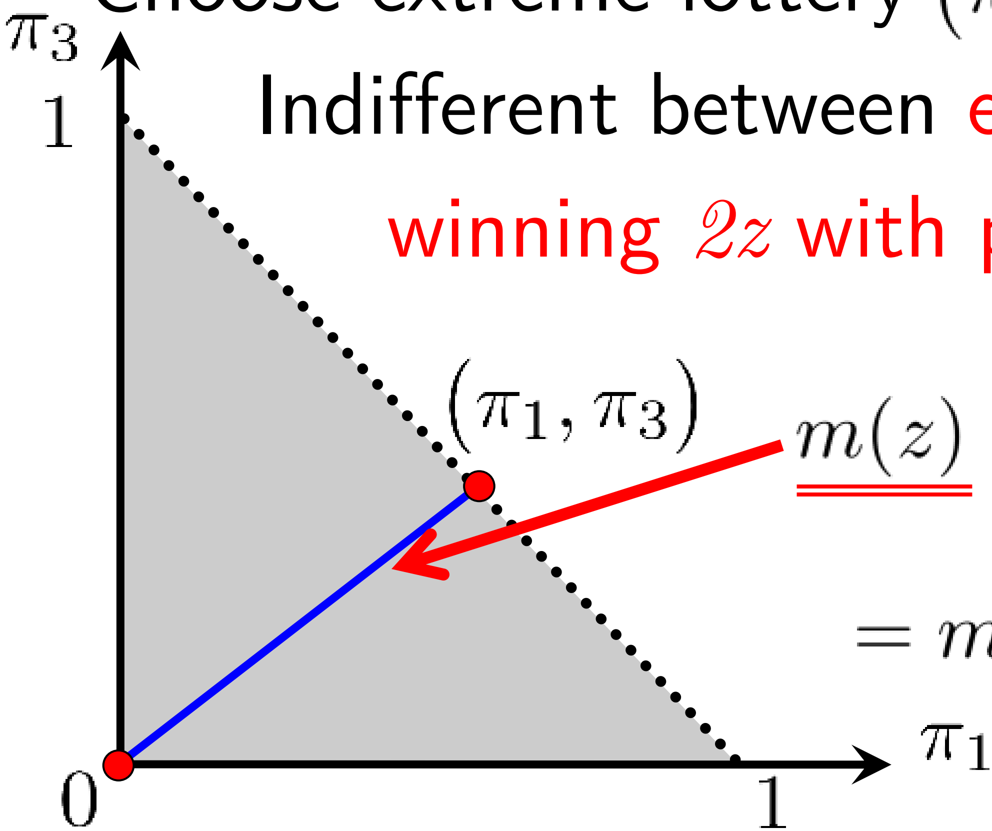


$$\underline{\underline{m(z)}} = \frac{u_2 - u_1}{u_3 - u_2} = \frac{\pi_3}{\pi_1} = \underline{\underline{\frac{n(z)}{d(z)}}}$$

$= m^0(z) = 1$ if risk neutral

> 1 if risk averse

Small Risk and Absolute Risk Aversion (ARA)

$$m(z) = \frac{u(w+z) - u(w)}{u(w+2z) - u(w+z)} = \frac{n(z)}{d(z)}$$

- Use L'Hospital's Rule to show $m'(0) = -\frac{u''(w)}{u'(w)}$:

$$m(0) = \lim_{z \to 0} \frac{n'(z)}{d'(z)} = \lim_{z \to 0} \frac{u'(w+z)}{2u'(w+2z) - u'(w+z)} = 1$$

$$\Rightarrow m'(0) = \lim_{z \to 0} \frac{m(z) - m(0)}{z} = \lim_{z \to 0} \frac{m(z) - 1}{z}$$

$$= \lim_{z \to 0} \frac{2u(w+z) - u(w) - u(w+2z)}{z(u(w+2z) - u(w+z))}$$

Small Risk and Absolute Risk Aversion (ARA)

- Use L'Hospital's Rule again:

$$\underline{\underline{m'(0)}} = \lim_{z\to 0} \frac{2u(w+z) - u(w) - u(w+2z)}{z(u(w+2z) - u(w+z))}$$

$$= \lim_{z\to 0} \frac{2u'(w+z) - 2u'(w+2z)}{u(w+2z) - u(w+z) + z\left(2u'(w+2z) - u'(w+z)\right)}$$

$$= \lim_{z\to 0} \frac{2u''(w+z) - 4u''(w+2z)}{2\left(2u'(w+2z) - u'(w+z)\right) + z\left(4u''(w+2z) - u''(w+z)\right)}$$

$$= \frac{-2u''(w)}{2u'(w) + 0\cdot(3u''(w))} \underline{\underline{= -\frac{u''(w)}{u'(w)}}}$$

Absolute vs. Relative Risk Aversion

- Absolute Risk Aversion at w $A(w) = -\frac{u''(w)}{u'(w)}$

$$\Rightarrow m(z) \approx m(0) + m'(0)z = 1 + A(w)z$$

$=$ measure of aversion to small absolute risk

- Consider $z = \theta w, m_R(\theta) = m(\theta w)$

$$\Rightarrow m'_R(\theta) = w \cdot m'(\theta w)$$

- Relative Risk Aversion at w

$$R(w) = m'_R(0) = -w \cdot \frac{u''(w)}{u'(w)}$$

State Claims

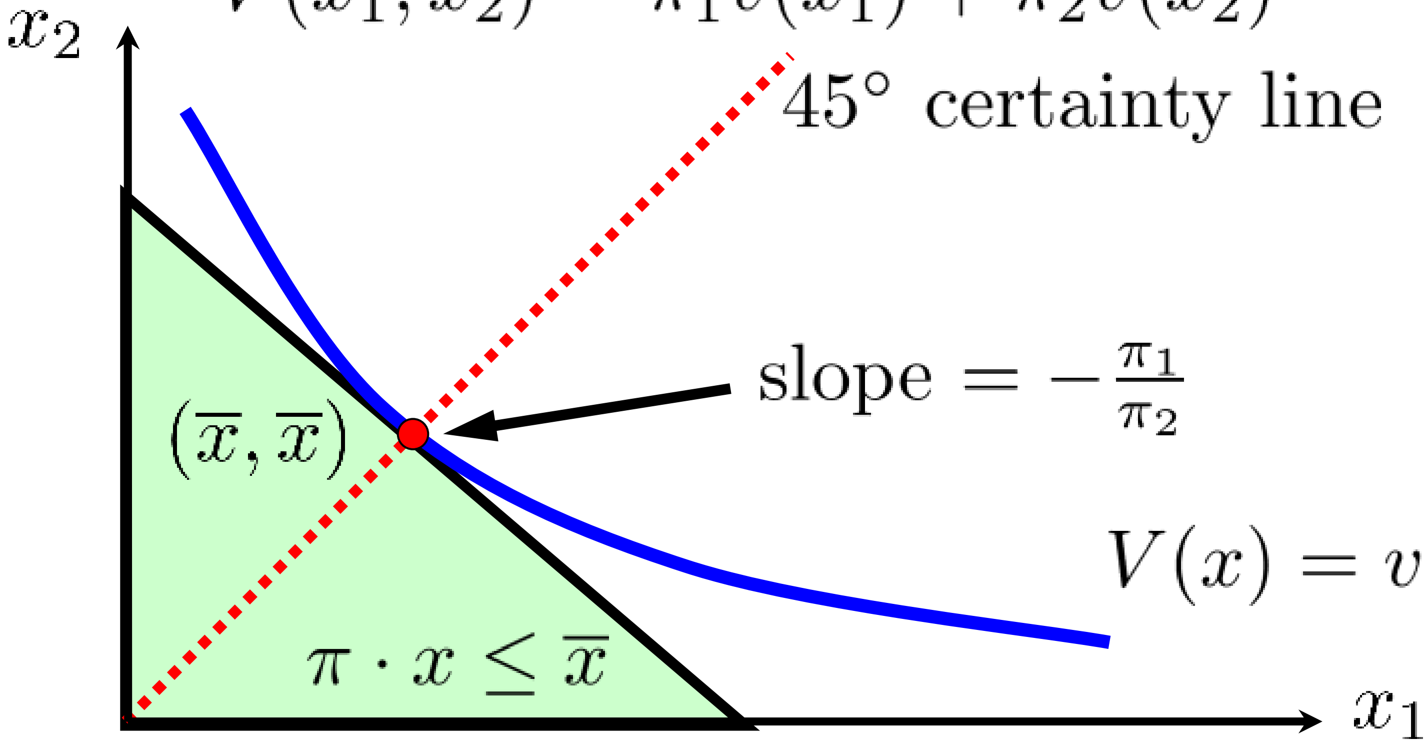
- Consequence x_s happens in state $s = 1, \cdots, S$
- Assign (subjective) probability π_s to state s
- A prospect $(\pi; x) = ((\pi_1, \cdots, \pi_S); (x_1, \cdots, x_S))$
  - People have preferences for these prospects
- Now focus on State Claims, or consumption (consequences) in each state
- EU:

$$U(\pi; x) = U(x) = \sum_{s=1}^{S} \pi_s v(x_s)$$

Example: State Claim Market for Election

- Two states: s=1: KMT wins; s=2: DPP wins
- π_s: Prob. of state s x_s: consumption in state s

$$V(x_1, x_2) = \pi_1 v(x_1) + \pi_2 v(x_2)$$

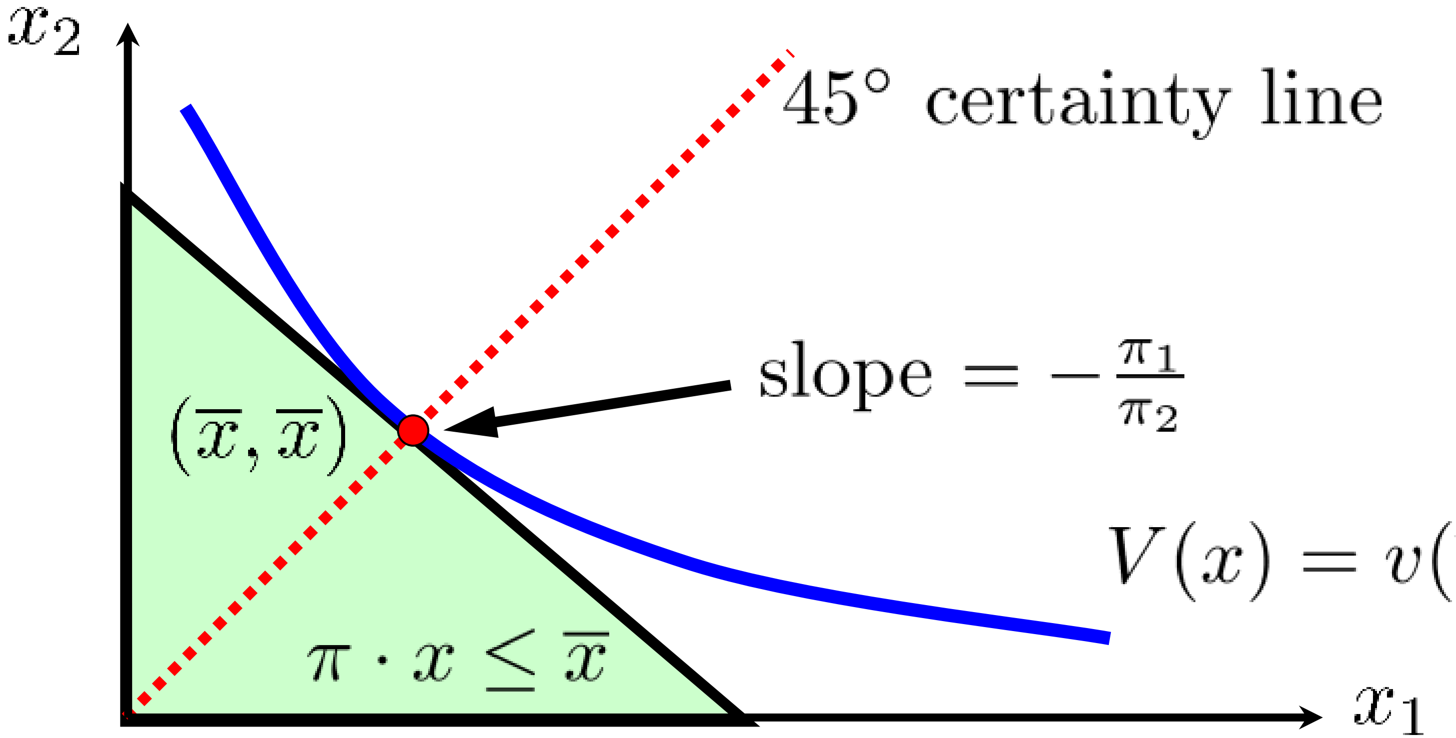


Risk Aversion: Concave $v(x)$

- Upper contour sets of *V(.)* is convex

$$V(x_1, x_2) = \pi_1 v(x_1) + (1 - \pi_1) v(x_2) \leq v(\overline{x})$$

  - Prefers certain bundle to risky ones with same EV



Jensen's Inequality

- For any probability vector π and consumption vector x , if $u(x)$ is strictly concave, then

$$\sum_{s=1}^{S} \pi_s u(x_s) \leq u\left(\sum_{s=1}^{S} \pi_s x_s\right)$$

- And inequality is "strict" unless $x_1 = \cdots = x_S$
- Proof: For S=2, strict concavity $\Rightarrow$ (if $x_1 \neq x_2$)

$$\frac{\pi_1}{\pi_1 + \pi_2} u(x_1) + \frac{\pi_2}{\pi_1 + \pi_2} u(x_2) < u\left(\frac{\pi_1 x_1 + \pi_2 x_2}{\pi_1 + \pi_2}\right)$$

Jensen's Inequality

1) For S=3, we also have (if $x_1 \neq x_2$)

$$\frac{\pi_1}{\pi_1+\pi_2}u(x_1)+\frac{\pi_2}{\pi_1+\pi_2}u(x_2)<u\left(\frac{\pi_1 x_1+\pi_2 x_2}{\pi_1+\pi_2}\right)$$

2) Concavity ⇒ $(\pi_1+\pi_2)u\left(\frac{\pi_1 x_1+\pi_2 x_2}{\pi_1+\pi_2}\right)+\pi_3 u(x_3)$

$$\leq u(\pi_1 x_1+\pi_2 x_2+\pi_3 x_3)$$

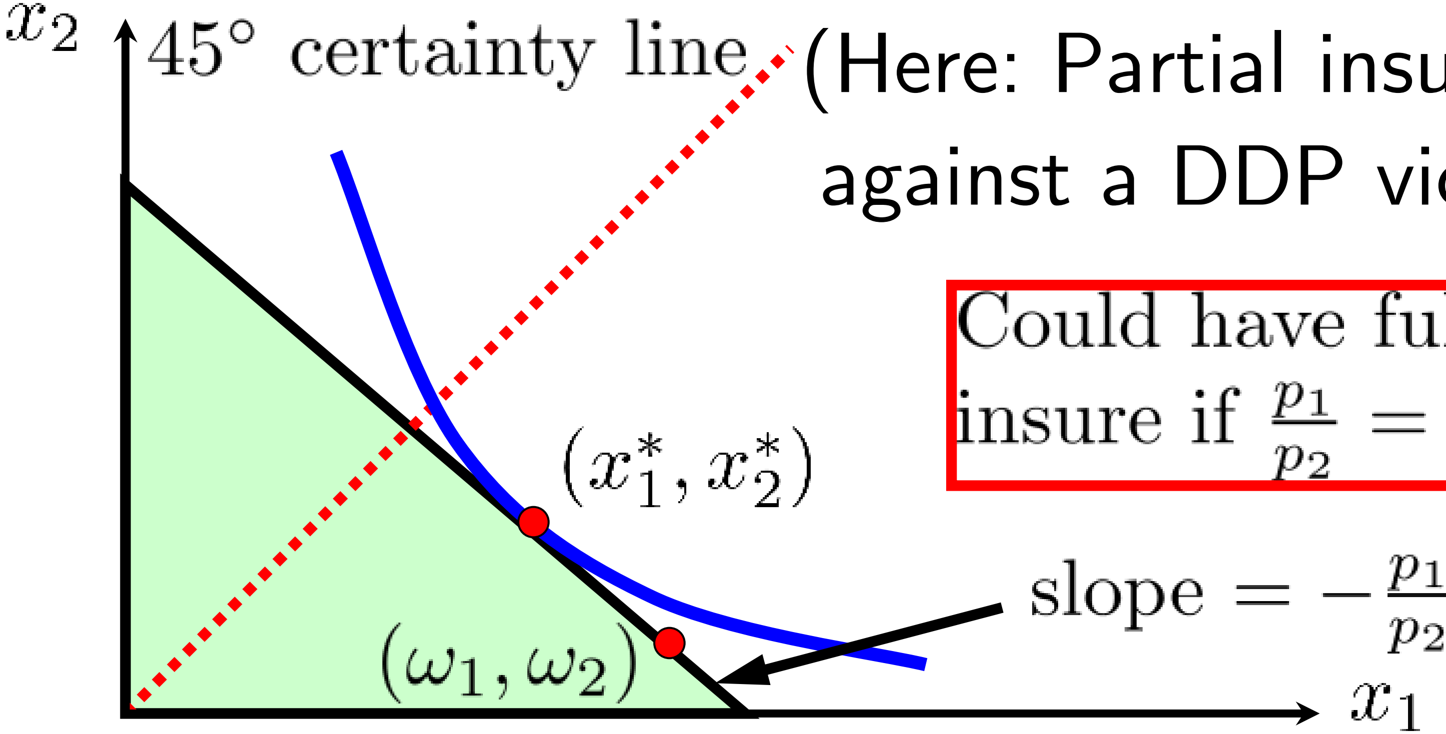
- Hence, (2) + (1) x $(\pi_1+\pi_2)$ yields:

$$\pi_1 u(x_1)+\pi_2 u(x_2)+\pi_3 u(x_3)<u(\pi_1 x_1+\pi_2 x_2+\pi_3 x_3)$$

- Similar inductive argument extends to S>3…

Trading in State Claim Markets

- ω_s: Endowment in state *s*, $\omega_1 > \omega_2$
- p_s : current price of unit consumption in state *s*
- Budget Constraint: $p_1x_1 + p_2x_2 = p_1\omega_1 + p_2\omega_2$



(Here: Partial insurance against a DDP victory)

Could have fully insure if $\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}$

Wealth↑, Will Riskiness of Optimal Choice ↑?

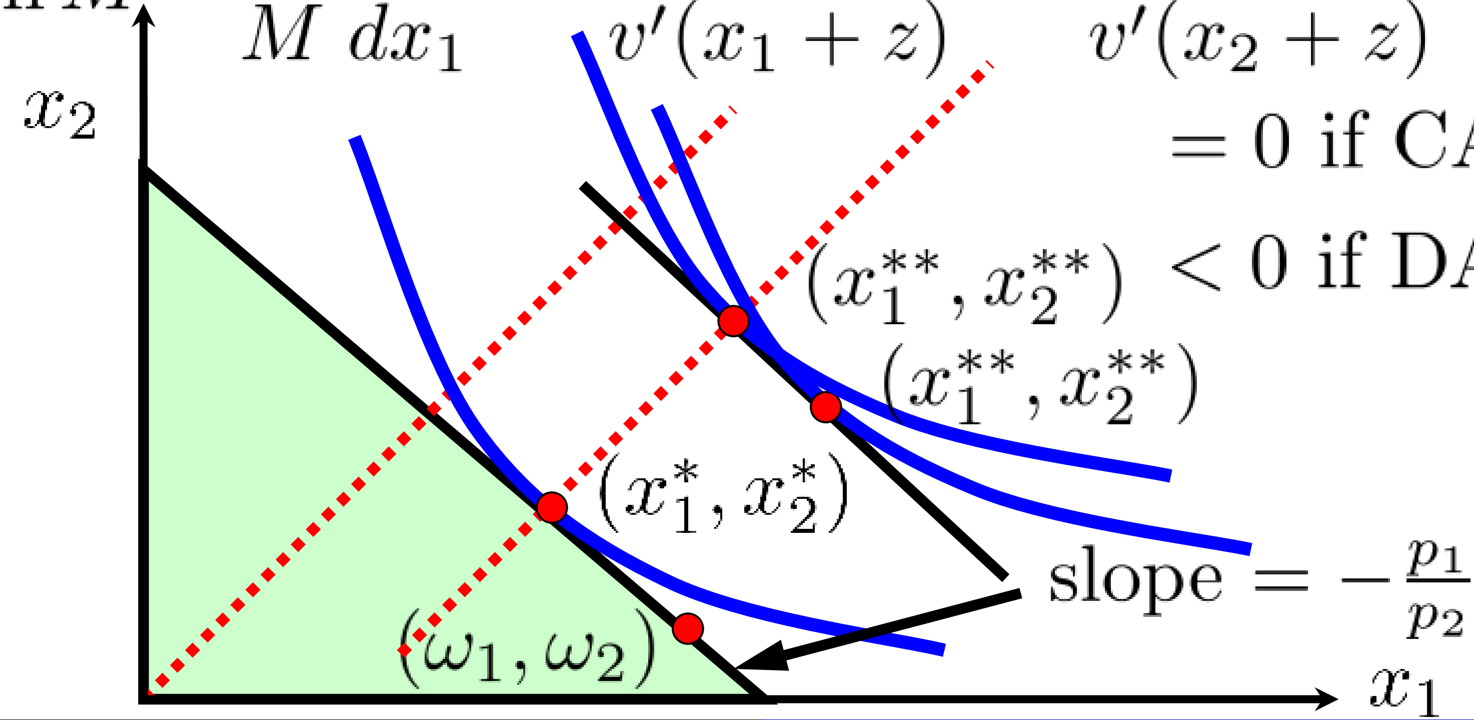
- Move from (x_1, x_2) to (x_1+z, x_2+z), log-MRS

$$\ln M = \ln v'(x_1+z) - \ln v'(x_2+z) + \ln\left(\frac{\pi_1}{\pi_2}\right)$$

$$\frac{\partial}{\partial x}\ln M = \frac{1}{M}\frac{dM}{dx_1} = \frac{v''(x_1+z)}{v'(x_1+z)} - \frac{v''(x_2+z)}{v'(x_2+z)}$$

$$= 0 \text{ if CARA}$$

$$< 0 \text{ if DARA}$$



Wealth$\uparrow$, Will Riskiness of Optimal Choice $\uparrow$?

- In words, with CARA,
- Wealth $\uparrow$ implies parallel shift; MRS same!
  - Optimal choice is as risky as original choice

- With DARA,
- Wealth $\uparrow$: Point lower than CARA; MRS $\uparrow$
  - Optimal choice is more risky than original choice
- Similar for IARA...

Simple Portfolio Choice: Riskless vs. Risky

- Ursula can invest in either:
  - Riskless asset: Certain rate of return $1 + r_1$
  - Risky asset: Gross rate of return $1 + r_2$

- If Ursula is risk averse, how high would the "risk premium" $(r_2 - r_1)$ need to be for Ursula to invest in the risky asset?

- Zero! (But risk premium affect proportions)

Simple Portfolio Choice: Riskless vs. Risky

- Using state claim formulation:
  - Risky asset yields $1 + r_{2s}$ in state s
  - Probability of state s is $\pi_s, \; s = 1, \cdots, S$
- Invests q in risky asset, $(W - q)$ in riskless one
- Final consumption in state s is

$$x_s = W(1 + r_1) + q\theta_s \qquad (\theta_s = r_{2s} - r_1)$$

- Ursula's utility:

$$U(q) = \sum_{s=1}^{S} \pi_s u(W(1 + r_1) + q\theta_s)$$

Simple Portfolio Choice: Riskless vs. Risky

- Marginal Gains from increasing q

$$U'(q) = \sum_{s=1}^{S} \pi_s u'\big(W(1+r_1) + q\theta_s\big) \cdot \theta_s$$

- Should choose q so that $U'(0) = 0$
- Since there is a single turning point by:

$$U''(q) = \sum_{s=1}^{S} \pi_s u''\big(W(1+r_1) + q\theta_s\big) \cdot \theta_s^2 < 0$$

Simple Portfolio Choice: Riskless vs. Risky

Since

$$U'(0) = u'(W(1+r_1)) \sum_{s=1}^{S} \pi_s \theta_s > 0 \Leftrightarrow \sum_{s=1}^{S} \pi_s \theta_s > 0$$

- Ursula will always buy some risky asset (unless infinitely risk averse)! The intuition is

$$U'(q) = \sum_{s=1}^{S} \pi_s \underline{u'(W(1+r_1)+q\theta_s)} \cdot \theta_s$$

- When taking no risk, each MU weighted with the same $u'(W(1+r_1))$, as if risk neutral!
- Not true for any $q > 0$
  - Depends on degree of risk aversion…

More Risk Averse Person Invest Less Risky?

- Yes!
  - Choose smaller q if everywhere more risk averse
- Proof:
- Consider Victor with utility $v(x) = g(u(x))$
  - g is increasing strictly concave
- If Ursula's optimal choice and consumption be

$$q^* \text{ and } x_s^* = W(1 + r_1) + \theta_s q^*$$

- Then, $U'(q^*) = \sum_{s=1}^{S} \pi_s u'(x_s^*) \cdot \theta_s = 0$

More Risk Averse Person Invest Less Risky?

- Claim: $V'(q^*) < 0$ (And we are done!)
- Proof:
- Order states so $\theta_1 > \theta_2 > \cdots > \theta_S$
- Let t be the largest state that $\theta_s = r_{2s} - r_1 > 0$
- Then, $u(x_s^*) \geq u(x_t^*)$ for all $s \leq t$

  $u(x_s^*) < u(x_t^*)$ for all $s > t$
- And, (by strict concavity of g)

$$g'\left(u(x_s^*)\right) \geq g'\left(u(x_t^*)\right),\ \text{for all } s \leq t$$

$$g'\left(u(x_s^*)\right) < g'\left(u(x_t^*)\right),\ \text{for all } s > t$$

More Risk Averse Person Invest Less Risky?

Hence,

$$V'(q^*) = \sum_{s=1}^{S} \pi_s g'\big(u(x_s^*)\big)u'(x_s^*) \cdot \theta_s$$

$$< \sum_{s=1}^{t} \pi_s g'\big(u(x_{\underline{\underline{t}}}^*)\big)u'(x_s^*) \cdot \theta_s$$

$$- \sum_{s=t+1}^{S} \pi_s g'\big(u(x_{\underline{\underline{t}}}^*)\big)u'(x_s^*) \cdot (-\theta_s)$$

$$= g'\big(u(x_t^*)\big) \sum_{s=1}^{S} \pi_s u'(x_s^*) \cdot \theta_s = g'\big(u(x_t^*)\big)U'(q^*) = 0$$

Summary of 7.2

- Victor is more risk verse than Ursula implies:
  - Mapping from u to v is concave
  - Victor will not accept gambles that Ursula rejects
- Absolute Risk Aversion (ARA) vs. RRA
- State Claim Market
  - Jensen's Inequality
  - Wealth effect
  - Risk averse people invest less risky
- Homework: Exercise-7.2-4 (Optional 7.2-5)

In-class Homework: Exercise 7.2-6

- $u(c), c \in \mathbb{R}$ is strictly concave if and only if for any $c_2 = (1-\lambda)c_1 + \lambda c_3 \in (c_1, c_3), 0 < \lambda < 1$

$$\Rightarrow u(c_2) = (1-\lambda)u(c_1) + \lambda u(c_3)$$

a. Rearrange and show that $u(c)$ is concave if

$$\lambda(c_3 - c_2) = (1-\lambda)(c_2 - c_1), 0 < \lambda < 1$$

$$\Rightarrow \lambda\big(u(c_3) - u(c_2)\big) = (1-\lambda)\big(u(c_2) - u(c_1)\big)$$

b. Hence show that concavity of $u(c)$ is equivalent to $\dfrac{u(c_2) - u(c_1)}{c_2 - c_1} > \dfrac{u(c_3) - u(c_2)}{c_3 - c_2}$

In-class Homework: Exercise 7.2-2

- Relative Risk Aversion at x is $R(x) = -x \cdot \dfrac{v''(x)}{v'(x)}$

a. Show that a CRRA individual's MRS $M(x_1, x_2)$ is constant along a ray from the origin. Assume he can trade state claims, show that the risk he takes rises proportionally with w.

b. Show that an individual with $v'(x) = x^{-1/\sigma}, \sigma > 0$ exhibits CRRA. Hence solve for the CRRA utility function.

c. Individuals are usually IRRA and DARA. What does this mean for the wealth expansion paths?