Recall: (M, d) is a metric space if

① $d(x,x) = 0$

② $d(x,y) > 0$ if $x \neq y$ (Positivity)

③ $d(x,y) = d(y,x)$ (Symmetry)

④ $d(x,y) + d(y,z) \geq d(x,z)$ (Triangle inequality)

Definition :

(1) A open ball (with center at x and radius r) is the set

$$N_r(x) := \{ y \in M : d(x,y) < r \}$$

A open ball $N_r(x)$ is also called a neighborhood (abbr. nbhd) of x

(2) A closed ball (with center at x and radius r) is the set

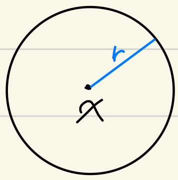
$$\overline{N_r(x)} := \{ y \in M : d(x,y) \leq r \}$$

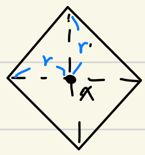
Example: ① $\mathbb{R}^1$ with Euclidean metric

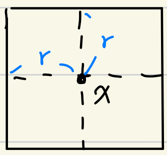
$$N_r(x) = (x-r, x+r)$$
$$\overline{N_r(x)} = [x-r, x+r]$$

② $\mathbb{R}^2$ with

(1) Euclidean metric

$N_r(x) =$ 

(2) ℓ^1 - metric 

(3) ℓ^∞ - metric 

The definition of open/closed balls depend on the metric

Now we fix a metric space (M, d), and introduce more concepts related to open/closed sets.

Definition: Let E be a subset of M

(1) A point $p \in M$ is called a limit point of E if every nbhd of p contains a point $q \neq p$ such that $q \in E$. ($\forall r > 0$, $\exists q \in N_r(p)$ s.t. $q \neq p$, $q \in E$)

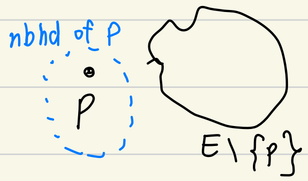
Example: $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}^n$, 0 is a limit point

(Use Archimedean property to show it)

Therefore, a point p is not a limit point of E

if $\exists$ nbhd N of p s.t. N does not contain any point $\underbrace{\text{other}}_{}$ in E other than p.

(2) A point $p \in M$ is called an isolated point of E
if $p \in E$ and p is not a limit point of E.



Example: (1) Any point in E if E is finite is isolated
(2) Any point in $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is isolated

(3) A point $p \in M$ is called an interior point of E

if $\exists$ nbhd N of p s.t. $N \subseteq E$.
($\exists\, r > 0$ s.t. $N_r(p) \subseteq E$)

Example: $E = [1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$
(1) Any $1 < p < 2$ is an interior point
(2) $p = 1$ is not an interior point.

Thm: If p is a limit point of E, then every nbhd of p contains infinitely points of E

Proof: Suppose not, $\exists$ nbhd N of p contains only $\{a_1, \cdots, a_n\} \subseteq E$.
Let $r = \min_{1 \leq j \leq n} d(p, a_j)$, then $N_{\frac{r}{2}}(p)$ contains no point in E,
so p is not a limit point.

Remark (Optional): There is another definition using the notion of the "limit" of a sequence.

Def 2: A point $p \in M$ is a limit point of E if there is a sequence $\{a_1, \cdots a_n\}$ in $E \setminus \{p\}$
such that $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $d(p, a_n) < \varepsilon$ for all $n > N$. (Notation $\lim_{n \to \infty} p_n = p$)
Exercise: Check this new definition is equivalent to the previous one we used)

Definition (open sets and closed set)

(1) A set E is <u>open</u> if every point of E is an interior point

(2) A set E is <u>closed</u> if E contains all the limit points of E.

Propositions: Let E be one of the sets $(a,b]$, $[a,b)$, $[a,b]$, (a,b).

(1) The set of all limit points of E is $[a,b]$

(2) The set of all interior points of E is (a,b)

Corollary:

(1) Interval (a,b) is open in $\mathbb{R}$ (So (a,b) is an "open interval")

(2) Interval $[a,b]$ is closed in $\mathbb{R}$ (So $[a,b]$ is a "closed interval")

(3) Intervals $(a,b]$ and $[a,b)$ are neither closed nor open.

Corollaries (1), (2) are special cases of the following theorem.

Theorem: (1) The open ball $N_r(x)$, where $r>0$, is an open set

(2) The closed ball $\overline{N_r(x)}$ is a closed set

Proof: (1) $\forall\ p \in N_r(x)$, $d(p,x) < r$

Claim: $N_\varepsilon(p) \subseteq N_r(x)$ if $\varepsilon = r - d(p,x)$

Proof of claim: $\forall\ q \in N_\varepsilon(p)$, we have

$d(q,x) \leq d(q,p) + d(p,x) < \varepsilon + d(p,x) = r \Rightarrow q \in N_r(x)$

(2) $\forall\ p \notin \overline{N_r(x)}$, $d(p,x) > r$

Claim: $N_\varepsilon(p)$ is disjoint from $\overline{N_r(x)}$ if $\varepsilon + r < d(p,x)$

Proof of claim: $\forall q \in N_\varepsilon(p)$, $d(q,x) > d(p,x) - d(q,p) > d(p,x) - \varepsilon > r$

$\Rightarrow p$ is not a limit point of $\overline{N_r(x)}$

So all the limit points of $\overline{N_r(x)}$ is in $\overline{N_r(x)}$

Definition: Let E be a subset of M, (E does not need to be closed or open.) Then

(1) $\bar{E}$, the **closure** of E, is $E \cup E'$, where E' is the set of all limit points of E.

(2) $\text{int } E$, the **interior** of E is the set of all interior points of E.

Theorem: For any set $E \subseteq M$,

(1) $\bar{E}$, the closure of E, is closed

(2) $\text{int } E$, the interior of E, is open

etch of proof:

(1) If p is a limit point of $\bar{E}$ $\Leftrightarrow$ $\forall r>0$, $\exists q \in \bar{E}$ s.t. $q \neq p$, but $d(q,p) < \frac{r}{2}$

Claim: p is also a limit point of E (so $p \in \bar{E}$)

Case 1: $q \in E \Rightarrow d(q,p) < \frac{r}{2} < r$
So, $q \in N_r(p)$
$\Rightarrow$ p is a limit point of E

Case 2: $q \notin E \Rightarrow q$ is a limit point $\exists s \in E$ s.t. $d(q,s) < \frac{r}{2}$

(2) Let $x \in \text{int } E$, $\exists r > 0$ s.t. $N_r(x) \subseteq E$

Claim: $N_r(x) \subseteq \text{int } E$

(Use the fact that $N_r(x)$ is open).

Theorem: For any set $E \subseteq M$

(1) $\text{int } E \subseteq E \subseteq \bar{E}$

(2) $\text{int } E = E$ if an only if E is open

(3) $\bar{E} = E$ if and only if E is closed

Relationship between open and closed sets.

Theorem: If E is an open set, then $E^c = M \setminus E$ (the complement of E) is closed, and vice versa.

pf: E is open

$\Longleftrightarrow$ any point $p \in E$ has a neighborhood $N_r(p) \subset E$

$\Longleftrightarrow$ $\forall\ p \in E$, $\exists$ nbhd $N_r(p)$ disjoint from E^c

$\Longleftrightarrow$ $\forall\ p \in E$, p is not a limit point of E^c

$\Longleftrightarrow$ all the limit points of E^c are in E^c

$\Longleftrightarrow$ E^c is closed

Union and Intersections of open sets / closed sets

Lemma (De Morgan's laws on sets)

Let $\{E_\alpha\}$ be a collection of sets, the following properties hold

(1) $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

(2) $\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha$

Theorem

(1) Arbitrary (finite or infinite) union of open sets is open
(2) Finite intersection of open sets is open

By lemma and the previous theorem, these can be stated in terms of open sets

(3) Arbitrary intersection of open sets is open
(4) Finite union of closed sets is open

Proof: (1) Let $\{S_\alpha\}$ be open sets

$\forall x \in \bigcup_\alpha S_\alpha$, $x \in S_\alpha$ for some α

$\Rightarrow \exists$ nbhd N of x, $N \subseteq S_\alpha$ (then $N \subseteq \bigcup_\alpha S_\alpha$)

Therefore, $\bigcup_\alpha S_\alpha$ is open

(2) Let $S_1, \cdots, S_n$ be opent sets

$\forall x \in \bigcap_i S_i$, $x \in S_i$ for all $1 \leq i \leq n$

$\Rightarrow \exists$ nbhd $N_{r_i}(x)$ of x, $N_{r_i}(x) \subseteq S_i$ for all $1 \leq i \leq n$

Let $r = \min_{1 \leq i \leq n} r_i$, $N_r(x) \subseteq N_{r_i}(x) \subseteq S_i$ for all $i \Rightarrow N_r(x) \subseteq \bigcap_i S_i$

Remark: A more general notion of the open sets in metric space is topology. A topology on a set X is a collection of subset of X, called open sets and satisfying the following axioms.

(1) X, ϕ are open sets.
(2) Any arbitrary union of open sets is open.
(3) Any finite intersection of open sets is open.

The above theorem shows that every metric space can be given a topology, in which the open set in this topology are open sets defined by the metric and neighborhoods

Definition: A set E is dense in metric space X if every point of X is a limit point of E or in E

$\Leftrightarrow$ $\overline{E} = X$

$\Leftrightarrow$ Every open set of X contains $p \in E$

Example: $\underbrace{\mathbb{Q}}_{\text{countable}}$ is dense in $\underbrace{\mathbb{R}}_{\text{uncountable}}$