# § Countable and uncountable sets

Question: How do we count ?

We can count from 1 to n, where n is a natural number

But there are numbers which are too many to list all of them one by one, such as $\mathbb{Z}$, $\mathbb{Q}$.

Observations: ① Counting is to associate a set $S$ with a natural number $|S|$.

For those set that is impossible to do this, we can say this set is infinite.

② We can compare the size of two sets $A, B$ by comparing the numbers $|A|$, $|B|$

Question: ① Is there any concept beyond infinity ?

② How can we compare numbers that are infinitely large ? Are their different concepts of infinity ?

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- Counting is to assign an index to each object.

$$S = \{a, b, c, d, e\}$$
$$\;\;\;\;\;1\;\;2\;\;3\;\;4\;\;5$$

- In mathematics, a function from a set $X$ to a set $Y$ assigns each element of $X$ to an element of $Y$.

Terminology: $f: X \to Y$

$f$: function / mapping
$X$: domain
$Y$: codomain
$f(A) := \{ f(x) : x \in A \} \subseteq Y$ : image of $A$ (for any $A \subseteq X$)
$f^{-1}(B) := \{ x : f(x) \subseteq B \}$ : preimage of $B$ (for any $B \subset Y$)
(The preimage can be defined even when $f$ does not have a inverse function)

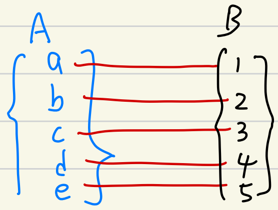
Remark: Some people distinguish between a "function" and a "map (mapping)". The use the word "function" if $X, Y$ are subsets of $\mathbb{C}, \mathbb{R}$, and use the word "map" if $X$ and $Y$ are general sets, sometimes with a function with additional structures or specific properties.

- Counting can not be done using arbitrary functions to $\{1,\cdots,n\}$,

  We must use a "bijection".

**Definitions:** If $f: A \to B$ is a function.

- We say $f$ is surjective (onto), if $f(A)=B$ $\twoheadrightarrow$
- We say $f$ is injective (one-to-one, 1-1), if $f(x)=f(y)$ implies $x=y$ $\hookrightarrow$
- We say $f$ is bijective ($f$ is a bijection), if $f$ is 1-1 and onto. $\hookrightarrow\!\!\!\!\twoheadrightarrow$
  , write $A \overset{f}{\sim} B$ or $A \sim B$.




A bijection gives a pairing of elements of two sets.
In mathematics, we also call it an "1-1 correspondence".

- Existence of a bijection is an equivalence relation on sets.
  (Check: (1) $A\sim A$ (2) $A\sim B \Rightarrow B\sim A$ (3) $A\sim B, B\sim C \Rightarrow A\sim C$)

**Definitions:** Call a set $A$ finite if $A \sim \{1,\cdots,n\}$ for some $n\in\mathbb{N}$.
or if $A$ is the empty set. ($A=\phi$)
Otherwise, $A$ is infinite

**Definitions:** An infinite set $A$ is countable if $A\sim\mathbb{N}$.
Otherwise, $A$ is uncountable.

Well-definedness: Show that $\mathbb{N}$ is infinite.

Idea of proof: Show that $\nexists$ bijection $[n] \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbb{N}$ by induction on $n$.

(See next page)

Question:
We know that a countable set exists ($\mathbb{N}$ is countable)
Is there any uncountable set? (and thus, there are different concepts regarding the infinity?)
We will answer it later.

<u>Claim</u>: $\nexists\, n$ such that $\{1, \cdots, n\} \xhookrightarrow{f} \mathbb{N}$ where $f$ is a bijection.

(pf) We prove by induction on $n$

① Base case $(n=1)$:

Assume $1 \longrightarrow f(1) \subseteq \mathbb{N}$,

then $f(1)+1 \neq f(x)$ for some $x \in \{1\}$

② Inductive step:

Assume $\{1, \cdots, n\} \not\hookrightarrow \mathbb{N}$, (proof by contradiction)

Suppose $\exists\, f$ such that $\{1, \cdots, n+1\} \xhookrightarrow{f} \mathbb{N}$

Then, $\bar{f} : \{1, \cdots, n+1\} \setminus \{n+1\} \hookrightarrow \mathbb{N} \setminus \{f(n+1)\}$

$$\Rightarrow \mathbb{N} \setminus \{f(n+1)\} \xhookrightarrow{g} \mathbb{N} \text{ by } \begin{cases} g(x) = x, & x < f(n+1) \\ g(x) = x-1, & x > f(n+1) \end{cases}$$

Thus, $g \circ \bar{f}$ is a bijection from $\{1, \cdots, n\} \longrightarrow \mathbb{N}$

Contradiction ($\rightarrow\!\leftarrow$)

③ Hence, by induction on $n$,

$\nexists\, n$ such that $\{1, 2, \cdots, n\} \xhookrightarrow{f} \mathbb{N}$,

so $\mathbb{N}$ is infinite. #

**Definition**: A sequence is a collection of objects that allows repetition and with an order.
$(a_n)_{n=1}^{\infty} = a_1, a_2, \cdots, a_n, \cdots\cdots$

Equivalently, it is a function with domain $\mathbb{N}$.

Note: ① Any countable set can be listed in a sequence.
② If $a_m = a_n$ implies $m=n$, then the sequence $(a_n)_{n=1}^{\infty}$ is countable

More countable sets:

**Theorem**: ① $\mathbb{Z}$ is countable

② The union of countably many countable set is countable.
③ An infinite subset of a countable subset is countable.

④ If $A$ is countable, then the set $A^n = \{(a_1, \cdots, a_n) : a_i \in A \text{ for all } 1 \le i \le n\}$ is countable

Proof: ②: If $A_1, A_2 \cdots\cdots$ are countable sets.

$$\begin{array}{lllll} A_1 \to & a_{11} & a_{12} & a_{13} & \\ A_2 \to & a_{21} & a_{22} & a_{23} & \cdots \\ A_3 \to & a_{31} & a_{32} & a_{33} & \end{array}$$

Construct a new sequence in this order.

③ Let $n_1 = \inf\{i : x_i \in E\}$
$\vdots$
$n_k = \inf\{i : x_i \in E \text{ and } i \ge n_{k-1}\}$
Then $E = \{x_{n_1}, x_{n_2}, \cdots\}$ is a sequence, or $E = \{f(k) = x_{n_k} : k \in \mathbb{N}\}$

**Corollary**: $\mathbb{Q}$ is countable, since $\mathbb{Q} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$ is infinite.

**Definition**: A set is "at most countable" if it is

if it is either finite or countable

Note: The terms "countable" and "at most countable" is not universally used.
An alternative style uses countably infinite to mean countable
and countable to mean at most countable.
Here we use the former.

**Proposition**: ① Any sequence is at most countable
② Any subset of an at most countable set is at most countable.

**Theorem** (The corresponded version of at most countable sets)

After replacing countable by at most countable, the previous theorem is still true.

# Uncountable set

**Theorem (Cantor 1874):** $\mathbb{R}$ is uncountable (infinite but not countabe)

(pf) Suppose that $\mathbb{R}$ is countable, then $[0,1] \subseteq \mathbb{R}$ is at most countable.

Proof by Contradiction

$\Rightarrow$ We can list $[0,1]$ by a sequence $\begin{cases} x_1 = 0.134\cdots \\ x_2 = 0.245\cdots \\ x_3 = 0.436\cdots \\ \vdots \end{cases}$ $\Rightarrow$ Construct $x = 0.257\cdots$ ($1 \neq 2$, $4 \neq 5$, $6 \neq 7$)

so ith digit differs from ith digit of $x_i$

Then, $x \neq x_i$ $\forall x_i \in [0,1] \Rightarrow x \notin [0,1] \subseteq \mathbb{R}$ $\rightarrow\leftarrow$

Theorem (Cantor 1891) For any set $A$, we have $A \nsim 2^A$

Use the idea in the previous proof (Cantor's diagonal argument)

Recall that $A \sim B$ is an equivalence relation on set.

Definition: We say $A, B$ have the same cardinality if $A \sim B$.

Cardinality can be represented in two ways.

(1) $A \sim B$, bijections between $A$ and $B$

(2) The cardinal number assigned to sets having the same cardinality as $A$.

Notations: $|A|$, $\mathrm{card}(A)$, $\# A$

Examples. (1) finite set : $n$

(2) $\{f : \mathbb{N} \to \{0,1\}\}$ : $2^{\mathbb{N}}$

(3) $\{f : \mathbb{R} \to \{0,1\}\}$ : $2^{\mathbb{R}}$

## § Metric spaces

**Definition**: A metric space is a set $M$ together with a distance function $d: M \times M \to \mathbb{R}$ satisfying the following axioms for all points $x, y, z \in M$.

① $d(x,x) = 0$

② (Positivity) $d(x,y) > 0$ if $x \neq y$

③ (Symmetry) $d(x,y) = d(y,x)$

④ (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

Main Example: ① Euclidean space $\mathbb{R}^n$ with $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

, where $\|\vec{z}\| = \sqrt{z_1^2 + \cdots + z_n^2}$ is the Euclidean norm in $\mathbb{R}^n$

The distance $d$ is called the Euclidean distance

When $n=1$, $d(x,y) = |x-y|$ is the absolute value of the difference $|x-y|$

Proof that $(\mathbb{R}^n, d)$ is a metric space

- Check ①, ②, ③ directly
- The axiom ④ follows from the triangle inequality of the Euclidean norm.

$$d(\vec{x},\vec{y}) + d(\vec{y},\vec{z}) = \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\| \geq \|(\vec{x}-\vec{y}) + (\vec{y}-\vec{z})\| = \|\vec{x}-\vec{z}\| = d(\vec{x},\vec{z})$$

② Complex space $\mathbb{C}^n$ with $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

where $\|\vec{z}\| = \sqrt{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n}$ and $\bar{\cdot}$ denote the complex conjugate.

Other examples of metric space

① Euclidean space $M = \mathbb{R}^n$

a. $\mathbb{R}^n$ with $\ell^1$-distance: $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

b. $\mathbb{R}^n$ with $\ell^\infty$-distance: $d(x,y) = \max_{1 \leq i \leq d} |x_i - y_i|$

(Sup-norm $\|x\| = \max_{1 \leq i \leq d} |x_i|$ )

② Function space :

Let $X = \mathbb{R}^n$ or $\mathbb{C}^n$

$M = \{\text{Integrable function } f: X \to \mathbb{R}\}$ ( usually $f$ is continuous with finite integral, $\int_X f(x)\,dx < \infty$

( All the integrable functions on $X$ )

a. $M$ with $L^2$-distance :

$$d(f,g) = \|f-g\| \text{ , where } \|h\| = \sqrt{\int_X h(x)\overline{h(x)}\,dx}$$

b. $M$ with $L^1$-distance :

$$d(f,g) = \|f-g\| \text{ , where } \|h\| = \int_X h(x)\,dx$$

c. $M$ with $L^\infty$-distance :

$$d(f,g) = \sup_{x\in X} |f(x)-g(x)|$$

③ discrete space (Optional)

a. $M = S^n = \{(x_1,\cdots,x_n), x_i \in S\}$ ( string of equal length with characters in $S$ )

$d(x,y) = \#\{1 \le i \le n : x_i \ne y_i\}$ ( number of positions at which the corresponding symbols are different )

( Hamming distance)

b. $M = \left\{ \square \text{ in a maze } \begin{array}{|c|c|c|c|} \hline & \times & \times & \\ \hline & & & \\ \hline \times & \times & \times & \\ \hline & & & \\ \hline \end{array} \right\}$

$d(\square_1, \square_2)$ = The length of shortest path between $\square_1$ and $\square_2$ .

c. For any undirected graph $G$, the set $V$ of vertices of $G$

$M = V$, $d(x,y)$ = The length of shortest path between $x$ and $y$ is a (disrete) metric space.

Exercise : Check a. b. are instances of c.

Remark: These spaces are "disrete" in the sense that there exist a real number $\varepsilon > 0$ such that $d(x,y) > \varepsilon$ for all $x, y$. Obviously, every finite set is disrete.

It can be shown that every subset of a disrete space is an open set (to be defined in the next lecture), so the topologies on these spaces are completely characterized and can not be further studied. We do not discuss discrete space in this class.

Systematic methods to construct a vector space

(1) If $\|\cdot\|$ is a norm on a vector space $V$, then $d(x,y) = \|x-y\|$ is a metric on $V$

(2) If $\langle\ ,\ \rangle$ is an inner product on a vector space $V$,

then $\|v\| = \sqrt{\langle v,v\rangle}$ is a norm.

Corollary: metric spaces $\subseteq$ normed spaces $\subseteq$ inner product spaces

Examples: (1) $\ell^1$-distance, $\ell^\infty$-distance is induced from $\ell^1$-norm, $\ell^\infty$-norm
$(L^1)$ $(L^\infty)$ $(L^1)$ $(L^\infty)$

(2) Euclidean norm on $\mathbb{R}^n$, $\mathbb{C}^n$.