Review:

Our ways constructed rational numbers and real numbers share a similar process

Step 1. There is a smaller set $\underline{A}$ and operations on it. We want to extend the operations to a bigger set.

Step 2. Construct a set $\underline{B}$ and operations on it

Step 3. Show that there is a subset $\underline{A'}$ of B, such that operations on B restricted to A' is equivalent to the operation on B.

Example: (1) Construct rational numbers from integers

$$A = \{m : m \in \mathbb{Z}\}, \quad B = \mathbb{Q} = \left\{\frac{m}{n} : m,n \in \mathbb{Z} \text{ and } n \neq 0\right\}, \quad A' = \left\{\frac{m}{1} : m \in \mathbb{Z}\right\}.$$

$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = cb$

(2) Construct real numbers from rational numbers.

$$A = \mathbb{Q}, \quad B = \text{all the cuts}, \quad A' = \{q \in \mathbb{Q} : q < r \text{ for a } r \in \mathbb{Q}\}.$$

Remark: There might be step 4 as follows.

Step 4. Establish some kinds of uniqueness that B have. For instance, show that there is a property shared by any constructions of B.

In example 2, we construct real number by cuts. Even though there might be different ways constructing the real numbers, the resulting sets share the same properties: $\mathbb{R}$ is an <u>ordered field</u> and have the <u>least upper bound property</u>. Therefore, <u>we can forget that</u> real numbers are cuts, but an extension of rational numbers characterized by these properties.

Today:

① Complex numbers ② principle of induction

- Complex numbers (denoted by $\mathbb{C}$)
  - a number system that extends <u>real numbers</u> with a specific element i, satisfying the equation $i^2 = -1$.
  - Every complex number can be expressed in the form $a + bi$.
  - Addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$

    Multiplication $(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i$

We assert that i exists implicitly in the definition, does such i really exist?

It requires a rigorous definition.

The complex numbers can be define in a new way.

- a complex number is a vector (a,b) in $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$.
- Addition: $(a,b)+(c,d) = (a+c, b+d)$
  Multiplication: $(a,b)\times(c,d) = (ac-bd, ad+bc)$
  (compare to $(a+bi)+(c+di) = (a+c)+(b+d)i$
  $(a+bi)\times(c+di) = (ac-bd)+(ad+bc)i$)
- Exercise: Check that $\mathbb{R}^2$ with these $+, \times$ is a field = a set with structure we can define $+, \times$, satisfying axiom A1 ~ A5, M1 ~ M5, D
- R $\mathbb{R}$ is a subfield of $\mathbb{C}$;
  $R \longrightarrow \mathbb{C}$
  $a \longmapsto (a,0)$
  $1 \mapsto (1,0)$
  $0 \mapsto (0,0)$
  $(a,0)+(b,0) = (a+b, 0)$
  $(a,0)\times(b,0) = (ab, 0)$

- $(0,1)\cdot(0,1) = (-1,0)$
  denote it by i.

Suppose that $z = a+bi$

- real part: $Re(z) = a$, imaginary part: $Im(z) = b$
- complex conjugate: $\bar{z} = a - bi$
- Check that for any two complex numbers z, w
  $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{z\cdot w} = \bar{z}\cdot\bar{w}$
  $z\cdot\bar{z} = a^2+b^2 \geq 0$ if $z = a+bi$ $z\cdot\bar{z}$ is a non-negative real number
- Define $|z| = \sqrt{z\cdot\bar{z}} \geq 0$.
- $|z\cdot w| = |z|\cdot|w|$ ($|(a+bi)(c+di)|^2 = (ac-bd)^2+(ad+bc)^2 = (a^2+b^2)(c^2+d^2) = |a+bi|^2|c+di|^2$)

Euclidean space

- $\mathbb{R}^k = \{(x_1,\cdots,x_k) : x_i \in \mathbb{R}$ for all $1\leq i\leq k\}$, $(x_1,\cdots,x_k)$ is a vector
  (1) Addition: $(x_1,\cdots,x_k)+(y_1,\cdots,y_k) = (x_1+y_1,\cdots,x_k+y_k)$
  (2) Scalar Multiplication: $c(x_1,\cdots,x_k) = (cx_1,\cdots,cx_k)$ for $c\in\mathbb{R}$
  (3) Inner product: $\vec{x}\cdot\vec{y} = \sum_{i=1}^{k} x_i y_i$ if $\vec{x} = (x_1,\cdots x_k)$, $\vec{y} = (y_1,\cdots y_k)$
  (or $\langle x,y\rangle$)
  (4) Norm: $|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} = (\vec{x}\cdot\vec{x})^{\frac{1}{2}}$, $|\vec{x}| \geq 0$ for all $\vec{x}$, $|\vec{x}| = 0$ if and only if $\vec{x} = 0$.
  ↳ length of a vector

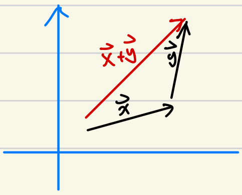
- $\mathbb{C}^k = \{(x_1,\cdots,x_k) : x_i \in \mathbb{C}$ for all $1\leq i\leq k\}$
  (1), (2) Addition, Multiplication: The same as $\mathbb{R}^k$.
  (3): $\vec{x}\cdot\vec{y} = \sum_{i=1}^{k} x_i \bar{y_i}$ if $\vec{x} = (x_1,\cdots x_k)$, $\vec{y} = (y_1,\cdots y_k)$,
  (or $\langle x,y\rangle$)
  where $\bar{y_i}$ is the complex conjugate of y_i
  (4) $|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} = (\vec{x}\cdot\vec{x})^{\frac{1}{2}}$, $|\vec{x}| \geq 0$ for all $\vec{x}$, $|\vec{x}| = 0$ if and only if $\vec{x} = 0$

  Check that $\vec{x}\cdot\vec{x}$ is a real number.

- When $k=1$, $|\vec{x}| = |a+bi| = \sqrt{a^2+b^2}$ if $x = a+bi$, the same as the norm defined on the complex numbers.
- When $\vec{x}, \vec{y}$ are real vectors, $\vec{x}\cdot\vec{y}$ and $\vec{x}$ are the same as the ones in $\mathbb{R}^k$.

- The norms in the Euclidean spaces satisfying the "triangle inequality".

$$|\vec{x}+\vec{y}| \le |\vec{x}|+|\vec{y}| \quad \text{for all } \vec{x},\vec{y}\in \mathbb{R}^k \text{ or } \mathbb{C}^k.$$



- **Corollary:** The Euclidean spaces are instances of 'metric space', a space equipped with distance function between any two points in the space (The complete definition will be given in the next lecture).

- Proof: For any vectors $\vec{x},\vec{y}$, we have

$$
\begin{aligned}
|\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})\cdot\overline{(\vec{x}+\vec{y})} && \text{(definition)}\\
&= \vec{x}\cdot\vec{x}+\vec{y}\cdot\vec{x}+\vec{x}\cdot\vec{y}+y\cdot\vec{y} && \text{(axiom D: distributive law)}\\
&= |\vec{x}|^2+|\vec{y}|^2+2\,\mathrm{Re}|\vec{x}\cdot\vec{y}|\\
&\le |\vec{x}|^2+|\vec{y}|^2+2\,|\vec{x}\cdot\vec{y}|\\
&\le |\vec{x}|^2+|\vec{y}|^2+2|\vec{x}||\vec{y}| \quad \cdots \quad (*)\\
&= (|\vec{x}|+|\vec{y}|)^2 .
\end{aligned}
$$

The $(*)$ inequality is the following inequality.

- **Cauchy-Schwarz inequality:**

If $\vec{x},\vec{y}\in\mathbb{C}^k$, then $|\vec{x}\cdot\vec{y}| \le |\vec{x}|\cdot|\vec{y}|$ (vector form)

. The equality holds if $\vec{x}=c\cdot\vec{y}$ for a complex number c.

Equivalently, if $x_1,\cdots,x_k,y_1,\cdots y_k$ are complex numbers, then

$$\left|\sum_{i=1}^{k} x_i\bar{y}_i\right| \le \left(\sum_{i=1}^{k} x_i\cdot\bar{x}_i\right)\left(\sum_{i=1}^{k} y_i\cdot\bar{y}_i\right) = \left(\sum_{i=1}^{k}|x_i|^2\right)\left(\sum_{i=1}^{k}|y_i|^2\right)$$

The equality holds if there is a c in $\mathbb{C}$ such that $x_i = cy_i$ for all $1\le i\le k$ (number form)

Proof: Let $\vec{x},\vec{y}\in\mathbb{C}^k$.

For all $t\in\mathbb{C}$, we have

$$
\begin{aligned}
0 &\le \|\vec{x}-t\vec{y}\|^2\\
&= \langle x-ty, x-ty\rangle\\
&= \langle x,x\rangle - t\langle y,x\rangle - \bar{t}\langle x,y\rangle + |t|^2\langle y,y\rangle
\end{aligned}
$$

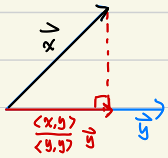
Choose $t=\frac{\langle x,y\rangle}{\langle y,y\rangle}$, then the inequality become

$$0\le \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$$

$$\Rightarrow \quad 0 \le |\vec{x}|^2|\vec{y}|^2 - |\langle x,y\rangle|^2$$

- **Geometrical interpretation:**

$$|\vec{x}|^2 = \left|\vec{x}-\frac{\langle x,y\rangle}{\langle y,y\rangle}\vec{y}\right|^2+\left|\frac{\langle x,y\rangle}{\langle y,y\rangle}\vec{y}\right|^2 \ge \left|\frac{\langle x,y\rangle}{\langle y,y\rangle}\vec{y}\right|^2 \quad \text{(Pythagorean theorem)}$$



$$\Rightarrow \quad \langle x,x\rangle \ge \frac{\langle x,y\rangle^2}{\langle y,y\rangle}$$

§ Induction

- **Principle of (mathematical) induction (POI)**

  Let S be a subset of $\mathbb{N}$, such that
  ① $1 \in S$
  ② If $k \in S$, then $k+1 \in S$ for all k.
  Then $S = \mathbb{N}$

- **Proofs by induction**

  Let $P(n)$ be statement indexed by $n \in \mathbb{N}$

  base case : Show that $P(1)$ is true

  inductive step: Show that if $P(k)$ true, then $P(k+1)$ is true.

  Then $P(n)$ holds for all $n \in \mathbb{N}$.

  Proof : Let $S = \{ n \in \mathbb{N} : P(n) \text{ is true} \}$, and use POI .

- **Writing proofs by induction.**

  - Indicate you're using induction and which variable you will induct on.
  - Show that the base case is true.
  - Show the inductive step
  - State your conclusion by using POI.

  Exercise : Prove that for any natural number n
  $1 + 3 + \cdots + 2n-1 = n^2$.

  - We prove by induction on n
  - Base case : $n = 1$: $1 = 1^2$.
  - Inductive step: If $1 + \cdots + (2n-1) = n^2$

    $1 + \cdots + (2n-1) + [2(n+1) - 1] = n^2 + 2(n+1) + 1$
    $= (n+1)^2$
  - By POI, $1 + \cdots + (2n-1) = n^2$ is true for all $n \in \mathbb{N}$.

- **Variants of mathematical induction**

  (1) **Strong mathematical induction**

  The inductive step needs truth of $P(1) \ldots, P(k)$ to prove $P(k+1)$.
  That is, if $P(1), \cdots, P(k)$ are true, then $P(k+1)$ is true.

  Exercise : A prime is a number p that for any integers m, n such that $a, b \geq 2$, we have $p \neq ab$.
  Prove that any integer $n \geq 2$, n is a product of prime numbers.

(2) Base case other than 0, 1

Base case can start from any natural numbers k,
such that $P(k)$ is true

(3) Induction on more than one indices.

The statement we want to prove that $P(m,n)$ is true for all m,n.

① First, use induction on m

Base case : $P(1,1)$ is true

Inductive step : $P(m,1) \longrightarrow P(m+1,1)$

By POI, $P(m,1)$ is true for any m

② Then, use induction on n.

Base case : $P(m,1)$ is true for any m

Inductive step : $P(m,n) \longrightarrow P(m,n+1)$ for any m.

By POI, $P(m,n)$ is true for any m,n

Exercise : $\sum_{i=1}^{m} \sum_{j=1}^{n} (i+j) = \frac{mn(m+n+2)}{2}$