Last time :

- Definition of sets: A collection of objects or numbers.

  Operation on sets: $A \cap B$, $A \cup B$, $A \setminus B$, $A \times B$

- Equivalence Relation $=$, $\sim$ ... on a set

  Reflexivity $a \sim a$

  Symmetry $a \sim b \Rightarrow b \sim a$

  Transitivity $a \sim b, b \sim c \Rightarrow a \sim c$

- rational numbers : fractions or ratios of two integers

  $$\frac{a}{b} \quad \text{or} \quad a : b$$

  Addition $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

  Multiplication $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Today, we are going to introduce

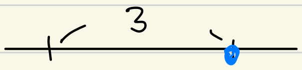
(1) Constrution of real numbers ($\mathbb{R}$) from rational numbers ($\mathbb{Q}$)

(2) Well-defined arithmetic operations and orders on $\mathbb{R}$.

(3) Least upper bound property (the completeness of the real numbers)

What is a number ?

- Greeks : There is a correspondance between lengths and numbers.
  geometrical quantity — algebraic objects



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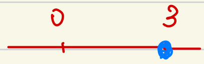
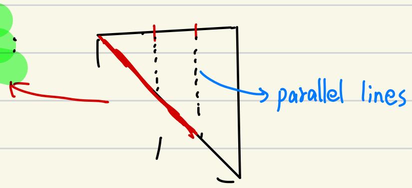


$\Rightarrow$ A number is a point on the line.

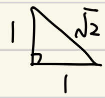
- Points in interest : Can be constructed by straightedge and compass.

  - Construct $\frac{2}{3}$:



  - Rational numbers can be constructed from similar triangles.

  - Construct $\sqrt{2}$:



Exercise : There is no rational solution of the equation $x^2 = 2$, so $\sqrt{2}$ is not rational.

Does $\sqrt{2}$ really exists ?

- As a length, it is constructable.
- As a number, it should be compatible with the arithmetic operations $+ , - , \times , \div \cdots\cdots$.
  But how we can we define these operations ? We only know how to do these computations in rational numbers

The problem was solved in 19-th century, by giving the formal definitions of "real numbers".

These numbers is fundamental in real analysis (the study of real numbers and the functions defined on it) , and also provide rigorous foundation of calculus

Why we need real numbers?

- Foundation of real analysis and calculus, and also all the science built on them.
- A mathematical model for continuous physical quantities, such as the position in the space, time, etc.
- Good mathematical properties that can used to define functions and solutions of equations $x^2=2$

Things need to be known before we explicitly construct real numbers.

- The real numbers are pure and abstract mathematical objects, and the construction seems artificial. But some properties that was used in the construction, such as the least upper bound property, are also useful afterwards.

- There are other ways constructing the real numbers, and it can be shown that they are all equivalent.
  - Cauchy sequences of rational numbers.

Construction of real numbers $\mathbb{R}$

- Idea: Every real number is associated with a cut

- Dedekind: A cut α is a subset of $\mathbb{Q}$ s.t.

  ① $\alpha \neq \phi, \mathbb{Q}$

  ② If $p \in \alpha$, $q \in \mathbb{Q}$ and $q<p$, then $q \in \alpha$ (closed downward) $\Rightarrow$ All lower ration numbers are in the set α

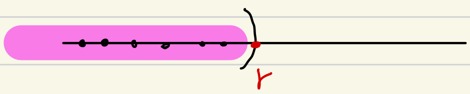
  ③ If $p \in \alpha$, then there is a $q \in \alpha$ s.t. $p<q$ (no largest number)

  $\Rightarrow$ For every rational number, there is always a larger rational number in the set!

Example: (1) $\alpha = \{x \in \mathbb{Q} : -1 < x < 1\}$ is not a cut (② fails)

(2) $\beta = \{x \in \mathbb{Q} : x \leq -1\}$ is not a cut (③ fails)

(3) For any $r \in \mathbb{Q}$, $r^* = \{x \in \mathbb{Q} : x < r\}$ is a cut.



Proposition: The set $\{x \in \mathbb{Q} : x<0\} \cup \{x \in \mathbb{Q} : x^2 < 2\}$ is a cut.

This might be the construction of $\sqrt{2}$

Theorem: ① $\mathbb{R}$ is an ordered field
(Rudin 1.19) ② $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

- Roughly speaking, a field is a mathematical structure that you can the following operations: addition, subtraction, multiplication, division.

- **Formal definition:**

A field is a set F with two operation $\underset{\text{addition}}{+}$, $\underset{\text{multiplication}}{\times}$, satisfying the following axioms.

- **Addition:**
  (A1) $x+y \in F$ for all $x, y \in F$ (F is closed to addition)
  (A2) $x+y = y+x$ for all $x, y \in F$ (Can change the order of addition)
  (A3) $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$ (Can combine in addition)
  (A4) F contains an element 0 such that $0+x = x+0 = x$ for all $x \in F$ (zero!)
  (A5) For every $x \in F$, there is an element $(-x) \in F$ such that $x+(-x) = 0$ (negation)
  (Then the subtraction $x-y$ can be defined as $x+(-y)$)
  $-x$ is called the additive inverse (of x).

- **Multiplication:**
  (M1) $xy \in F$ for all $x, y \in F$ (F is closed to multiplication)
  (M2) $xy = yx$ for all $x, y \in F$ (Can change the order of multiplication)
  (M3) $(xy)z = x(yz)$ for all $x, y, z \in F$ (Can combine in multiplication)
  (M4) F contains an element 1 such that $1x = x1 = x$ for all $x \in F$ (one!)
  (M5) For every $x \in F$, there is an element $1/x \in F$ such that $x \cdot (1/x) = 1$ (inverse)
  (Then the division x/y can be defined as $x \cdot (1/y)$)
  $1/y$ is called the multiplicative inverse (of y)

- **distributive law**
  (D) $x(y+z) = xy + yz$ for all $x, y, z \in F$ (Interaction between addition & multiplication)

Examples: (1) Rational numbers (last lecture) $\mathbb{Q}$

$(\mathbb{Q}, +, \times)$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

(2) Real numbers (this lecture) $\mathbb{R}$

(3) Complex numbers (next lecture) $\mathbb{C}$

Question: We defined that every real number is a "cut" a subset of $\mathbb{Q}$.
What is the addition and multiplication on it?

If α, β are two cuts,

Addition $\alpha+\beta := \{r+s : r\in\alpha, s\in\beta\}$

Check that ① $\alpha+\beta$ is a cut (closed downward, no largest element) (See next page)

② The axioms A1 – A5 is satisfied.

additive identity : $0^* = \{x\in\mathbb{Q} : x<0\}$

additive inverse of α : $\beta = \{p\in\mathbb{Q} : \exists\, r>0 \text{ s.t. } -p-r\notin\alpha\}$

there is a

Check that $\alpha+\beta = 0^*$

Multiplication : (1) If $\alpha,\beta > 0^* = \{x\in\mathbb{Q} : x<0\}$ (We will define the order on $\mathbb{R}$)

$\alpha\beta := \{p : p<rs \text{ for some } r\in\alpha, s\in\beta\}$

(2) Extend the definition to $\alpha,\beta\in\mathbb{Q}$ by

$$\begin{cases} (-\alpha)\cdot\beta = -\alpha\beta \\ \alpha\cdot(-\beta) = -\alpha\beta \\ (-\alpha)\cdot(-\beta) = \alpha\beta \\ 0^*\cdot\alpha = \alpha\cdot 0^* = 0^* \end{cases}$$

Check that : ① $\alpha\cdot\beta$ is a cut

③ The axioms M1 – M5, D are satisfied.

The details checking these axioms can be found in Rudin Chapter 1 Appendix.

Definition: An <u>order</u> on set is a relation $<$ satisfies that

① trichotomy : If $x,y\in S$, then exactly one of these are true.

$x<y$, $x=y$, $x>y$.

② Transitivity : If $x<y$ and $y<z$, then $x<z$

Example: (1) In $\mathbb{Q}$, we say $\frac{m}{n}$ is <u>positive</u> if $mn>0$

Then define that $\frac{a}{b} < \frac{c}{d}$ if $\frac{a}{b} - \frac{c}{d}$ is positive.

(2) Given a set S, we can define an order on all the subsets as follows :

$A<B$ if and only if $A \subsetneq B$

Definition : The order on $\mathbb{R}$ is defined as follows:

$\alpha<\beta$ if and only if $\alpha \subsetneq \beta$

① $\alpha+\beta \neq \phi, \mathbb{Q}$ since $\alpha \neq \mathbb{Q}, \beta \neq \mathbb{Q}$.

(pf) $\alpha \neq \mathbb{Q} \Rightarrow \exists\, a \notin \alpha,\ a \in \mathbb{Q}$.

$\beta \neq \mathbb{Q} \Rightarrow \exists\, b \notin \beta,\ b \in \mathbb{Q}$.

② For $r+s \in \alpha+\beta$, for all $q < r+s$, $q \in \mathbb{Q} \Rightarrow q \in \alpha+\beta$

(pf) $q-s < r \Rightarrow q-s \in \alpha$ (since α is a cut)

$\Rightarrow q = \underbrace{(q-s)}_{\in \alpha} + \underbrace{s}_{\in \beta} \Rightarrow q \in \alpha+\beta$ #

Theorem: $\mathbb{Q}$ is a ordered subfield of $\mathbb{R}$, in the sense that under the map

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{R} \\ \cup & & \cup \\ x & \longmapsto & x^* \end{array}$$

, the addition, multiplication, order in $\mathbb{Q}$ are preserved.

Proposition: (1) If $a \leq b$, then $a+c \leq b+c$

(2) If $r > 0$ and $a < b$, then $ra < rb$. for all $a, b, c, r \in \mathbb{R}$.

Remark: This kind of field is called ordered field.

least upper bound

Def: Let $E \subset S$, S is ordered.
If there exists a $\beta \in S$ s.t.
for all $x \in E$ we have $x \leq \beta$,
then β is called an upper bound for E.

Def: Let $E \subset S$, if $\exists\ \alpha \in S$ s.t.
(1) α is an upper bound of E.
and (2) if $r < \alpha \Rightarrow r$ is not an upper bound of E
((2) is equivalent to that r is an upper bound of $E \Rightarrow r \geq \alpha$)
Then α is called the least upper bound (l.u.b.) of E or supremum of E.
In this case, we write $\alpha = \sup E$.

Example: Let $S = \mathbb{Q}$
(1) E is a set with finite elements, $\sup E$ = largest element in E.
(2) $E = \{1 - \frac{1}{n},\ n \in \mathbb{N}\}$, $\sup E = 1$
(3) $E = \{x \in \mathbb{Q} : x^2 < 2\}$, $\sup E$ does not exist.

Least upper bound property

Thm: $\mathbb{R}$ has the least upper bound property.
That is, for every non-empty subset A of $\mathbb{R}$
if A has an upper bound, then it also has a l.u.b. in S.

Sketch of proof: A is a collection of cuts, with upper bound β.
Let $\gamma = \bigcup \{\alpha : \alpha \in A\}$ (Notice that α is a cut, so it is a subset of $\mathbb{Q}$).
Check that ① γ is an cut
② $\gamma = \sup A$.

Example: $E = \{x \in \mathbb{Q} : x^2 < 2\}$, $\sup E$ exists in $\mathbb{R}$.

Exercise: Let $\alpha = \sup E$, then $\alpha^2 = 2$. In this sense $\alpha = \sqrt{2}$.

Similarly, the lower bound and greatest lower bound (infimum) of a set can be defined.

Def: Let $E \subset S$, S is ordered.
If there exists a $r \in S$ s.t.
for all $x \in E$ we have $x \geq r$,
then r is called a lower bound for E.

Def: Let $E \subset S$, if $\exists\, \alpha \in S$ s.t.

(1) α is a lower bound of E.

and (2) if $r > \alpha \Rightarrow r$ is not a lower bound of E

((2) is equivalent to that r is a lower bound of $E \Rightarrow r \leq \alpha$)

Then α is called the greatest lower bound of E or infimum of E.

I this case, we write $\alpha = \inf E$.

If a set $E \subset \mathbb{R}$ has a lower bound, then $\inf E$ always exist due to the following fact and the least upper bound property

Proposition: $\inf E = -\sup(-E)$.

Useful fact:

$$\inf\left\{\frac{1}{n},\ n \in \mathbb{N}\right\} = 0$$

$\Leftrightarrow$ $\forall\, y > 0$, there is a $n \in \mathbb{N}$ s.t. $y > \frac{1}{n}$

This is a corollary of the following property

Proposition (Archimedean property)

$$\forall\, x, y > 0,\ \underset{\text{there is a}}{\exists}\, n \in \mathbb{N} \text{ such that } nx > y.$$

Useful fact ($\mathbb{Q}$ is dense in $\mathbb{R}$)

$$\forall\, x, y \in \mathbb{R},\ x < y \Rightarrow \exists\, q \in \mathbb{Q} \text{ s.t. } x < q < y.$$

Next time: complex numbers

the principle of induction.