Introduction to Quantitative Methods, Quiz 2

1. Let $\mathbb{N}$ denote the set of positive integers. Answer the following questions.

   a. (20 points) Give the definitions of least upper bound of a set $S \subset \mathbb{R}$.

   b. (20 points) What is the value of $\inf\{\frac{1}{x^n} : n \in \mathbb{N}, x \in \mathbb{R} \text{ and } x > 1\}$

2. (30 points) Let $A \subseteq B \subseteq \mathbb{R}$, prove that $\sup A \overset{\leq}{\cancel{\geq}} \sup B$ and $\inf A \overset{\geq}{\cancel{\leq}} \inf B$

3. Recall that the mathematical construction of a real number is a cut. A cut α is a subset of $\mathbb{Q}$ satisfying the following conditions.

   (1) $\alpha \neq \emptyset, \mathbb{Q}$

   (2) If $p \in \alpha, q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$ (closed downwards).

   (3) If $p \in \alpha$, then there is a $q \in \alpha$ such that $p < q$ (no largest number).

   Answer the following questinos:

   a. (30 points) If a relation $\leq$ defined on cuts is defined such that $\alpha < \beta$ if and only if $\alpha \subsetneq \beta$ for any two cuts α, β. Show that $<$ is an order on cuts.

   b. (20 points) If p is a rational number and $p^2 < 2$, let $q = \frac{2p+2}{p+2}$. Show that $p < q$ and also $q^2 < 2$. Then deduce that the set $\{x \in \mathbb{Q} : x < 0\} \cup \{x \in \mathbb{Q} : x^2 < 2\}$ is a cut.

1. Let $\mathbb{N}$ denote the set of positive integers. Answer the following questions.

   a. (20 points) Give the definitions of least upper bound of a set $S \subset \mathbb{R}$.

1.(a)

An upper bound x of a set S is a number x such that $x \geq r$ for all $r \in S$

A Least Upper Bound x of a set S is a number $x \in \mathbb{R}$, s.t

1. x is an upper bound of S

2. If $r < x$, r is not an upper bound of S.

   b. (20 points) What is the value of $\inf\{\frac{1}{x^n} : n \in \mathbb{N}, x \in \mathbb{R} \text{ and } x > 1\}$

1.(b) Claim $0 = \inf\left\{\frac{1}{x^n} : n \in \mathbb{N}, \begin{matrix} x \in \mathbb{R} \\ x > 1 \end{matrix}\right\}$

i.e. For all $y > 0$, $\exists\, n$ s.t. $y \geq \frac{1}{x^n}$

Lemma $x^n = [1+(x-1)]^n \geq 1 + n(x-1)$

(pf) $n=1$: $x = 1+(x-1)$

$n=2$: $x^2 = [1+(x-1)]^2 = 1+2(x-1)+(x-1)^2 \geq 1+2(x-1)$

$n-1 \to n$: $x^n = [1+(x-1)]^n = [1+(x-1)]\cdot[1+(x-1)]^{n-1}$

$\geq [1+(x-1)]\cdot[1+(n-1)(x-1)] = 1+n(x-1)+(n-1)(x-1)^2$

$\geq 1+n(x-1)$ #

Archimedean Property:

If $a, b > 0$, $a, b \in \mathbb{R}$ $\exists\, n$ s.t. $na > b$.

Let $\begin{matrix} a = x-1 \\ b = \frac{1}{y} - 1 \end{matrix}$ $\Rightarrow \exists\, n$ s.t. $\underline{n(x-1)} + 1 > \underline{\left(\frac{1}{y} - 1\right)} + 1 = \frac{1}{y}$

Hence, $\frac{1}{y} \leq n(x-1)+1 \leq x^n$, $\Rightarrow y \geq \frac{1}{x^n}$ #

2. (30 points) Let $A \subseteq B \subseteq \mathbb{R}$, prove that $\sup A \leq \sup B$ and $\inf A \geq \inf B$

Claim: We only need to prove that

(1) $\sup B$ is an upper bound of A $\Rightarrow \sup B \geq \sup A$ ← least U. B.

(2) $\inf B$ is a lower bound of A $\Rightarrow \inf B \leq \inf A$ ← largest L. B.

pf of (1): For all $x \in B$, $x \leq \sup B = b$

$\Rightarrow$ For all $y \in A \subseteq B$, $y \leq b$

$\Rightarrow$ b is an U. B. of A.

pf of (2) is similar. #

3. Recall that the mathematical construction of a real number is a cut. A cut α is a subset of $\mathbb{Q}$ satisfying the following conditions.

(1) $\alpha \neq \emptyset, \mathbb{Q}$

(2) If $p \in \alpha, q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$ (closed downwards).

(3) If $p \in \alpha$, then there is a $q \in \alpha$ such that $p < q$ (no largest number).

Answer the following questinos:

a. (30 points) If a relation $\leq$ defined on cuts is defined such that $\alpha < \beta$ if and only if $\alpha \subsetneq \beta$ for any two cuts α, β. Show that $<$ is an order on cuts.

For all cuts α, β, r, an order on cut satisfies:

(1) Either $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$ is true.

(2) If $\alpha < \beta$. $\beta < r$, then $\alpha < r$.

pf of (1): Given two cuts α, β.

Case 1: $\alpha = \beta$ (as sets) $\Rightarrow$ $\alpha = \beta$ (as order) ok.

Case 2: $\alpha \neq \beta$, so $\exists x \in \alpha$ such that $\underline{x \notin \beta}$

Need to show that $\alpha > \beta$ or $\alpha < \beta$.

Claim: For all $y \in \beta$, we have $y < x$.

(1) If $y = x$: $x \in \beta$ $(\rightarrow\leftarrow)$

(2) If $y > x$: since β is a cut, $y \in \beta$, $x \in \mathbb{Q}$, $y > x \Rightarrow x \in \beta$ $(\rightarrow\leftarrow)$

(3) So, $y < x$ is the only possibility. #

Then, since α is a cut, $\Rightarrow y \in \alpha$ for all $y \in \beta$. $\Rightarrow \beta \subseteq \alpha$.

pf of (2): $\left.\begin{array}{l}\alpha < \beta \Rightarrow \alpha \subseteq \beta. \\ \beta < r \Rightarrow \beta \subseteq r\end{array}\right\} \Rightarrow \alpha \subseteq \beta \subseteq r \Rightarrow \alpha < r.$ #

(For all $x \in \alpha$, $x \in \alpha \subseteq \beta \subseteq r \Rightarrow x \in r$)

b. (20 points) If p is a rational number and $p^2 < 2$, let $q = \frac{2p+2}{p+2}$. Show that $p < q$ and also $q^2 < 2$. Then deduce that the set $\{x \in \mathbb{Q} : x < 0\} \cup \{x \in \mathbb{Q} : x^2 < 2\}$ is a cut.

(1) $p < q \iff p < \frac{2p+2}{p+2} \iff p^2 + 2p < 2p+2 \iff p^2 < 2.$ #

(2) $q^2 < 2$: $q^2 - 2 = \frac{(2p+2)^2}{(p+2)^2} - 2 = \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p+2)^2}$

$= \frac{2(p^2-2)}{(p+2)^2} < 0$ #

Let $\alpha = \{x \in \mathbb{Q} : x < 0\} \cup \{x \in \mathbb{Q} : x^2 < 2\}$

(1) $\alpha \neq \phi, \mathbb{Q}$;

(pf) $0 \in \alpha$, but $2^2 > 2 \Rightarrow 2 \notin \alpha$. #

(2) If $p \in \alpha$, $q \in \mathbb{Q}$, then $p < q$.

(pf) If $q < 0$: $q \in \alpha$.

If $q \geq 0$; $p \geq q \geq 0$. Since $p^2 < 2$, $q^2 \leq p^2 < 2$. So, $q \in \alpha$. #

(3) If $p \in \alpha$, then $q = \frac{2p+2}{p+2}$ satisfies (1) $q \in \mathbb{Q}$.

(2) $q > p$

(3) $q^2 < 2$.