Online Math Camp (24F)

Quiz 1 (8/12)

1. Let $A = \{2, 3, 4\}$, $B = \{1, 4\}$

a. (10 points) List all the subset of A (including the empty set).

$A = \{2,3,4\}$, $\{2,3\}$, $\{3,4\}$, $\{2,4\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\phi = \{\}$. 8 subsets.

b. (20 points) What is $A \cap B, A \cup B, A \setminus B$, $A \times B$

$A \cap B = \{4\}$, $A \cup B = \{1,2,3,4\}$, $A \setminus B = \{2,3\}$

$A \times B = \{(2,1),(2,4),(3,1),(3,4),(4,1),(4,4)\}$

c. (10 points) Let $C = \{1, 2, \cdots n\}$ for a positive integer n. How many subsets are there in C.

Let $S \subset C$, for each $1 \leq x \leq n$, either $x \in S$ or $x \notin S$.
Hence, there are 2^n subsets.

2. The ***rational numbers*** $\mathbb{Q}$ are the set of numbers that can expressed by fractions. Formally,

$$\mathbb{Q} := \{\frac{m}{n} \,|\, m, n \in \mathbb{Z}, n \neq 0\}$$

where $\mathbb{Z}$ is the set of integers.

a. (20 points) Define a relation $=$ between rational numbers, such that $\frac{a}{b} = \frac{c}{d}$ if $ad = cb$. Show that this relation is an equivalence relation. (You will need to state what conditions should be satisfied if a relation is an equivalence relation.)

① $\frac{a}{b} = \frac{a}{b}$ since $ab = ab$, so the reflexive condition $x = x$ is true.

② $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc \Rightarrow \frac{c}{d} = \frac{a}{b}$, so the symmetry condition. $x = y \Rightarrow y = x$ is true.

③ $\frac{a}{b} = \frac{c}{d}, \frac{c}{d} = \frac{e}{f} \Rightarrow ad = bc, cf = de \Rightarrow a\cancel{d}\cancel{c}f = b\cancel{c}\cancel{d}e \Rightarrow af = be \Rightarrow \frac{a}{b} = \frac{e}{f}$

if $c, d \neq 0$.

We know that $b, d, f \neq 0$,

(1) If $c \neq 0$: Done.

(2) If $c = 0$: $bc = ad = 0$, $cf = de = 0$ $\Rightarrow a = 0, e = 0$, and $\frac{a}{b} = \frac{0}{b} = \frac{0}{f} = \frac{e}{f}$ #

b. (30 points) Define the addition of rational numbers and check that your definition is well defined, that is, $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$, then $\frac{a}{b} + \frac{c}{d}$ is a rational number and $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$.

Want to show that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'} = \frac{a'}{b'} + \frac{c'}{d'}$

$$\Leftrightarrow (ad+bc)\,b'd' = (a'd'+b'c')\,bd$$

Since $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$, we have $\begin{cases} ab' = a'b, \\ cd' = c'd \end{cases}$

So $ab' \cdot dd' + cd' \cdot bb' = a'b \cdot dd' + c'd \cdot bb'$

$$\Rightarrow \underbrace{ad \cdot (b'd') + cb(b'd')}_{(ad+bc)\cdot b'd'} = \underbrace{a'd' \cdot (bd) + c'b'(b \cdot d)}_{(a'd'+b'c')\cdot bd}$$

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c. (10 points) Define 0 to be the equivalence class $\mathbb{Q} := \{\frac{0}{n} \,|\, n \in \mathbb{Z}, n \neq 0\}$. For any rational number $\frac{a}{b}$, show that there exist a unique rational number $\frac{c}{d}$, such that $\frac{a}{b} + \frac{c}{d} = 0$.

$$\frac{a}{b} + \frac{c}{d} = \frac{0}{n} \Rightarrow n(ad+bc) = 0 \Rightarrow bc = (-a)\cdot d \Rightarrow \frac{c}{d} = \frac{-a}{b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

is unique #