1. (20 points) Let X and Y be two metric spaces. Give the definition of a function $f\colon X \to Y$ being continuous. You may give any equivalent definition.

Definition 1

For any $x \in X$ and $\varepsilon > 0$, there exist a real number δ such that for all $|y-x| < \delta$, we have $|f(y)-f(x)| < \varepsilon$

Definition 2

If $\{x_n\}$ is a sequence in X and $\lim_{n\to\infty} x_n = y$, then $\lim_{n\to\infty} f(x_n) = f(y)$

Definition 3

For all open sets $U \subseteq Y$, the set $f^{-1}(Y) : \{x \in X : f(x) \in U\}$ is open.

2. (20 points) Find the set S such that the function $f(x) = \frac{x+1}{x^2}$ is continuous on S but not continuous on $\mathbb{R} \setminus S$.

We claim that $S = \mathbb{R} \setminus \{0\}$

(1) $\mathbb{R} \setminus \{0\} \subseteq S$:

$x+1$ and x^2 are continuous on $\mathbb{R} \setminus \{0\}$,

so $\frac{x+1}{x^2}$ is continuous on $\mathbb{R} \setminus \{0\}$.

(2) $0 \notin S$:

Let $x_n = \frac{1}{n}$, we have $\lim_{n\to\infty} x_n = 0$

We have $f(x_n) = \frac{x_n+1}{x_n^2} > \frac{1}{x_n^2} = n^2$ for all n.

The sequence $\{f(x_n)\}_{n=1}^{\infty}$ diverges. Hence $f(x)$ is not continuous at $x = 0$.

3. Let $f(x) = 3x^3 + 2x^2 + x + 1$. Explain why the following statements are true:

(a) (10 points) The set $X = \{x \in \mathbb{R} : 0 < f(x) < 1\}$ is open.

(b) (10 points) The set $Y = \{f(x) : 0 \le x \le 1\}$ is closed and bounded.

$f(x)$ is a polynomial, f is continuous on $\mathbb{R}$

(a) For any continuous function, the preimage of a open set is open. Since $(0,1)$ is open, we have $X = f^{-1}((0,1))$ is open

(b) For any continuous function, the image of a compact set is compact.

By Heine-Borel theorem, $[0,1]$ is compact

Then $Y = f([0,1])$ is compact, and therefore it is closed and bounded.

4. (20 points) Prove that if $f: \mathbb{R} \to \mathbb{R}$ is strictly monotone (strictly increasing or decreasing) and ~~one-to-one,~~ bijective then f is continuous. Deduce that x^k is a continuous function on $\mathbb{R}$ for all k.

Let Y be the image of f.

We can define inverse function of f on Y since f is one to one.

For any $x \in \mathbb{R}$ and $\varepsilon > 0$, we want to find $\delta > 0$ s.t. for all $|z - x| < \delta$, we have

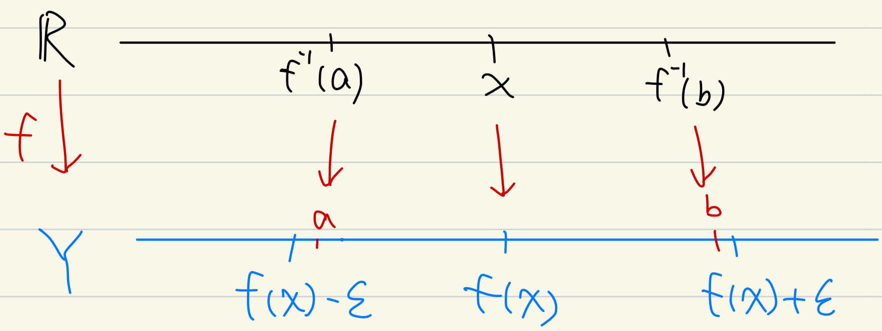
$f(x) - \delta < f(z) < f(x) + \delta$

let $a = \sup Y \cap [f(x) - \varepsilon, f(x) + \varepsilon]$

$b = \inf Y \cap [f(x) - \varepsilon, f(x) + \varepsilon]$

Because f is strictly increasing, we have

$z \in (f^{-1}(a), f^{-1}(b))$ if only if $f(x) - \varepsilon \leq f(z) \leq f(x) + \varepsilon$



$\delta = \min(|x - f^{-1}(a)|, |x - f^{-1}(b)|)$ is desired.

Let $f(x) = x^k$

① If $k > 0$, $f(x)$ is strictly increasing

② If $k < 0$, $f(x)$ is strictly decreasing

③ If $k = 0$, $f(x) = 1$ is a constant function.

Hence $f(x)$ is continuous for all $k \in \mathbb{R}$.

5. (a) (10 points) Let M be a metric space and $f\colon M \to \mathbb{R}^n$ be a function, f can be represented by functions in each coordinate, i.e., $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Prove that f is continuous if and only if f_i is continuous for all $1 \le i \le n$.

(b) (10 points) Let f be a function from $\mathbb{R}^m$ and let $f(x) = Ax + b$, where A is an $n \times m$ matrix and b is a vector in $\mathbb{R}^n$. Prove that f is continuous.

(a) For any sequence $\{x_n\}$

we have $\lim_{m\to\infty} f(x_m)$ exists if and only if $\lim_{n\to\infty} f_i(x_m)$ exist. If one of them are true,

then $\lim_{m\to\infty} f(x_m) = \left(\lim_{m\to\infty} f_i(x_m), \cdots, \lim_{m\to\infty} f_i(x_m)\right)$

Suppose $\{x_m\}$ converges to $p = (p_1, \cdots p_n)$

Then $\lim_{m\to\infty} f(x_m) = f(p)$ if and only if

$\lim_{m\to\infty} f_i(x_m) = f_i(p_i)$

Therefore, f is continuous if and only if f_i is continuous.

(b) Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ and $b = (b_1, \cdots, b_n)$

Let $A_i = (a_{i1}, \cdots, a_{im})$ be the i-th row of A.

We have $f_i(x) = A_i x + b_i = \sum_{k=1}^{m} a_{ik} x_k + b_i$.

Now we show that f_i is continuous.

For any $\varepsilon > 0$, let $\delta = \frac{1}{\max_k |a_{ik}|}$

Then for any $x, y \in X^m$ such that $\sum_{k=1}^{m} |x_k - y_k| < \delta$

we have

$$|f_i(x) - f_i(y)| = |A_i(x-y)|$$

$$= \left|\sum_{k=1}^{m} a_{ik}(x_k - y_k)\right|$$

$$< \left(\max_k a_{ik}\right) \sum_{k=1}^{m} |x_k - y_k|$$

$$< \varepsilon$$

Therefore f_i is continuous.

By 5.(a), f is continuous.