Introduction to Quantitative Methods, Quiz 9

1. (20 points) Let X and Y be two metric spaces. Give the definition of a function $f\colon X \to Y$ being continuous. You may give any equivalent definition.

2. (20 points) Find the set S such that the function $f(x) = \frac{x+1}{x^2}$ is continuous on S but not continuous on $\mathbb{R} \setminus S$.

3. Let $f(x) = 3x^3 + 2x^2 + x + 1$. Explain why the following statements are true:

   (a) (10 points) The set $X = \{x \in \mathbb{R}\colon 0 < f(x) < 1\}$ is open.

   (b) (10 points) The set $Y = \{f(x)\colon 0 \leq x \leq 1\}$ is closed and bounded.

4. (20 points) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is strictly monotone (strictly increasing or decreasing) and one-to-one, then f is continuous. Deduce that x^k is a continuous function on $\mathbb{R}$ for all k.

5. (a) (10 points) Let M be a metric space and $f\colon M \to \mathbb{R}^n$ be a function, f can be represented by functions in each coordinate, i.e., $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Prove that f is continuous if and only if f_i is continuous for all $1 \leq i \leq n$.

   (b) (10 points) Let f be a function from $\mathbb{R}^m$ and let $f(x) = Ax + b$, where A is an $n \times m$ matrix and b is a vector in $\mathbb{R}^n$. Prove that f is continuous.