1. Let $\{p_n\}$ be a sequence.

(a) (15 points) What is the meaning of ' $\lim_{n\to\infty} p_n = p$'?

(b) (15 points) What is the meaning of "$\{p_n\}$ is a Cauchy sequence."?

1. (a) For any $\varepsilon > 0$, there exists a positive integer N such that

$$d(p_n, p) < \varepsilon \quad \text{for all } n > N$$

(b) For any $\varepsilon > 0$, there exists a positive integer N such that

$$d(p_n, p_m) < \varepsilon \quad \text{for all } n, m > N.$$

We can also define Cauchy sequence in terms of diameters.

Let $E_N = \{p_n\}_{n \geq N} = \{p_N, p_{N+1}, \cdots\}$

For the same ε, it's equivalent to say

$$\operatorname{diam}(E_N) < \varepsilon.$$

2. Find the limit of each of the following sequences. You should provide an $N-\epsilon$ argument rather than just write out the answers.

(a) (10 points) $\{a_n\} \subset \mathbb{R}$ and $a_n = \frac{2n^2}{1+n^2}$

(b) (10 points) $\{b_n\} \subset \mathbb{R}^2$ and $b_n = (\frac{1}{n}, \frac{n+2}{3n})$

(c) (10 points) $\{c_n\} \subset \mathbb{R}$ and $c_n = 1 + \frac{1}{3} + \ldots + \frac{1}{3^{n-1}}$

2. (a) The sequence $\{a_n\}$ converges to 2.

Because for any $\varepsilon > 0$, we have

$$\left| \frac{2n^2}{1+n^2} - 2 \right| = \left| \frac{2}{1+n^2} \right| < \varepsilon \quad \forall\ n > \sqrt{\frac{\varepsilon}{2} - 1}.$$

(b) We first show that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n+2}{3n} = \frac{1}{3}.$$

For any $\varepsilon > 0$, we have

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \qquad \forall\ n > \frac{1}{\varepsilon}$$

and $\left| \frac{n+2}{3n} - \frac{1}{3} \right| = \left| \frac{2}{3n} \right| < \varepsilon \quad \forall\ n > \frac{2}{3\varepsilon}$

Then we know that $\{b_n\} = \{(\frac{1}{n}, \frac{n+2}{3n})\}$ converges

and $\lim_{n\to\infty} \left(\frac{1}{n}, \frac{n+2}{3n}\right) = \left(\lim_{n\to\infty} \frac{1}{n}, \lim_{n\to\infty} \frac{n+2}{3n}\right) = \left(0, \frac{1}{3}\right).$

(3) We claim that $\{c_n\}$ converges to $\frac{3}{2}$.

First, notice that

$$\left|\frac{3}{2}-c_n\right| = \left|\frac{3}{2}-\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}\right)\right|$$

$$= \frac{3}{2}\left|1-\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}\right)\right|$$

$$= \frac{3}{2}\left|1-\left(1-\frac{1}{3^n}\right)\right|$$

$$= \frac{2}{3^{n-1}}.$$

For any $\varepsilon>0$, we have

$$\left|\frac{2}{3^{n-1}}\right| < \varepsilon \quad \text{for all } n > \log_3\frac{2}{\varepsilon}-1$$

Therefore, $\{c_n\}$ converges to $\frac{3}{2}$.

3. (20 points) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ that converges to a real number d, prove that

$$\lim_{n\to\infty}(2x_n^2+3x_n+4)=2d^2+3d+4.$$

Since $\lim_{n\to\infty} x_n = d$, we have

$$\lim_{n\to\infty}(2x_n^2+3x_n+4) = 2\lim_{n\to\infty} x_n^2 + 3\lim_{n\to\infty} x_n + 4$$

$$= 2\left(\lim_{n\to\infty} x_n\right)^2 + 3\lim_{n\to\infty} x_n + 4$$

$$= 2d^2 + 3d + 4$$

4. (a) (10 points) Let $\{a_n\}$ be a sequence in a metric space (X, d). Assume that there is a subsequence of $\{a_n\}$ converges, does $\{a_n\}$ always converge? Explain your answer.

(b) (10 points) If we further assume $\{a_n\}$ is a Cauchy sequence, does $\{a_n\}$ always converge? Explain your answer. Note that if X is a general metric space that is not complete, then a Cauchy sequence in X does not always converge.

a) We construct a sequence $\{a_n\}$ and its subsequence $\{b_n\}$ by letting

$$a_n = \begin{cases} 1 & \text{for } n = 2m \\ 2 & \text{for } n = 2m-1 \end{cases} \quad \text{and} \quad b_k = a_{2k} \ \forall k$$

If $\{a_n\}$ converges to a number P, then $\lim_{n\to\infty} \{b_n\} = P$ because every subsequence of a converging sequence will converges to the same limit. But we know that $b_k = 1 \ \forall k$, so $p = 1$. This leads to a contradiction since $a_{2m-1} = 2 \ \forall m \in \mathbb{N}$. Thus $\{a_n\}$ does not converge.

(b) Let $\{b_k\}$ be a subsequence of a Cauchy sequence $\{a_n\}$ such that $b_k = a_{f(k)}$ and $f(1) < f(2) < \cdots < f(k) < \cdots$.

Assume that $\lim_{k\to\infty} b_k = p$, we show that $\lim_{n\to\infty} a_n = p$.

$\forall \varepsilon > 0$, by definition $\exists N_1\ N_2$ such that

$d(b_k, p) < \frac{\varepsilon}{2} \ \forall k \geq N_1$ and $d(a_n, a_m) < \frac{\varepsilon}{2} \ \forall n, m \geq N_2$

Let $N = \max(f(N_1), N_2)$, for all $n \geq N$ we have

$d(a_n, p) \leq d(a_n, a_{f(N)}) + d(a_{f(N)}, p) \leq \frac{\varepsilon}{2} + d(b_N, p) \leq \varepsilon$

Therefore $\{a_n\}$ converges to p.