1. (20 points) State the *Heine-Borel theorem.*

1. A subset K of $\mathbb{R}^n$ (or $\mathbb{R}$) is compact if and only if K is closed and bounded.

2. Let K be a compact subset of $\mathbb{R}^n$, x is a point in $\mathbb{R}^n$.

(a) (10 points) Let A be the set $\{z \in \mathbb{R}\colon z = d(x, y) \text{ for some } y \in K\}$, where $d(x, y)$ is the Euclidean distance in R^n. Prove that A is a compact subset of $\mathbb{R}$.

(b) (10 points) Show that there exist $y^* \in K$ such that $d(x, y^*) \leq d(x, y)$ for all $y \in K$.

(a) Proof 1:

By Heine-Borel theorem, K is closed and bounded and we want to show that A is closed and bounded.

Boundedness: Since K is bounded,

$\exists\, n \in \mathbb{R}, \bar{x} \in K$ such that $d(\bar{x}, y) \leq n \quad \forall y \in K$

Then $d(x, y) \leq d(y, \bar{x}) + d(\bar{x}, y) \leq n + d(x, \bar{x})$

$\begin{cases} d(x, \bar{x}) + n \text{ is an upper bound of } A \\ 0 \text{ is a lower bound of } A \end{cases} \Rightarrow A$ is bounded.

Closedness: Let l be a limit point of A, we want to show that $l \in A$.

For any $n \in \mathbb{N}$, there exist $y_n \in K$ s.t. $|d(x, y_n) - l| < \frac{1}{n}$

$\{y_n\}$ is a infinite subset of $\mathbb{R}^n$ $\{y_n\}$ is bounded since K is bounded

By Bolzano-Weierstrass theorem $\{y_n\}$ has a limit point y in $\mathbb{R}^n$.

Claim: $d(x, y) = l$ (thus $l \in A$)

Proof of claim: For any $\varepsilon > 0$, $\exists\, n$ s.t $d(y_n, y) < \frac{\varepsilon}{2}$ and $\frac{1}{n} < \frac{\varepsilon}{2}$

$(d(x, y_n) - l) - d(y_n, y) \leq d(x, y) - l \leq (d(x, y_n) - l) + d(y_n, y)$

$\Rightarrow \quad |d(x, y)| \leq \frac{1}{n} + \frac{\varepsilon}{2} \leq \varepsilon$

Hence $d(x, y) = l$, $l \in A$. Therefore A is compact

(a) Proof ⊇. Let $\{U_\alpha\}$ be an open cover of A

For each U_α, let $W_\alpha = \{ y \in \mathbb{R}^n : d(y,k) \in U_\alpha \}$

Check: W_α is open in $\mathbb{R}^n$

Then $\{W_\alpha\}$ is an open cover of K.
Since K is compact, there exist a finite subcover

$W_{\alpha_1} \cdots W_{\alpha_n}$ of $\{W_\alpha\}$:

Then $U_{\alpha_1}, \cdots, U_{\alpha_n}$ is a finite subcover of $\{U_\alpha\}$.

(b) Let $d^* = \inf A = \inf \{ d(y,x) : y \in K \}$

$d^* \in A$ since A is closed

By definition of A, $\exists\, y^* \in K$ s.t. $d(y^*,x) = \inf A$.

Therefore, $d(y^*,x) \leq d(y,x)$ for all $y \in K$.

3. (20 points) Let $X_{\alpha \alpha \in S}$ be a collection of connected sets such that $\bigcap_{\alpha \in S} X_a \neq \emptyset$, prove that $\bigcup_{\alpha \in S} X_a$ is connected.

3. We prove by contradiction

Suppose $\bigcup_{\alpha \in S} X_\alpha$ is not connected,

there exist non-empty separated sets A and B such that

$$\bigcup_{\alpha \in S} X_\alpha = A \cup B \quad \text{and} \quad \bar{A} \cap B = A \cap \bar{B} = \phi$$

Let x be any element in $\bigcap_{\alpha \in S} X_\alpha$, WLOG let $x \in A$.

For any $\alpha \in S$, $X_\alpha = X_\alpha \cap (\bigcup_{\alpha \in S} X_\alpha) = (X_\alpha \cap A) \cup (X_\alpha \cap B)$

Since $X_\alpha \cap A \subseteq A$ and $X_\alpha \cap B \subseteq B$, they are separated

Note that X_α is connected and $x \in X_\alpha \cap A$.

Therefore, $X_\alpha \cap B = \phi$ and $X_\alpha \subseteq A$ for all α

Then $B = B \cap (\bigcup_{\alpha \in S} X_\alpha) = \bigcup_{\alpha \in S} (X_\alpha \cap B) = \phi$, a contradiction

Hence $\bigcup_{\alpha \in S} X_\alpha$ is connected.

4. (20 points) Two sets A and B in a metric space are called separated if $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. A set S in a metric space is called *totally disconnected* if for any distinct $x, y \in S$, there exist separated sets A and B such that $x \in A$, $y \in B$ and $A \cup B = S$. Prove that the Cantor set is totally disconnected in $\mathbb{R}$.

The Cantor set is $C = \bigcap_{n=1}^{\infty} K_n$, where K_n is union of some disjoint closed intervals with the same length 3^{-n}.

Let $x, y \in C$ and $x \neq y$. There exist $n \in \mathbb{N}$ such that $3^{-n} < |x-y|$

Then x, y are not in the same interval in K_n

Suppose $x \in \left[\frac{\alpha}{3^n}, \frac{\alpha+1}{3^n}\right]$ and $y \in \left[\frac{\beta}{3^n}, \frac{\beta+1}{3^n}\right]$

We have $\alpha+1 \neq \beta$. Then the intervals $\left(-\infty, \frac{\alpha+1}{3^n} + \varepsilon\right)$ and $\left(\frac{\alpha+1}{3^n} + 2\varepsilon, \infty\right)$ are separated sets when $\varepsilon < \frac{1}{3^{n+1}}$.

Let $A = \left(-\infty, \frac{\alpha+1}{3^n} + \varepsilon\right) \cap S$ and $B = \left(\frac{\alpha+1}{3^n} + 2\varepsilon, \infty\right) \cap S$

A, B are desired separated sets.

5. (a) (10 points) Let S be a non-empty connected of $\mathbb{R}$. If there are two distinct real numbers a and b in S such that $a < b$, prove that for any $a < c < b$ we have $c \in S$

(b) (10 points) Use 5.a to show that any open connected subset S of $\mathbb{R}$ is an open interval (a, b) where $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. Here $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$ and $(a, \infty) = \{x \in \mathbb{R} : x > a\}$

5. (a) If there exist $c \in (a,b)$ such that $c \notin E$.

Let $A = (-\infty, c) \cap S$, $B = (c, \infty) \cap S$

we have $A \cup B = S$. A and B are non-empty

because $a \in A$ and $b \in B$,

Since $A \subseteq (-\infty, c)$, $B \subseteq (c, \infty)$, A and B are separated

Since S is connected, such c does not exist

, we have $c \in S$. for all $a < c < b$.

(b) Let $a = \inf S$ and $b = \sup S$, we claim that $S = (a,b)$

We want to show that $\forall\ x \in (a,b)$ we have $x \in S$.

Since $x < \sup S$, there exist $y \in S$ such that $x < y$

Since $x > \inf S$, there exist $z \in S$ Such that $x > z$

Then $z < x < y$ and we have $x \in S$ by 5.(a).

Therefore $(a,b) \subseteq S$, S is one of (a,b), $[a,b)$, $(a,b]$, $[a,b]$.

S is open $\Rightarrow S = (a,b)$.