Introduction to Quantitative Method: Quiz 6

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Brief Introduction

Some of you may be surprised by the grades on Quiz 6. This is not only due to the challenging problems but also the strict grading criteria. As the grader responsible for this quiz, I will provide you with the solutions, along with tips on improving your proof-writing skills and avoiding common pitfalls.

Problem 1. The definition of compact sets in the metric space.

Solution. The correct answer for stating that a set $X \subset M$ is compact is as follows:

- Every open covering of X has[1] a **finite** subcover.
- Every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ has a **convergent** subsequence.

The open covering definition of compactness is more general and applies to topological spaces. It can be proven equivalent to sequential compactness in the metric space, so I accepted both definitions. Below are some key points to keep in mind:

- Clearly indicate **who** the covering and subcover refer to, and avoid informal explanations.

□

[1] reduces to

Problem 2. Prove that the union of compact sets is still compact.

Solution. Using the *Heine–Borel theorem* will earn at most 70% of the full points (this applies to Problems 4 and 5 as well),[2] although no one used it correctly in this problem.

The correct approach is as follows: Take a cover of $X \cup Y$, say $\mathcal{U} = \{U_\alpha\}$. Since X is compact, $\mathcal{U}$ has a finite subcover, $\mathcal{U}_X$, that covers X. Similarly, we can define $\mathcal{U}_Y$ for Y in the same way. The union $\mathcal{U}_X \cup \mathcal{U}_Y$ forms the desired finite cover, hence $X \cup Y$ is compact

Many students received 15 points for this problem and had a large arrow marked on their solutions. This is because they took two covers, $\{U_\alpha\}$ and $\{V_\alpha\}$, for X and Y respectively, and then claimed that these could be reduced to two countable covers, thus concluding that $X \cup Y$ is covered by a countable cover.

However, this reasoning is incorrect. You cannot show that **every cover of $X \cup Y$ can be written in this form**, so this deduction is flawed.[3] $\square$

Problem 3. A continuous function maps a compact set to the compact set.

Solution. Anyone who does not use the condition that the preimage of f preserves openness will receive 0 points, as this is the critical aspect of the problem.[4]

The preservation of openness is a key characteristic of continuity in more general contexts, and you will explore this in future courses. I will provide additional notes on this after the solution.

The correct answer is as follows:[5] Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of the image, $f(X)$. By the given condition, $f^{-1}(\mathcal{U})$ is open and covers X. Using compactness, we obtain a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), \cdots, f^{-1}(U_n)\}$ that covers X. Therefore, $\{U_1, U_2, \cdots, U_n\}$ forms an open subcover[6] that covers $f(X)$ as desired. Hence, $f(X)$ is compact. $\square$

[2] According to the syllabus, you are not allowed to use it without proof.

[3] In fact, more points could have been deducted if I had graded more strictly.

[4] If you're unsure why this is so crucial, feel free to ask me in person.

[5] This problem is straightforward, and you may find it helpful to sketch a graph to solidify your understanding.

[6] *sub* to $\mathcal{U}$

Option: Definition of continuous function

As we learned in calculus, a continuous function is a function that preserves limiting action in the domain and range simultaneously, i.e., $\lim_{x\to a} f(x) = f(a)$. However, the ϵ-δ argument may be hard to apply in topological spaces. Here, we outline a proof that ***in a metric space, if the preimage of an open set under f is always open, then f is continuous***.

($\Rightarrow$) Let $f : X \to Y$ be a continuous function and $U \subset Y$ be an open set. For any point x in U, there exists a neighborhood $B_\epsilon(x) \subset U$.

By the ϵ-δ condition, we know that there is a δ-ball in X that lies entirely within $f^{-1}(U) \subset X$, which matches the definition of openness. Since open balls characterize open sets in metric spaces, this result is not surprising.

($\Leftarrow$) It's clear enough.[7]

Problem 4. Show the following 2 sets are not compact in $\mathbb{R}$

- $\mathbb{Q}$
- $A = \{\frac{1}{2^n} : n = 1, 2, 3 \cdots\}$

Solution. Many students attempted to find a counterexample-a cover that has no finite subcover-but their explanations contained several flaws.[8] I graded these attempts with 0 points because they provided no valid reasoning for their claims.

I will present both the sequentially compact and covering compact versions for you, and you can determine which one you prefer.

- **seq.** Let $\{1, 2, 3 \cdots\}$ be a sequence in $\mathbb{Q}$. It is obviously divergent, hence has no convergent subsequence.
- **cov.** Consider $\mathcal{U} = \{B_{\frac{1}{2}}(q) : q \in \mathbb{Q}\}$ be an cover of $\mathbb{Q}$. Suppose $\mathbb{Q}$ can be covered by $n < \infty$ patches of $\mathcal{U}$, says $\mathcal{U}' = \{U_1, U_2 \cdots U_n\}$. Then the diameter[9] of $\mathcal{U}'$ is at most n, which contradicts to the unboundedness of $\mathbb{Q}$.[10]
- **seq.** Clearly, 0 is the only limit point of A and every subsequence converges to it, however, $0 \notin A$ and exists some sequences has no convergent subsequence. As a result, A is not compact.
- **cov.** For $\frac{1}{\{}2^n\}$, we cover it by $U_n = B_{\frac{1}{2^{n+1000}}}(\frac{1}{2^n})$. $\mathcal{U}$ is a disjoint union and covers A. Hence, it has no finite subcover. (we can't discard any patch in the disjoint union and keep it still a cover of A)

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[7] You may try to verify every point in Y.

[8] For example, singleton is not closed, $(0, \frac{1}{2})$ is not closed, $\mathbb{R}$ covers $\mathbb{Q}$ but is finite......

[9] $\sup d(p, q)$, where $p, q \in \mathbb{Q}$.

[10] Or you can cheaply say $\mathcal{U}'$ can't cover $\mathbb{Q} \cap (-2n, 2n)$.

□

Problem 5. Any closed subset, F, of a compact set, K, is still compact.

Solution. In $\mathbb{R}^n$, we can directly use *Heine Borel Theorem* to conclude this result since a subset of a bounded set is still bounded. However, this problem is in the general metric space, you can't deduce it in this way.

It's easy to prove it in both of two definitions of the compactness, we show the covering version.

- **cov.** Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of F. Since F is closed, its complement F^c is open, and $F^c \cup \mathcal{U}$ forms a cover of K. By the compactness of K, there exists a finite subcover, $F^c \cup \{U_1, U_2, \ldots, U_n\}$, that covers K. Notice that $F \cap F^c = \emptyset$, so $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$ that covers F. Thus, F is compact.

- **seq.** Let $\{a_n\}_{n=1}^{\infty} \subset F$ be a sequence. By the compactness of K, the sequence $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence, $\{a_{n_k}\}_{k=1}^{\infty}$. Since F is closed, the limit of this subsequence lies in F. By the definition of sequential compactness, where *every sequence has a convergent subsequence*, F is indeed compact.

□

Epilogue

This is a good example of how to properly use the **given conditions**. Every condition in a mathematical theorem, problem, or experiment has meaning and purpose. When reading a proof in a textbook or online, **understanding the reasoning behind it and knowing where each condition is applied are the most important aspects.** The correctness of the logic is less significant,[11] as you are unlikely to remember all the details, but you'll retain the intuition and techniques used in the proof.

Thank you for reading this 4-page note. I hope it is useful for you in this class and in your future mathematical studies.

[11] When you're a freshman, it's better to approach math as a kind of belief to foster deeper exploration.