Introduction to Quantitative Methods, Quiz 6

1. (20 points) Define a set X is a *compact set* in metric space (M, d).
2. (20 points) Let X, Y be compact sets in a metric space M. Prove that $X \cup Y$ is compact.
3. (20 points) Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for any open set $X \subseteq \mathbb{R}$, the set $f^{-1}(X) = \{a \in \mathbb{R}\colon f(a) \in X\}$ is open. Prove that for any compact subset K of $\mathbb{R}$, its image $f(K) = \{f(b)\colon b \in K\}$ is compact.
4. For each of the following subsets of $\mathbb{R}$, prove that it is not compact in $\mathbb{R}$.
   (a) (10 points) The set of rational numbers $\mathbb{Q}$.
   (b) (10 points) The set $S = \{\frac{1}{2^n}\colon n \in \mathbb{N}\}$.
5. (20 points) Let X be a compact set in a metric space M. Prove that any closed subset of X is compact.